***N*-Tupling Transformations and Invariant Definite Integrals**

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Abstract Functions on the real number line of the type $\psi(x) = c + x - \frac{b}{x-\mu}$, with $b > 0$, have the interesting property that for any continuous, absolutely integrable function F on $\mathbb{R}$, the graph of $F(\psi(x))$ is a “doubling” of the graph of $F(x)$, while the integral over $\mathbb{R}$ remains invariant, $\int_{-\infty}^{\infty} F(\psi(x)) dx = \int_{-\infty}^{\infty} F(x) dx$. In this paper, we discover new families of n -to-1 mappings on $\mathbb{R}$ that have the same invariance property.

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Introduction

Functions on the real number line of the type

$$\psi(x) = c + x - \frac{b}{x - \mu}, \quad (1)$$

with b, c, μ real numbers, $b > 0$, called *doubling* transformations, have the interesting property that for any continuous, absolutely integrable function F on $\mathbb{R}$, we have

$$\int_{-\infty}^{\infty} F(\psi(x)) dx = \int_{-\infty}^{\infty} F(x) dx. \quad (2)$$

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Fig. 1 Graph of $y = e^{-(x-2/x)^2}$

Thus, for example,

$$\int_{-\infty}^{\infty} e^{-(x-\frac{b}{x})^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \quad (3)$$

Note that the integrand on the left-hand side can be made into a continuous function on $\mathbb{R}$ by defining its value at $x = 0$ to be 0. This result can be found for example in Wilson's *Advanced Calculus* [7, p. 386], and makes a nice exercise for a student of advanced calculus (Fig. 1).

The name *doubling* comes from the fact that ψ is a 2-to-1 function on $\mathbb{R} - \{\mu\}$, and thus the graph of $F \circ \psi$ is a *doubling* of the graph of F , as one can see in the figure above for the case $F(x) = e^{-x^2}$. The graph is doubled, while the area bounded by the graph remains the same. Hagler [4,5] and [6] made use of doubling transformations in the study of orthogonal systems of polynomials such as those of Jacobi, Hermite and Laguerre.

Our interest here is in obtaining further examples of functions satisfying the invariance property in (2). Specifically, we seek n -to-1 mappings ψ on $\mathbb{R}$ such that the identity in (2) holds. We call n the *tupling number* of ψ , and call ψ an *n-tupling transformation*. This investigation has led us to an interesting blend of calculus, linear algebra and abstract algebra.

For any n we succeed in obtaining a class of n -tupling transformations satisfying (2), specifically, the class of rational functions of the type

$$\psi(x) = c + x - \lambda^2 \sum_{k=1}^{n-1} \frac{\frac{\sin^2(j\pi/n)}{\sin^2(k\pi/n)}}{x - \mu - \lambda \frac{\sin((k-j)\pi/n)}{\sin(k\pi/n)}}, \quad (4)$$

where $c, \lambda, \mu \in \mathbb{R}, \lambda \neq 0$, and j is an integer with $1 \leq j \leq n/2, \gcd(j, n) = 1$; see “Explicit formula for $\psi(x)$ ” section. When $n = 1$, the ψ in (4) is just a translation, $\psi(x) = x + c$, while for $n = 2$, ψ is just the doubling transformation (1) given above. It is not hard to show that compositions of functions of type (4) also satisfy the invariance property (2). More interestingly, functions of type (4) admit composition-factorizations for certain composite n . For example, if $n = 6$ and ψ_6 denotes a 6-tupling transformation of type (4), then we can write $\psi_6 = \psi_2 \circ \psi_3$ for some doubling transformation ψ_2 and tripling transformation ψ_3 , each of type (4); see Examples 6, 7, 8.

To state our main result, let $L^1(\mathbb{R})$ denote the set of real valued functions F on $\mathbb{R}$ such that $|F|$ is integrable on $\mathbb{R}$, and $C_0(\mathbb{R})$ denote the set of continuous functions on $\mathbb{R}$ that vanish at infinity, that is, $\lim_{x \rightarrow \pm\infty} F(x) = 0$.

Theorem 1 *For any $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, and any ψ of the type (4), we have $F \circ \psi \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, and the equality in (2) holds. Moreover, the same conclusion follows if ψ is replaced by any composition of functions of type (4).*

It is easy to see that translations $\psi(x) = x + c$ satisfy (2) for any $F \in L^1(\mathbb{R})$. In Lemma 11 we show in fact that they are the only 1-to-1 rational functions satisfying (2) for all $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$. For values of $n \geq 2$, we do not know if the functions of type (4),

Fig. 2 Graph of $y = F \circ \psi_1 \circ \psi_2(x)$, $a = 1$, $b = 2$

Fig. 3 Graph of $y = F \circ \psi_1 \circ \psi_2(x)$, $a = 1.7$, $b = 2$

or compositions thereof, are the only n -to-1 rational functions satisfying (2). Theorem 1 is an immediate consequence of Theorem 4, and the fact that the set Ψ , defined in (5), is closed under composition. A number of open problems for further investigation are given in “Factorization in Ψ ” section.

What makes the functions in (4) special is that they satisfy their own algebraic invariance property, $\psi(\phi(x)) = \psi(x)$, for some rational function $\phi(x)$ of finite order (see (9)). For example, if ψ is the doubling function $\psi(x) = x - \frac{1}{x}$ and $\phi(x) = \frac{-1}{x}$, then ϕ is of order 2, that is $\phi(\phi(x)) = x$, and $\psi(\phi(x)) = \psi(x)$. Before getting into the discussion of constructing such functions let's look at another example.

Example 2 Let a, b be distinct positive real numbers and ψ_1, ψ_2 be doubling transformations given by $\psi_1(x) = x - \frac{(a-b)^2}{x}$, $\psi_2(x) = x - \frac{ab}{x}$. The composition $\psi_1 \circ \psi_2$ is a *quadrupling* (4-tupling) transformation given by

$$\psi(x) := \psi_1 \circ \psi_2(x) = \frac{(x^2 - a^2)(x^2 - b^2)}{x(x^2 - ab)},$$

an odd function with poles at $0, \pm\sqrt{ab}$ and zeros at $\pm a, \pm b$. Let $F(x) = \frac{1}{1+x^2}$. Then $F(\psi(x))$ is an even function with local maxima at the zeros of ψ and zeros at the poles of ψ , and, by Theorem 1, we have that

$$\pi = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} F(\psi(x)) dx = \int_{-\infty}^{\infty} \frac{x^2(x^2 - ab)^2 dx}{x^2(x^2 - ab)^2 + (x^2 - a^2)^2(x^2 - b^2)^2}.$$

It is interesting to note that the value of the integral is independent of the choice of a and b . Below is a plot of $F(\psi(x))$ with $a = 1$, $b = 2$, a *quadrupling* of the graph of $f(x) = \frac{1}{1+x^2}$.

As a approaches b , the valley between the peaks at $x = a$ and $x = b$ narrows, until we reach $a = b$ whence the graph is just a doubling of the graph of $F(x)$. Indeed, when $a = b$, $\psi_1(x) = x$ and $\psi(x) = \psi_2(x)$. Fixing $b = 2$ and letting a increase, one can animate the graph and show the peak at a passing through the peak at b , all the while conserving the total area bounded below the graph (Figs. 2, 3).

Setting the Stage for the Investigation

In seeking functions ψ satisfying the integral invariance property (2), we restrict our attention to rational functions ψ that are strictly increasing on $\mathbb{R}$, that is, strictly increasing on each open interval where they are defined, and satisfy $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$, (and consequently $\psi(x) \rightarrow -\infty$ as $x \rightarrow -\infty$). Any such ψ has a tupling number n equal to one more than the number of its real poles, where, as usual, the poles of ψ are the zeros of the denominator of ψ (written in reduced form). Strictly speaking, we can view ψ as either an n -to-1 mapping on its domain, or as an n -to-1 mapping on the extended real number line $\mathbb{R} \cup \{\infty\}$, taking the value ∞ at each of its poles, including the pole at ∞ . Further, we restrict F to be a function in $C_0(\mathbb{R}) \cap L^1(\mathbb{R})$. For any such F and ψ , we have that $F \circ \psi \in C_0(\mathbb{R})$, the continuity achieved by defining the value of $F \circ \psi$ at any pole of ψ to be zero, and that $F \circ \psi \in L^1(\mathbb{R})$ (Lemma 14). In particular, both integrals in (2) exist.

Thus our set of interest is,

$$\Psi := \left\{ \psi(x) \in \mathbb{R}(x) : \begin{array}{l} \psi(x) \text{ is strictly increasing, } \lim_{x \rightarrow \infty} \psi(x) = \infty \text{ and } \\ (1.2) \text{ holds for any } F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R}) \end{array} \right\}, \quad (5)$$

where $\mathbb{R}(x)$ is the field of rational functions. We may view the elements of Ψ as either formal expressions in an indeterminate symbol x or as functions of x , the context making it clear which vantage point we are taking. As formal expressions, the field $\mathbb{R}(x)$ is a monoid with respect to function composition (closed under compositions, associative, possessing an identity) with identity element x . Thus, we will also use the symbol x to denote the identity function on $\mathbb{R}$.

We say that an element $f \in \mathbb{R}(x)$ is invertible (in $\mathbb{R}(x)$) and has inverse g if $f \circ g = x = g \circ f$ for some $g \in \mathbb{R}(x)$. It is not hard to see (Lemma 10) that an element $\phi \in \mathbb{R}(x)$ is invertible if and only if it is an invertible linear fractional transformation,

$$\phi(x) = \frac{ax + b}{cx + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc \neq 0.$$

Note that ϕ is increasing if $ad - bc > 0$ and decreasing if $ad - bc < 0$. As functions on the complex plane, such transformations are called Möbius transformations, and play an important role in the study of analytic mappings on the complex plane; see eg. [3, Chapter 3] or [1, Chapter 3].

Any invertible rational function is certainly 1-to-1 on its domain, but the converse (oddly) is not true. For instance the polynomial $\psi(x) = x^3$ is 1-to-1 on $\mathbb{R}$ but it is not invertible as an element of $\mathbb{R}(x)$. If we restrict our attention to rational functions in Ψ this subtle distinction goes away. In Lemma 11 we show that the only 1-to-1 functions in Ψ are just the translations $\psi(x) = x + r$, $r \in \mathbb{R}$.

Note that the set Ψ is closed under function composition. Indeed, if $\psi_1, \psi_2 \in \Psi$, then $\psi_1 \circ \psi_2$ is increasing, $\psi_1 \circ \psi_2(x) \rightarrow \infty$ as $x \rightarrow \infty$, and for any $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, we have $F \circ \psi_1, F \circ \psi_1 \circ \psi_2 \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ (Lemma 14), and so applying (2) to ψ_2 with F replaced by $F \circ \psi_1$, we see that

$$\int_{-\infty}^{\infty} F(\psi_1 \circ \psi_2(x)) \, dx = \int_{-\infty}^{\infty} (F \circ \psi_1)(\psi_2(x)) \, dx = \int_{-\infty}^{\infty} F(\psi_1(x)) \, dx = \int_{-\infty}^{\infty} F(x) \, dx.$$

Thus, Ψ is a submonoid of $(\mathbb{R}(x), \circ)$.

If ψ_1 is an n -to-1 mapping in Ψ and ψ_2 is an m -to-1 mapping in Ψ , then $\psi_1 \circ \psi_2$ is an mn -to-1 mapping in Ψ . In this manner, one can construct n -to-1 mappings in Ψ by factoring

n , say $n = n_1 \cdot n_2 \cdots n_r$, and expressing ψ as a composition of functions in Ψ with tupling numbers $n_1, \dots, n_r$.

We noted above that any $\psi \in \Psi$ must have $n - 1$ distinct real poles $r_1, \dots, r_{n-1}$, for some n . In Lemma 12, we show in addition that $\psi(x) \sim x$ as $x \rightarrow \infty$, that is, $\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1$. Thus, it is reasonable to seek transformations ψ having partial fraction expansions of the type

$$\psi(x) = c + x - \frac{b_1}{x - r_1} - \cdots - \frac{b_{n-1}}{x - r_{n-1}}, \quad (6)$$

for some $c, b_i \in \mathbb{R}$ with $b_i > 0$, $1 \leq i \leq n - 1$ (although we do not know if ψ must have this form). The b_i must be positive to make ψ increasing on $\mathbb{R}$. We shall construct functions of this type in Ψ by building them out of elements of finite order.

For any rational function ϕ and positive integer k , let $\phi^k = \phi \circ \phi \circ \cdots \circ \phi$, the composition of ϕ with itself k -times. We say that ϕ has finite order if $\phi^n = x$ (the identity function) for some positive integer n , and the minimal such n is called the order of ϕ . If ϕ has order n , then ϕ is invertible ($\phi^{-1} = \phi^{n-1}$) and so, as noted above, it must be an invertible linear fractional transformation.

Let $\mathcal{G} = \mathcal{G}(\mathbb{R})$ denote the group of invertible linear fractional transformations over $\mathbb{R}$, the group operation being composition. It is easy to see (and well known) that $\mathcal{G}$ is isomorphic to $GL_2(\mathbb{R})/\Lambda$, the multiplicative group of invertible 2 by 2 matrices over the reals modulo the subgroup Λ of nonzero multiples of the identity matrix, via the correspondence

$$\phi(x) := \frac{ax + b}{cx + d} \quad \rightarrow \quad A_\phi := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Using this correspondence we can readily characterize all rational functions ϕ of order n .

Theorem 3 *Let ϕ be an increasing rational function over $\mathbb{R}$. Then ϕ is of order $n \geq 2$ if and only if ϕ is a linear fractional transformation of the form*

$$\phi(x) = 2\lambda \cos(j\pi/n) + \mu - \frac{\lambda^2}{x - \mu}, \quad (7)$$

for some $\lambda, \mu \in \mathbb{R}$, $\lambda \neq 0$, and integer j with $1 \leq j \leq n/2$, $\gcd(j, n) = 1$.

Given any increasing linear fractional transformation ϕ of order $n \geq 2$ we define

$$\psi(x) = c + x + \phi(x) + \phi^2(x) + \cdots + \phi^{n-1}(x), \quad (8)$$

with $c \in \mathbb{R}$. It is plain that $\psi(x)$ is increasing on $\mathbb{R}$, satisfies $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$, and that ψ is invariant under composition by ϕ , that is,

$$\psi(\phi(x)) = \psi(x). \quad (9)$$

This invariance property is the key ingredient needed to prove that any such ψ satisfies the integral invariance property (2).

Theorem 4 *If $\phi(x)$ is an increasing linear fractional transformation of order $n \geq 2$ and $\psi(x)$ is defined by (8), then $\psi \in \Psi$ and $\psi(x)$ is an n -to-1 mapping on the extended real number line.*

The proofs of Theorem 3 and Theorem 4 are given in “Proof of Theorem 3” and “Proof of Theorem 4” sections respectively. Putting the two theorems together yields the explicit class of n -to-1 mappings $\psi \in \Psi$, given in (4). The details of this derivation are provided in “Explicit formula for $\psi(x)$ ” section.

Example 5 In this example we make explicit the formula in (4) for $n = 2, 3$ and 4. If $n = 2$, then $j = 1$, $\phi(x) = \mu - \frac{\lambda^2}{x-\mu}$, and

$$\psi(x) = c + x - \frac{\lambda^2}{x - \mu}, \quad \lambda \neq 0, c, \mu \in \mathbb{R}. \quad (10)$$

This recaptures the original transformation in (1). If $n = 3$, then $j = 1$, $\phi(x) = \lambda + \mu - \frac{\lambda^2}{x-\mu}$, and

$$\psi(x) = c + x - \frac{\lambda^2}{x - \mu} - \frac{\lambda^2}{x - \mu - \lambda}, \quad \lambda \neq 0, c, \mu \in \mathbb{R}. \quad (11)$$

If $n = 4$, then $j = 1$, $\phi(x) = \sqrt{2}\lambda + \mu - \frac{\lambda^2}{x-\mu}$, and

$$\psi(x) = c + x - \frac{\lambda^2}{x - \mu} - \frac{\lambda^2/2}{x - \mu - \lambda \frac{\sqrt{2}}{2}} - \frac{\lambda^2}{x - \mu - \sqrt{2}\lambda}, \quad \lambda \neq 0, c, \mu \in \mathbb{R}. \quad (12)$$

Factorization in Ψ

Recall, every transformation ψ in Ψ is a strictly increasing n -to-1 mapping on $\mathbb{R}$ for some n , called the tupling number of ψ . Also, the set of invertible transformations in Ψ is just the set of translations $x + a$, $a \in \mathbb{R}$. These are the only elements of Ψ with tupling number 1. For $\alpha, \beta \in \Psi$, we say that α is a *factor* of β if $\beta = \alpha \circ \gamma$ or $\beta = \gamma \circ \alpha$ for some $\gamma \in \Psi$. We call a noninvertible transformation $\psi \in \Psi$ *irreducible* if it cannot be expressed as a composition of two non-invertible transformations in Ψ . Otherwise it is called *composite*. In the latter case, $\psi = \psi_1 \circ \psi_2$ for some non-invertible $\psi_1, \psi_2 \in \Psi$. Certainly, if ψ has a prime tupling number, then ψ is irreducible. For composite tupling numbers, transformations of type (4) need not be irreducible as the following examples show for the cases $n = 4, 6$ and 8.

Before stating the examples let us note that the set Ψ is invariant under translations, $\psi(x) \rightarrow \psi(x+a)$ and $\psi(x) \rightarrow \psi(x)+b$, and invariant under the dilation $\psi(x) \rightarrow \lambda\psi(x/\lambda)$. Indeed, for $\lambda > 0$, $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, $\psi \in \Psi$,

$$\int_{-\infty}^{\infty} F(\lambda\psi(x/\lambda)) dx = \lambda \int_{-\infty}^{\infty} F(\lambda\psi(u)) du = \lambda \int_{-\infty}^{\infty} F(\lambda u) du = \int_{-\infty}^{\infty} F(v) dv,$$

where in the second to last step we applied (2) to the function $F(\lambda x)$. A similar argument holds for $\lambda < 0$. Thus, in the study of transformations ψ of type (4), we may assume that $c = 0$, $\mu = 0$ and $\lambda = 1$, that is, ψ takes on a *generic* form,

$$\psi(x) = x - \sum_{k=1}^{n-1} \frac{\frac{\sin^2(j\pi/n)}{\sin^2(k\pi/n)}}{x - \frac{\sin((k-j)\pi/n)}{\sin(k\pi/n)}}, \quad (13)$$

for some j with $1 \leq j \leq n/2$, $\gcd(j, n) = 1$. We omit the details in the examples since, for the most part, they were obtained by rather brute force methods, and it would be nice to establish instead a theoretical framework for the existence of such factorizations. The guiding principle we used was to match the poles and residues of the two sides.

Example 6 Factoring a 4-tupling transformation. Let $\psi_4(x)$ be a generic 4-tupling transformation as in (13) ($n = 4$, $j = 1$), so that,

$$\psi_4(x + \frac{\sqrt{2}}{2}) = x + \frac{\sqrt{2}}{2} - \frac{1}{x + \frac{\sqrt{2}}{2}} - \frac{1}{2x} - \frac{1}{x - \frac{\sqrt{2}}{2}}.$$

Put

$$\alpha(x) := x + \frac{\sqrt{2}}{2} - \frac{2}{x}, \quad \beta(x) := x - \frac{1}{2x}.$$

Then α, β are doubling transformations of type (10) and

$$\begin{aligned} \alpha(\beta(x)) &= x + \frac{\sqrt{2}}{2} - \frac{1}{2x} - \frac{2}{x - \frac{1}{2x}} = x + \frac{\sqrt{2}}{2} - \frac{1}{2x} - \left(\frac{1}{x - \frac{\sqrt{2}}{2}} + \frac{1}{x + \frac{\sqrt{2}}{2}} \right) \\ &= \psi_4\left(x + \frac{\sqrt{2}}{2}\right), \end{aligned}$$

that is, $\psi_4(x) = \alpha \circ \beta(x - \frac{\sqrt{2}}{2})$.

Example 7 Factoring a 6-tupling transformation. Suppose that $n = 6$ and that $\psi_6(x)$ is the generic 6-tupling transformation (13) ($j = 1$). Then

$$\psi_6(x) = x - \frac{1}{x} - \frac{1}{3\left(x - \frac{\sqrt{3}}{3}\right)} - \frac{1}{4\left(x - \frac{\sqrt{3}}{2}\right)} - \frac{1}{3\left(x - \frac{2\sqrt{3}}{3}\right)} - \frac{1}{x - \sqrt{3}}.$$

Let ψ_2, ψ_3 be doubling and tripling transformations of types (10), (11), given by

$$\psi_2(x) := x + \frac{\sqrt{3}}{2} - \frac{9}{4x}, \quad \psi_3(x) := x - \frac{\sqrt{3}}{6} - \frac{1}{3x} - \frac{1}{3\left(x - \frac{\sqrt{3}}{3}\right)}.$$

Then

$$\psi_6(x) = \psi_2 \circ \psi_3\left(x - \frac{\sqrt{3}}{3}\right).$$

Example 8 Factoring an 8-tupling transformation. Suppose that $n = 8$ and that $\psi_8(x)$ is the generic 8-tupling transformation with $j = 1$,

$$\psi_8(x) = x - \sum_{k=1}^7 \frac{\frac{\sin^2(\pi/8)}{\sin^2(k\pi/8)}}{x - \frac{\sin((k-1)\pi/8)}{\sin(k\pi/8)}}.$$

Let α, β, γ be doubling transformations given by

$$\begin{aligned} \alpha(x) &:= x - \frac{(2 - \sqrt{2})/4}{x}, \\ \beta(x) &:= x - \frac{2 - \sqrt{2}}{x}, \\ \gamma(x) &:= x + \frac{1}{2}\sqrt{2 + \sqrt{2}} - \frac{4(2 - \sqrt{2})}{x}. \end{aligned}$$

Then,

$$\psi_8(x) = \gamma \circ \beta \circ \alpha\left(x - \frac{1}{2}\sqrt{2 + \sqrt{2}}\right).$$

The next example gives families of composite 4-tupling transformations in Ψ that are not of type (12).

Example 9 In [2] the authors give criteria for determining when a rational function is a composition of two doubling transformations, from which one can readily create examples having particularly nice sets of zeros and poles, such as Example 2 and the following; see [2, Examples 4.1, 4.2, 4.3].

$$\alpha \circ \beta(x) = \frac{(x-a)(x+a^2)(x-a^3)(x+a^4)}{x(x-a^2)(x+a^3)}, \quad (a > 1); \quad (14)$$

$$\gamma \circ \delta(x) = \frac{(x^2-a^2)(x^2-a^8)}{x(x+a^2)(x-a^3)}, \quad (a > 1); \quad (15)$$

where

$$\begin{aligned} \alpha(x) &= x + a(a-1)^3 - \frac{2a^3(a-1)^2(a^2+1)}{x}, & \beta(x) &= x + (a^3-a^2) - \frac{a^5}{x}, \\ \gamma(x) &= x + 2(a^3-a^2) - \frac{a^2(a^4-1)(a^2-1)}{x}, & \delta(x) &= x + (a^2-a^3) - \frac{a^5}{x}. \end{aligned}$$

It is plain that these examples are not transformations of the type (12). Indeed, the poles of transformations of type (12) are always in arithmetic progression.

There remain a number of open problems:

Open Problem 1. Characterize the set of all irreducible transformations in Ψ . In particular, which elements of type (4) are irreducible? The examples above show that any ψ of type (4) with tupling number 4, 6 or 8 is composite. Can further examples of this type be found?

Open Problem 2. Does every transformation in Ψ have an essentially unique decomposition as a composition of irreducible transformations? By essentially unique, we mean up to multiples of invertible factors. For instance if $\psi = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ then we also have $\psi = (\alpha_1 \circ \phi) \circ (\phi^{-1} \circ \alpha_2 \circ \phi) \cdots (\phi^{-1} \circ \alpha_k)$ for any translation ϕ . In [2] the authors proved that such is the case if one restricts Ψ to compositions of doubling transformations.

Open Problem 3. Let Ψ_0 be the semigroup of transformations generated (under composition) by the transformations of type (4). Since Ψ is closed under compositions, $\Psi_0 \subseteq \Psi$. Does $\Psi_0 = \Psi$?

Open Problem 4. Does every element of Ψ have a partial fraction expansion of the type (6)? In Theorem 6 we show that every element of Ψ_0 has such an expansion.

Open Problem 5. Are there any rational functions satisfying the invariance property (2) for any $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, that are not in Ψ or $-\Psi$?

Lemmas

Lemma 10 If $\phi(x) \in \mathbb{R}(x)$ is an invertible rational function with respect to composition, then $\phi(x)$ is an invertible linear fractional transformation.

Proof Say $\phi(x) = \frac{p(x)}{q(x)}$, with $p(x), q(x)$ relatively prime polynomials over $\mathbb{R}$. Since $\phi(x)$ is invertible (in $\mathbb{R}(x)$), it is 1-to-1 on $\mathbb{C}$ and so the equation $\phi(x) = 0$ has at most one solution in $\mathbb{C}$. Thus $p(x)$ has at most one complex zero, and its multiplicity is at most one since, being an invertible function, $\phi'(x)$ is nonzero on $\mathbb{C}$ and consequently the system $p(x) = p'(x) = 0$

has no solution. Thus $p(x)$ is of degree at most 1. Since $1/\phi(x)$ is also invertible, we see that $q(x)$ also is of degree at most one. $\square$

Lemma 11 *If ψ is a 1-to-1 rational function on $\mathbb{R}$ such that the integral invariance property (2) holds for any $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, then ψ is a translation.*

Proof Let $\psi \in \Psi$ be a 1-to-1 rational function on $\mathbb{R}$. Let β denote its inverse (as a function on the reals). In particular, β is continuous on $\mathbb{R}$, wherever it is defined. Let $[a, b]$ be any interval contained in the domain of ψ . On this interval, ψ has no pole and must be either increasing or decreasing, so we'll assume the former. Let $1_{[a,b]}$ denote the characteristic function on the interval $[a, b]$, and for any $\epsilon > 0$, let F be a continuous function on $\mathbb{R}$ agreeing with $1_{[a,b]}$ on $[a, b]$, supported on $(a - \epsilon, b + \epsilon)$, and satisfying $0 \leq F(x) \leq 1$ for all x . Such a function is called a *majorizing* function for $1_{[a,b]}$. We assume that ϵ is sufficiently small so that no pole of ψ is in $(a - \epsilon, b + \epsilon)$. Plainly, $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$. Write $F(x) = 1_{[a,b]}(x) + E(x)$, where $E(x)$ is a function supported on $(a - \epsilon, a) \cup (b, b + \epsilon)$ with $0 \leq E(x) \leq 1$ for all x . Then by (2) we have

$$\begin{aligned} b - a &= \int_{-\infty}^{\infty} 1_{[a,b]}(x) dx \leq \int_{-\infty}^{\infty} F(x) dx = \int_{-\infty}^{\infty} F(\psi(x)) dx \\ &= \int_{-\infty}^{\infty} 1_{[a,b]}(\psi(x)) + E(\psi(x)) dx \\ &\leq \int_{-\infty}^{\infty} 1_{[a,b]}(\psi(x)) + 1_{[a-\epsilon,a]}(\psi(x)) + 1_{[b,b+\epsilon]}(\psi(x)) dx \\ &= (\beta(b) - \beta(a)) + (\beta(a) - \beta(a - \epsilon)) + (\beta(b + \epsilon) - \beta(b)), \end{aligned}$$

and so

$$b - a \leq \beta(b + \epsilon) - \beta(a - \epsilon).$$

Since this inequality holds for any $\epsilon > 0$ we conclude, by the continuity of β , that $b - a \leq \beta(b) - \beta(a)$. Using a *minorizing* function, one obtains in a similar manner the opposite inequality, and so $b - a = \beta(b) - \beta(a)$. Thus, for any such $a < b$ we have

$$\frac{\beta(b) - \beta(a)}{b - a} = 1.$$

Taking the limit as $b \rightarrow a$ we see that $\beta'(a) = 1$ for any real number a where ψ is defined. Thus $\beta(x) = x + c$ for some constant c on any open interval on which it is defined, the constant c a priori depending on the interval, and $\psi(x) = x - c$ on a corresponding interval. Being a rational function, we must then have $\psi(x) = x - c$ on $\mathbb{R}$. $\square$

Lemma 12 *If $\psi \in \Psi$ then $\psi(x) \sim x$ as $x \rightarrow \infty$, that is, $\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1$.*

Proof Let $\psi \in \Psi$ be an n -to-1 increasing function with poles $r_1 < r_2 < \dots < r_{n-1}$. Put $r_0 = -\infty, r_n = \infty$. Let β_i denote the inverse function of $\psi|_{(r_i, r_{i+1})}$, the restriction of ψ to the interval (r_i, r_{i+1}) , $0 \leq i \leq n - 1$. Note, β_i is continuously differentiable on $\mathbb{R}$. Then, arguing as in the previous lemma, we obtain

$$b - a = \sum_{i=0}^{n-1} (\beta_i(b) - \beta_i(a)).$$

Dividing by $b - a$ and taking the limit as $b \rightarrow a$, we see that $\sum_{i=0}^{n-1} \beta'_i(x) = 1$ for all x , and thus $\sum_{i=0}^{n-1} \beta_i(x) = x + c$, for some constant c . Inserting $\psi(x)$ for x , we have $\sum_{i=0}^{n-1} \beta_i(\psi(x)) = \psi(x) + c$. Now for any $x \in (r_{n-1}, \infty)$, we have $\beta_{n-1}(\psi(x)) = x$, and so for any such x , we have

$$x + \sum_{i=0}^{n-2} \beta_i(\psi(x)) = \psi(x) + c.$$

Observing that $\lim_{x \rightarrow \infty} \beta_i(\psi(x)) = r_{i+1}$, for $0 \leq i \leq n-2$, we obtain

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = \lim_{x \rightarrow \infty} \frac{x - c + \sum_{i=0}^{n-2} \beta_i(\psi(x))}{x} = 1.$$

□

Lemma 13 *For any $\psi \in \Psi$ there exists a positive constant B such that $\psi'(x) \geq B$ for all x in the domain of ψ .*

Proof Since $\psi(x) \sim x$ as $x \rightarrow \infty$, we have

$$\psi(x) = x + c + \frac{r(x)}{g(x)}, \quad \psi'(x) = 1 + \frac{g(x)r'(x) - r(x)g'(x)}{g(x)^2},$$

for some polynomials $r(x)$, $g(x)$ with either $r(x)$ identically zero, or $\deg(r(x)) < \deg(g(x))$. Thus, $\psi'(x) \rightarrow 1$ as $x \rightarrow \pm\infty$. In particular, $\psi'(x) > \frac{1}{2}$ on some neighborhood of $\pm\infty$, say $(-\infty, s) \cup (t, \infty)$, $s, t \in \mathbb{R}$. Also, at each real pole of ψ the slope goes to ∞ from the left and the right. Thus there exists an open interval about each pole on which $\psi'(x) > \frac{1}{2}$. The complement of the union of these open intervals is a compact set. Since, ψ' is continuous on this compact set and strictly positive, its minimum value on this set must be a positive number. □

Lemma 14 *Let $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, and $\psi \in \Psi$. Then $F \circ \psi \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$.*

Proof We noted earlier that for any such F and ψ , $F \circ \psi \in C_0(\mathbb{R})$, the continuity achieved by defining $F \circ \psi$ to be zero at any pole of ψ . We turn now to the integrability of $F \circ \psi$. Let $r_1 < r_2 < \dots < r_{n-1}$ be the distinct poles of ψ and put $r_0 = -\infty$, $r_n = \infty$. For $0 \leq i \leq n-1$, since ψ is an increasing, continuously differentiable function on the interval (r_i, r_{i+1}) with image $(-\infty, \infty)$, we see, upon making a substitution, that

$$\int_{-\infty}^{\infty} |F(x)| dx = \int_{r_i}^{r_{i+1}} |F(\psi(u))| \psi'(u) du. \quad (16)$$

To be precise, one can break up the integral on the left-hand side into two pieces, say over $(-\infty, 0]$ and over $[0, \infty)$. For $0 \leq i \leq n-1$, let z_i be the zero of ψ in the interval (r_i, r_{i+1}) . Then, for $z_i < t < r_{i+1}$, we have

$$\int_0^{\psi(t)} |F(x)| dx = \int_{z_i}^t |F(\psi(u))| \psi'(u) du.$$

Taking the limit as $t \rightarrow r_{i+1}$, yields $\int_0^{\infty} |F(x)| dx = \int_{z_i}^{r_{i+1}} |F(\psi(u))| \psi'(u) du$. Doing the same thing for the second piece, and then putting the two pieces together, yields the equality in (16). Since (16) holds for $i = 0, \dots, n-1$, we obtain

$$\int_{-\infty}^{\infty} |F(\psi(u))| \psi'(u) du = \sum_{i=0}^{n-1} \int_{r_i}^{r_{i+1}} |F(\psi(u))| \psi'(u) du = n \int_{-\infty}^{\infty} |F(x)| dx.$$

In particular, since $\psi'(u)$ is always positive and $F \in L^1(\mathbb{R})$, we see that $F(\psi(x))\psi'(x) \in L^1(\mathbb{R})$. Now, by Lemma 13, there exists a $B > 0$ such that $\psi'(x) \geq B$ for all x in the domain of ψ . Thus $|F(\psi(u))| \leq \frac{1}{B} |F(\psi(u))| \psi'(u)$ for all u in the domain of ψ , and therefore $F \circ \psi \in L^1(\mathbb{R})$. $\square$

Proof of Theorem 3

Let ϕ be an increasing rational function on $\mathbb{R}$ of order $n \geq 2$. Then, as noted earlier, ϕ is invertible and so by Lemma 10, ϕ is a linear fractional transformation. If ϕ is a linear polynomial then it is plain that ϕ has finite order if and only if ϕ is the identity function. Thus, since $n > 1$, ϕ is not a polynomial and so it must have a pole. Note that, since ϕ has order n , then so does any conjugate $\alpha^{-1} \circ \phi \circ \alpha$ of ϕ . In particular, letting $\alpha(x) = x - \mu$, $\mu \in \mathbb{R}$, we see that $\phi(x - \mu) + \mu$ has order n . Thus, by making a translation if necessary, we may assume that ϕ has a pole at 0. Any increasing, linear fractional transformation with pole at 0 has the form $\phi(x) = a - \frac{b}{x}$ with $a, b \in \mathbb{R}$, $b > 0$, and corresponds to a matrix A_ϕ as given below,

$$\phi(x) := a - \frac{b}{x} = \frac{ax - b}{x} \rightarrow A_\phi := \begin{bmatrix} a & -b \\ 1 & 0 \end{bmatrix}.$$

Since $\phi^n(x) = x$, we have $A_\phi^n = CI$ for some nonzero real number C , where I is the identity matrix. The minimal polynomial for A_ϕ is just the characteristic polynomial $\det(A_\phi - tI) = t^2 - at + b$, and so this polynomial must be a divisor of $t^n - C$. Thus $t^2 - at + b = (t - r_1)(t - r_2)$ for some distinct n -th roots r_1, r_2 of C . If r_1, r_2 are real then we must have n even, $r_1 = -r_2$ and $b = -r_2^2$, contradicting our assumption that b is positive. Therefore r_1 and r_2 must be complex conjugates. If $C > 0$, say $C = \lambda^n$ for some $\lambda > 0$, then $r_1 = \lambda e^{\frac{2j\pi i}{n}}$ for some integer j , while if $C < 0$, say $C = -\lambda^n$ with $\lambda > 0$, then $r_1 = \lambda e^{\frac{(2j+1)\pi i}{n}}$, for some integer j . Putting the two cases together, we can say that

$$r_1 = \lambda e^{\frac{j\pi i}{n}}, \quad r_2 = \lambda e^{\frac{-j\pi i}{n}}, \quad a = 2\lambda \cos(j\pi/n), \quad b = \lambda^2, \quad (17)$$

for some j with $1 \leq j \leq n-1$, and some $\lambda > 0$. Replacing j with $n-j$ has the effect of changing the sign of λ (and interchanging r_1 and r_2). Since we are assuming that ϕ has order n , we must have $\gcd(j, n) = 1$. Thus ϕ has the form

$$\phi(x) = 2\lambda \cos(j\pi/n) - \frac{\lambda^2}{x}, \quad 1 \leq j \leq n/2, \quad \gcd(j, n) = 1, \quad \lambda \neq 0. \quad (18)$$

To complete the proof of Theorem 3, we simply note that if ϕ has a pole at μ then $\phi(x + \mu) - \mu$ must have the form above.

Explicit formula for $\psi(x)$

Suppose now that $\psi(x) = c + x + \phi(x) + \dots + \phi^{n-1}(x)$, with $\phi(x)$ as in (18). The values r_1 and r_2 in (17) are eigenvalues of A_ϕ with corresponding eigenvectors $(r_1, 1)$, $(r_2, 1)$, and so,

$$A_\phi = PDP^{-1}, \quad \text{with } D = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, \quad \text{and } P = \begin{bmatrix} r_1 & r_2 \\ 1 & 1 \end{bmatrix}.$$

It follows that for any positive integer k , the transformation ϕ^k corresponds to the matrix

$$\begin{aligned} A_\phi^k &= PD^kP^{-1} = \frac{1}{r_1 - r_2} \begin{bmatrix} r_1 & r_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r_1^k & 0 \\ 0 & r_2^k \end{bmatrix} \begin{bmatrix} 1 & -r_2 \\ -1 & r_1 \end{bmatrix} \\ &= \frac{1}{r_1 - r_2} \begin{bmatrix} r_1^{k+1} - r_2^{k+1} & \lambda^2(r_2^k - r_1^k) \\ r_1^k - r_2^k & \lambda^2(r_2^{k-1} - r_1^{k-1}) \end{bmatrix}. \end{aligned}$$

Thus, for any positive integer k ,

$$\phi^k(x) = \frac{\lambda \sin(j(k+1)\pi/n)x - \lambda^2 \sin(jk\pi/n)}{\sin(jk\pi/n)x - \lambda \sin(j(k-1)\pi/n)}, \quad (19)$$

an increasing function with pole at

$$\alpha_k := \lambda \frac{\sin(j(k-1)\pi/n)}{\sin(jk\pi/n)}, \quad 1 \leq k \leq n-1. \quad (20)$$

Doing a long division and then using the trigonometric identity,

$$1 - \frac{\sin(j(k+1)\pi/n) \sin(j(k-1)\pi/n)}{\sin^2(jk\pi/n)} = \frac{\sin^2(j\pi/n)}{\sin^2(jk\pi/n)}, \quad (21)$$

we obtain

$$\begin{aligned} \phi^k(x) &= \lambda \frac{\sin(j(k+1)\pi/n)}{\sin(jk\pi/n)} - \frac{\lambda^2 \left(1 - \frac{\sin(j(k+1)\pi/n) \sin(j(k-1)\pi/n)}{\sin^2(jk\pi/n)} \right)}{x - \lambda \frac{\sin(j(k-1)\pi/n)}{\sin(jk\pi/n)}} \\ &= \lambda \frac{\sin(j(k+1)\pi/n)}{\sin(jk\pi/n)} - \lambda^2 \frac{\frac{\sin^2(j\pi/n)}{\sin^2(jk\pi/n)}}{x - \lambda \frac{\sin(j(k-1)\pi/n)}{\sin(jk\pi/n)}}. \end{aligned}$$

After absorbing constants into the value c one obtains from the definition of ψ ,

$$\psi(x) := c + x + \sum_{k=1}^{n-1} \phi^k(x) = c' + x - \lambda^2 \sum_{k=1}^{n-1} \frac{\frac{\sin^2(j\pi/n)}{\sin^2(jk\pi/n)}}{x - \lambda \frac{\sin(j(k-1)\pi/n)}{\sin(jk\pi/n)}}, \quad (22)$$

for some constant c' . Since $\gcd(j, n) = 1$, as k runs from 1 to $n-1$, jk also runs through a complete set of nonzero residues (mod n), and so making a change of variables we arrive at the formula in (4) (with $\mu = 0$).

Proof of Theorem 4

Let ϕ be an increasing linear fractional transformation of order $n \geq 2$ and ψ be defined by $\psi(x) = c + x + \phi(x) + \phi^2(x) + \dots + \phi^{n-1}(x)$, with $c \in \mathbb{R}$. As before, we may assume that the pole of ϕ is 0, and thus ϕ is as given (18), and ψ as given in (22). Plainly, ψ is a strictly increasing function on $\mathbb{R}$ with $n-1$ distinct real poles, $\alpha_1, \dots, \alpha_{n-1}$ as given in (20), and so it is an n -to-1 mapping on $\mathbb{R}$. Moreover, by (22), $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Put $\phi_0(x) = x$, and for $1 \leq k \leq n-1$, $\phi_k(x) = \phi^k(x)$. Then, since $\phi^n(x) = x$, we have

$$\psi(\phi_k(x)) = \psi(x), \quad \text{for } 1 \leq k \leq n-1. \quad (23)$$

Note also that

$$\psi(x) = c + \phi_0(x) + \phi_1(x) + \cdots + \phi_{n-1}(x),$$

and that ψ is continuously differentiable on its domain, with

$$\psi'(x) = \phi'_0(x) + \phi'_1(x) + \cdots + \phi'_{n-1}(x).$$

We order the poles,

$$\alpha_{i_1} < \alpha_{i_2} < \cdots < \alpha_{i_{n-1}},$$

and put $\alpha_{i_0} := -\infty$, $\alpha_{i_n} := \infty$.

Let $F \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$. In particular, F is integrable on $\mathbb{R}$ and, by Lemma 14, so is $F \circ \psi$. Noting that for $0 \leq \ell \leq n-1$, ψ is an increasing, continuously differentiable function on the interval $(\alpha_{i_\ell}, \alpha_{i_{\ell+1}})$, with image $(-\infty, \infty)$ we see upon making a u -substitution, that

$$\int_{-\infty}^{\infty} F(x) dx = \int_{\alpha_{i_\ell}}^{\alpha_{i_{\ell+1}}} F(\psi(u)) \psi'(u) du.$$

Thus, we obtain

$$\begin{aligned} n \int_{-\infty}^{\infty} F(x) dx &= \sum_{\ell=0}^{n-1} \int_{\alpha_{i_\ell}}^{\alpha_{i_{\ell+1}}} F(\psi(u)) \psi'(u) du \\ &= \sum_{\ell=0}^{n-1} \int_{\alpha_{i_\ell}}^{\alpha_{i_{\ell+1}}} F(\psi(u)) \sum_{k=0}^{n-1} \phi'_k(u) du \\ &= \sum_{\ell=0}^{n-1} \int_{\alpha_{i_\ell}}^{\alpha_{i_{\ell+1}}} \sum_{k=0}^{n-1} F(\psi(u)) \phi'_k(u) du. \end{aligned}$$

At this point we would like to express the integral of the sum as a sum of the integrals, but this requires each piece to be integrable. We claim, in fact, that each piece is absolutely integrable. Indeed, since $0 < \phi'_k(u) \leq \psi'(u)$ for all u (where defined) we have $|F(\psi(u)) \phi'_k(u)| \leq |F(\psi(u)) \psi'(u)|$, but the latter function is integrable on $(\alpha_{i_\ell}, \alpha_{i_{\ell+1}})$, since $F \circ \psi \in L^1(\mathbb{R})$. Thus, we have

$$\begin{aligned} n \int_{-\infty}^{\infty} F(x) dx &= \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} \int_{\alpha_{i_\ell}}^{\alpha_{i_{\ell+1}}} F(\psi(u)) \phi'_k(u) du \\ &= \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} \int_{\alpha_{i_\ell}}^{\alpha_{i_{\ell+1}}} F(\psi(\phi_k(u))) \phi'_k(u) du, \quad (\text{since } \psi(\phi_k(x)) = \psi(x)), \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} F(\psi(\phi_k(u))) \phi'_k(u) \, du \\
&= \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} F(\psi(t)) \, dt, \quad (\text{letting } t = \phi_k(u)) \\
&= n \int_{-\infty}^{\infty} F(\psi(t)) \, dt.
\end{aligned}$$

Partial Fraction Expansions

Let Ψ_0 be the semigroup generated by elements of type (4), under composition.

Theorem 15 Any transformation $\psi \in \Psi_0$ is an increasing n -tupling transformation (for some n) with partial fraction expansion of the type,

$$\psi(x) = c + x - \frac{b_1}{x - r_1} - \frac{b_2}{x - r_2} - \cdots - \frac{b_{n-1}}{x - r_{n-1}}, \quad (24)$$

for some real number c , positive real numbers b_i , and distinct real numbers r_i , $1 \leq i \leq n-1$.

Proof By definition, the generators of Ψ_0 have partial fraction expansions of type (24), and so it suffices to show that if ψ_1, ψ_2 are transformations type (24), then so is their composition. Clearly, any function of type (24) is a strictly increasing n -to-1 function on the extended real number line. Thus if ψ_1 is m -to-1 and ψ_2 is n -to-1 then $\psi_1 \circ \psi_2$ is a strictly increasing mn -to-1 function. In particular, it has $mn - 1$ distinct real poles, say $w_1, \dots, w_{mn-1}$. Moreover, the poles must be simple, and the residue of each pole w_i (that is, the coefficient of $\frac{1}{x-w_i}$) must be negative (in order for $\psi_1 \circ \psi_2$ to be increasing on a neighborhood of the pole.) Also note that, written in reduced form, $\psi_1 = \frac{f_1}{g_1}$, $\psi_2 = \frac{f_2}{g_2}$, for some polynomials f_1, g_1, f_2, g_2 of degrees $m, m-1, n, n-1$ respectively. Thus the composition has the form $\psi_1 \circ \psi_2 = \frac{f_3}{g_3}$ for some polynomials f_3, g_3 of degrees $mn, mn-1$ respectively, and so $\psi_1 \circ \psi_2$ has a partial fraction expansion of the type

$$\psi_1 \circ \psi_2(x) = c + bx - \frac{b_1}{x - w_1} - \cdots - \frac{b_{mn-1}}{x - w_{mn-1}},$$

for some real numbers c, b and positive real numbers $b_1, \dots, b_{mn-1}$. Since $\psi_1 \circ \psi_2(x) \sim x$ as $x \rightarrow \infty$ (by Lemma 12 or as seen directly), we must have $b = 1$. $\square$

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