Some Exact Solutions for Sediment Transport Equations

B. Bira · T. Raja Sekhar

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Abstract This paper presents some exact solutions for the nonlinear hyperbolic partial differential equations (PDEs), governing sediment transport in channels and rivers. The infinitesimal transformations for the governing system of PDEs are derived by using Lie group analysis. These are used to reduce the number of independent variables and consequently reduces the original system of PDEs into system of ordinary differential equations (ODEs). Moreover, the reduced system of ODEs are solved and new classes of exact group invariant solutions for the governing PDEs are constructed.

Keywords Hyperbolic PDEs · Sediment transport · Lie group analysis · Infinitesimal transformations · Exact solution

Introduction

In recent years, sediment transport modeling is playing a key role in realistic river hydraulic community. Many physical phenomena such as floods, meandering, sediment load computation, river bed aggradation, channel design and navigation which comes under the problems of the sediment transport in river having engineering interest are described by one dimensional hyperbolic PDEs. The understanding of how sand interact with the water flow in certain environments is crucial for both the environment and businesses. In order to quantify the interaction between sediment transport and water flow, it is necessary to solve the PDEs governing the sediment transport equations. Today engineering and science researchers routinely confront problems in finding the solutions of many mathematical models formulated

B. Bira (✉)

Department of Mathematics, National Institute of Science and Technology, Palur Hills, Berhampur-8,
India
e-mail: birabibekananda@gmail.com

T. R. Sekhar

Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-2, India
e-mail: trajasekhar@maths.iitkgp.ernet.in

in terms of nonlinear PDEs. Unfortunately, finding solutions for such nonlinear PDEs is an arduous task. In such context the researchers adopt some analytical and numerical procedure which set the scene to understand the physical phenomena completely.

Lie group analysis is one of the most widely used mathematical method for deriving exact solutions of non-linear system of partial differential equations with applications in different fields (see, [1,2]). Several researchers has illustrated the advantages of applications of Lie group analysis for investigating nonlinear differential equations. Huang and Ivanova [3] studied the group analysis of a class of variable co-efficient nonlinear telegraph equations and obtained exact solutions. Lie group transformations for self-similar shocks in a gas with dust particles has been discussed by Jena [4]. Sharma and Radha [5] obtained exact solutions of Euler equations of ideal gasdynamics via Lie group analysis. Symmetry groups and similarity solutions for the system of equations for a viscous compressible fluid has been discussed by Pandey et al. [6]. In [7] symmetry reductions and exact solutions has been studied extensively.

In the present work, we apply the symmetry method to the system of PDEs governing the sediment transport in channels and rivers. The symmetry groups, admitted by the governing system of PDEs, which are indeed continuous groups of transformations under which the system of PDEs invariant, are obtained. We then proceed to determine the similarity solutions, which in turn are used to transform the system of PDEs to a system of ODEs. Finally, we give some exact solutions to the system of PDEs.

Exact Solutions via Lie Group Analysis

The system of equations which governs the sediment transport in channels and rivers, can be written as follows ([8,9]):

$$\begin{aligned}h_t + uh_x + hu_x &= 0, \\u_t + uu_x + gh_x + gB_x &= 0, \\B_t + Au^2u_x &= 0\end{aligned}\quad (1)$$

where h is the total height above the bottom of the channel, u is the water velocity in the x direction and B is the height of the bed. Here A is a given experimental constant and g is acceleration due to gravity. The independent variables t and x denote time and space respectively.

Here we investigate the most general Lie group of transformations which leaves the system (1) invariant. Now, we consider Lie group of transformations (see, [1,2]) with the independent variables x and t , and with the dependent variables h , u , B for the current problem as

$$\begin{aligned}t^* &= t + \epsilon\phi_1(x, t, h, u, B) + O(\epsilon^2), & x^* &= x + \epsilon\phi_2(x, t, h, u, B) + O(\epsilon^2), \\h^* &= h + \epsilon\mu_1(x, t, h, u, B) + O(\epsilon^2), & u^* &= u + \epsilon\mu_2(x, t, h, u, B) + O(\epsilon^2), \\B^* &= B + \epsilon\mu_3(x, t, h, u, B) + O(\epsilon^2),\end{aligned}\quad (2)$$

where ϕ_1 , ϕ_2 , μ_1 , μ_2 and μ_3 are the generators to be determined such that the system of PDEs (1) remains invariant with respect to the transformations (2) and ϵ is very small group parameter.

Since the system of governing PDEs (1) is of first order, we apply the first order prolongation to the system of PDEs (1). Then we obtain the over-determined system of equations in terms of dependent variables and some derivatives of dependent variables. The determining

equations are solved by power series method and obtained the infinitesimal transformations as follows

$$\begin{aligned}\phi_1 &= \alpha_1 + \alpha_2 t, & \phi_2 &= \alpha_3 + \alpha_4 x, & \mu_1 &= 2(\alpha_4 - \alpha_2)h, \\ \mu_2 &= (\alpha_4 - \alpha_2)u, & \mu_3 &= 2(\alpha_4 - \alpha_2)B,\end{aligned}\quad (3)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are arbitrary constants. The similarity variables can be obtained from the characteristic equations given as below:

$$\frac{dt}{\phi_1} = \frac{dx}{\phi_2} = \frac{dh}{\mu_1} = \frac{du}{\mu_2} = \frac{dB}{\mu_3}.$$

i.e

$$\frac{dt}{\alpha_1 + \alpha_2 t} = \frac{dx}{\alpha_3 + \alpha_4 x} = \frac{dh}{2(\alpha_4 - \alpha_2)h} = \frac{du}{(\alpha_4 - \alpha_2)u} = \frac{dB}{2(\alpha_4 - \alpha_2)B}.$$

Now we consider different cases to obtain some exact solutions.

Case A: $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$.

In this case, the similarity variable and the new dependent variables are obtained as follows

$$\begin{aligned}\xi &= (\alpha_3 + \alpha_4 x)(\alpha_1 + \alpha_2 t)^{\frac{-\alpha_4}{\alpha_2}}, & h &= (\alpha_1 + \alpha_2 t)^{\frac{2(\alpha_4 - \alpha_2)}{\alpha_2}} H(\xi), \\ u &= (\alpha_1 + \alpha_2 t)^{\frac{(\alpha_4 - \alpha_2)}{\alpha_2}} U(\xi), & B &= (\alpha_1 + \alpha_2 t)^{\frac{2(\alpha_4 - \alpha_2)}{\alpha_2}} P(\xi).\end{aligned}\quad (4)$$

Substituting (4) in (1), we obtain the following reduced system of ODEs

$$\begin{aligned}(U - \xi) \frac{dH}{d\xi} + H \frac{dU}{d\xi} + \frac{2(\alpha_4 - \alpha_2)}{\alpha_4} H &= 0, \\ (U - \xi) \frac{dU}{d\xi} + g \frac{dH}{d\xi} + g \frac{dP}{d\xi} + \frac{(\alpha_4 - \alpha_2)}{\alpha_4} U &= 0, \\ -\xi \frac{dP}{d\xi} + AU^2 \frac{dU}{d\xi} + \frac{2(\alpha_4 - \alpha_2)}{\alpha_4} P &= 0.\end{aligned}\quad (5)$$

For $U = 1$ and $\alpha_4 = 2\alpha_2$, we get the solution of (5) as

$$H = \frac{1}{4g}(1 - \xi)^2, \quad P = -\frac{1}{4g}\xi^2. \quad (6)$$

Finally, the solution of (1) is found to be

$$h = \frac{1}{4g}((\alpha_1 + \alpha_2 t)^2 - (\alpha_3 + 2\alpha_2 x)), \quad u = (\alpha_1 + \alpha_2 t), \quad B = -\frac{1}{4g}(\alpha_3 + 2\alpha_2 x).$$

Case B: $\alpha_1 \neq 0$ and $\alpha_2 = 0$.

The similarity and the dependent variables for this case are found to be

$$\begin{aligned}\xi &= (\alpha_3 + \alpha_4 x) \exp\left(\frac{-\alpha_4}{\alpha_1} t\right), & h &= H(\xi) \exp\left(\frac{2\alpha_4}{\alpha_1} t\right), & u &= U(\xi) \exp\left(\frac{2\alpha_4}{\alpha_1} t\right), \\ B &= P(\xi) \exp\left(\frac{2\alpha_4}{\alpha_1} t\right),\end{aligned}\quad (7)$$

which reduces (1) to a system of ODEs as

$$\begin{aligned} (U - \frac{1}{\alpha_1}\xi) \frac{dH}{d\xi} + H \frac{dU}{d\xi} + \frac{2}{\alpha_1}H &= 0, \\ (U - \frac{1}{\alpha_1}\xi) \frac{dU}{d\xi} + g \frac{dH}{d\xi} + g \frac{dP}{d\xi} + \frac{1}{\alpha_1}U &= 0, \\ -\xi \frac{dP}{d\xi} + A\alpha_1 U^2 \frac{dU}{d\xi} + 2P &= 0. \end{aligned} \quad (8)$$

The system of Eq. (8) can be solved for $U = 1/\alpha_1$, which is given as

$$H = \frac{1}{2\alpha_1^2 g} (1 - \xi), \quad P = -\frac{1}{2\alpha_1^2 g} \xi^2, \quad (9)$$

and the corresponding solution of (1) is obtain as follows

$$\begin{aligned} h &= \frac{1}{2\alpha_1^2 g} \left(\exp\left(\frac{\alpha_4}{\alpha_1} t\right) - (\alpha_3 + 2\alpha_2 x) \right)^2, \quad u = \frac{1}{\alpha_1} \exp\left(\frac{\alpha_4}{\alpha_1} t\right), \\ B &= -\frac{1}{2\alpha_1^2 g} (\alpha_3 + 2\alpha_2 x)^2. \end{aligned}$$

Case C: $\alpha_1 = 0$ and $\alpha_2 = 0$.

This case produces the similarity and the new dependent variables as

$$\xi = t, \quad h = (\alpha_3 + 2\alpha_2 x)^2 H, \quad u = (\alpha_3 + 2\alpha_2 x) U, \quad B = (\alpha_3 + 2\alpha_2 x)^2 P. \quad (10)$$

Usage of (10) in (1) yields the reduced system of ODEs as follows

$$\begin{aligned} \frac{dH}{d\xi} + 3\alpha_4 U H &= 0, \\ \frac{dU}{d\xi} + 2g\alpha_4 (H + P) + \alpha_4 U^2 &= 0, \\ \frac{dP}{d\xi} + A\alpha_4 U^3 &= 0 \end{aligned}$$

which can be solved numerically.

Case D: $\alpha_3 = 0$ and $\alpha_4 = 0$.

Here the corresponding similarity and dependent variables are

$$\xi = x, \quad h = (\alpha_1 + \alpha_2 t)^{-2} H, \quad u = (\alpha_1 + \alpha_2 t)^{-1} U, \quad B = (\alpha_1 + \alpha_2 t)^{-2} P, \quad (11)$$

and the consequent reduced system of ODEs is

$$\begin{aligned} U \frac{dH}{d\xi} + H \frac{dU}{d\xi} - 2\alpha_2 H &= 0, \\ U \frac{dU}{d\xi} + g \frac{dH}{d\xi} + g \frac{dP}{d\xi} - \alpha_2 U &= 0, \\ AU^2 \frac{dU}{d\xi} - \alpha_2 P &= 0. \end{aligned} \quad (12)$$

We solve the above system of ODEs by assuming $H = k_2 = \text{constant}$, and we get

$$H = k_2, \quad U = \xi, \quad P = A\xi^2, \quad (13)$$

for $\alpha_2 = 1/2$, $A = -1/4$ and $g = 1$. Combining (11) and (13) we obtain the solution for (1) as follows

$$h = k_2 \left(\alpha_1 + \frac{1}{2}t \right)^{-2}, \quad u = x \left(\alpha_1 + \frac{1}{2}t \right)^{-1}, \quad B = -\frac{1}{4}x^2 \left(\alpha_1 + \frac{1}{2}t \right)^{-2}.$$

Conclusions

In this work we have presented some exact solutions of the system of nonlinear hyperbolic PDEs which governs sediment transport in one space dimension. For that Lie group analysis is performed and infinitesimal transformations which leave the equations in hand invariant are derived. The governing system of PDEs is reduced to system of ODEs via the transformations and some special exact solutions to the system of PDEs are obtained. The solutions so obtained are of great interest; these solutions play vital role to study realistic river hydraulic community, such as floods, meandering, sediment load computation and navigation which are crucial for both environment and business. These solutions are also useful in the proper understanding of qualitative features of many physical phenomena and processes in various areas of natural science. Even those special exact solutions that do not have a clear physical meaning can be used as test problems to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods.

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