Temperature dependent growth rate of phytoplankton and salinity induced grazing rate of zooplankton as determinants of realistic multi-delayed food chain model

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Abstract Impact of temperature dependent growth rate of phytoplankton, salinity induced grazing rate of zooplankton and gestation delay of phytoplankton and zooplankton differently are considered in the present study. Reproduction of predator after consuming prey is not instantaneous, but mediated by some time lag required for gestation. Here first gestation delay is considered in the numerical response term of phytoplankton and second one is at the numerical response term of zooplankton respectively. There is a very interesting interplay between these two delays. In accordance with previous studies, it is observed that delay destabilizes the system, in general and stability loss occurs via Hopf-bifurcation. In particular, we show that there exists critical values of the delay parameters below which the coexistence equilibrium is stable and above which it is unstable. Hopf bifurcation occurs when the delay parameters cross their critical values. All the stability results of the system significantly depend upon the temperature and salinity of the environment. Numerical examples are also support the model assumption and analytical results.

Keywords Temperature dependent growth rate · Salinity induced grazing rate · Gestation delay · Multiple delays · Hopf-bifurcation

Introduction

Understanding the dynamical relationship between resource and consumer through mathematical modeling in theoretical ecology has gained a lot of importance during the last few decades due to its universal existence and importance (Berryman 1992). Different food chain models are studied by many authors (Hassell 1971; Jana et al. 2014). Dynamical complexity arises in food chain model through the consideration of functional response of consumer. Consumer's (predator's) functional response which is defined as the amount of resource (prey) catch per consumer (predator) per unit of time is assumed to be one of the most important components of resource-consumer/prey-predator interaction because predation intensity may change the shape of the community structure and ecosystem properties (Sih et al. 1985). A brief review on prey-predator models with discrete delay for both Kolmogorov and Non-Kolmogorov type model system is presented by Ruan (2009) where he shows different mathematical criterions for different biological assumptions due to delay. Models with delay are much more realistic. In reality time delays occur in almost every biological situation (Jana et al. 2015; MacDonald 1989; Jana et al. 2014) and assumed to be one of the significant reasons of regular fluctuations in population density (Jana et al. 2015; MacDonald 1989; Jana et al. 2014; May 1981). Reproduction of predator after consuming prey is not instantaneous, but mediated by some time lag required for gestation. Therefore, to make a prey-predator model biologically more realistic, one has to consider this gestation delay in the model system. In food chain models, different consumers show different gestation time, so it is very important and significant observation to study impact of multiple time lags as gestation delays of different consumers in one food chain model. Jana (2014) studies the

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stabilizing effect of prey refuge and predator's interference on the dynamics of prey with delayed growth and generalist predator with delayed gestation. Jana and Samanta (2014) show the complex dynamical results of a multi-delayed ratio-dependent prey-predator system with prey harvesting under stochastic environment. Where as Jana et al. (2015) show the effect of multiple delays on the dynamics of generalist predator in a stochastic environment. Effect of age-selective harvesting of both prey and predator is studied by Jana et al. (2016).

Temperature plays a significant role to maintain the phytoplankton population at estuary because the temperature directly affects the rates of many reactions including photosynthesis and respiration (cf. Lomas et al. 2002) and indirectly affects phytoplankton through the activity of grazer (cf. Smayda 2004). The growth of phytoplankton species has an optimal temperature range (cf. Zhou 2006). Bintz et al. (2003) shows the phytoplankton blooms are completely absent from cool un-enriched systems and warmer systems promote phytoplankton blooms. Salinity is one of the most important factors for governing zooplankton population dynamics of any estuary (cf. Ghosh 2001; Ketchum 1951) which defines estuaries in such a way as to include any marine waters in which seawater was measurably diluted with fresh water from land drainage. Dube et al. (2010) conclude that salinity is uniform through the water column but varies between the riverine region (1 psu) and seaside (30 psu) in shallow lagoons with no distinct stratification. Growth models by using light and temperature effects on phytoplankton are developed by Baird et al. (2001), Flynn (2003), and Bissinger et al. (2008). Baird et al. (2001) use the interacting effects of temperature, light capture and nutrient uptake on phytoplankton growth model. Bissinger et al. (2008) uses quantile regression method and improve Eppley curve to detect the maximum growth rates of marine phytoplankton as a function of temperature. Different factors such as temperature (cf. Di Toro et al. 1975; Nicklisch et al. 2008; Sipkay 2010), body size [(cf. Ray et al. 2001), nutrient pulse (cf. Cottingham et al. 2004; Dube and Jayaraman 2008; Dube et al. 2010), top-down effect (cf. Morozov et al. 2005; Irigoien et al. 2005; Calbet and Saiz 2005)] are included in the NPZ models to know the effects of these factors on plankton dynamics.

In the present work, we have investigated the complex dynamical behavior of a nutrient-phytoplankton-zooplankton population with two delays both describes the gestation period of phytoplankton and zooplankton respectively. The paper is structured as follows: in "Mathematical model formulation", study area, the model system, its stability and Hopf-bifurcation has been discussed, numerical simulation results have been presented in

"Numerical simulation" and at last, in "Discussion", conclusions and discussions have been made.

Mathematical model formulation

Study area

The total length of Hooghly estuary is about 295 km and it lies between the latitude $21^{\circ}31'N$ to $23^{\circ}30'N$ and $87^{\circ}45'N$ to $88^{\circ}45'N$. The estuary bifurcates near Sagar island into main Hooghly in the north and northwestern side and river Moorriganga in the eastern side, which is connected to another river Thakuran and Matla, forming the Sundarban estuarine complex. The present study is restricted to mangrove dominated region (lower part) of the estuarine complex. The creeks traversing these regions belong to shallow, mesotidal regime (Fig. 1).

The seasons of this region are categorized as pre-monsoon (March–June), monsoon (July–October) and post-monsoon (November–February). The pre-monsoon is characterized by high air temperature ranging from 28 to $40^{\circ}C$ and occasional rains. The monsoon encounters high rainfall (average 185 cms.), when the south west wind enters the Indian subcontinent. The post-monsoon is

Fig. 1 Map showing the study area in Hooghly-Matla estuarine complex

characterized by cold weather (average 22°C) and negligible rainfall.

Model and analysis

Now we consider a NPZ model in the given form where N , P and Z represent nutrient, phytoplankton and zooplankton densities respectively.

$$\begin{aligned}\frac{dN}{dt} &= In - NL_r - \delta \frac{PN}{N + K_p}, \\ \frac{dP}{dt} &= \delta \frac{PN}{N + K_p} - Pr_1 - g_s \frac{PZ}{P + K_Z}, \\ \frac{dZ}{dt} &= g_s \frac{PZ}{P + K_Z} - Zr_2.\end{aligned}\quad (2.1)$$

In this present NPZ model temperature dependant phytoplankton growth rate and salinity induced grazing rate of zooplankton are incorporated this equation which are taken from the model of Mandal et al. (2012). In the present model, Dissolved Inorganic Nitrogen is considered as nutrient (N) for phytoplankton. In = input of nutrient through mineralization and leaching, L_r , K_p , N , P , δ are respectively loss rate of nutrient, half saturation constant for nutrient uptake by phytoplankton, nutrient, phytoplankton, temperature dependent phytoplankton growth rate.

$$\delta = \frac{P_{max} e^{\gamma T} L_{eff}}{\alpha}, \quad (2.2)$$

where P_{max} , T , L_{eff} are respectively maximum growth rate of Phytoplankton, water temperature, light effect on photosynthesis. Phytoplankton growth rate is T dependent and the dependence of T is exponential. $\gamma = 0.09$ and $\alpha = 6.05$ is define by Ray and Straskraba (2001).

$$L_{eff} = \frac{I}{I_{opt}} e^{1 - \frac{I}{I_{opt}}}, \quad (2.3)$$

where I = irradiance effect, I_{opt} is the optimum surface solar radiation for photosynthesis.

$$I = I_r e^{(-E_c - E_{xt})d}, \quad (2.4)$$

where I_r , E_c , d , E_{xt} , are respectively surface solar irradiance, extinction coefficient of water, depth of water and self shading of phytoplankton due to its own body size. Self shading is calculated by using the equation of Radtke and Straskraba (1980). The equation is based on the physical laws of particle sizes and as follows:

$$E_{xt} = 0.12V^{(-0.33)}, \quad (2.5)$$

where E_{xt} , V are respectively self shading of phytoplankton due to its own body size and average body volume of phytoplankton.

$$r_1 = (r_{ph} + r_s + M_{rp}), \quad (2.6)$$

where r_{ph} , r_s , M_{rp} are respectively respiratory rate of phytoplankton, rate of settling, mortality rate of phytoplankton.

$$g_s = g_z \frac{S_e}{S_e - S_r}, \quad (2.7)$$

where, g_s , S_r , S_e , g_z , S_r , K_z , are respectively salinity induced grazing of zooplankton, salinity of upstream, salinity of downstream, grazing rate of zooplankton, Half saturation constant of zooplankton on phytoplankton grazing by zooplankton.

$$r_2 = (r_{fp} + r_{zo} + M_{rz}), \quad (2.8)$$

where r_{fp} , r_{zo} , M_{rz} , are respectively rate of predation by fish, respiratory rate of zooplankton, mortality rate of zooplankton,

Delay is frequently used in NPZ model to make the model biologically more realistic. Recently, researchers are using more than one delay to study the effect on the system populations. In the second phase of this study, i consider two delays in the model system (2.1). One discrete delay τ_1 is considered in the specific growth rate of phytoplankton to incorporate the effect of gestation time which takes τ_1 units of time to respond to changes in the phytoplankton population. The second delay τ_2 is gestation time of zooplankton density. We thus obtain the following multi delayed NPZ model

$$\begin{aligned}\frac{dN}{dt} &= In - L_r N - \delta \frac{NP}{N + K_p}, \\ \frac{dP}{dt} &= \delta \frac{N(t - \tau_1)P}{N(t - \tau_1) + K_p} - r_1 P - g_s \frac{PZ}{P + K_Z}, \\ \frac{dZ}{dt} &= g_s \frac{P(t - \tau_2)Z}{P(t - \tau_2) + K_Z} - r_2 Z,\end{aligned}\quad (2.9)$$

where τ_1 represents the gestation period of phytoplankton and τ_2 gestation period of zooplankton respectively.

Feasibility The system (2.1) has only one interior equilibrium point given The equilibrium point E^* which is biologically feasible if (i) $(g_s - r_2) > 0$ and (ii) $(L_r K_p + \delta P^* - In) < 0$. Linearizing the system (2.9) at E^* , we get

$$\begin{aligned}\frac{dP}{dt} &= a_{11}P(t) + a_{12}Z(t), \\ \frac{dZ}{dt} &= a_{22}Z(t) + a_{23}F(t) + b_{21}P(t - \tau_1), \\ \frac{dF}{dt} &= c_{32}Z(t - \tau_2),\end{aligned}\quad (2.10)$$

where $a_{11} = -L_r - \delta \frac{P^*}{(N^* + K_p)^2}$, $a_{12} = -\delta \frac{N^*}{N^* + K_p}$, $a_{22} = \delta \frac{P^* Z^*}{(P^* + K_Z)^2}$, $a_{23} = \delta \frac{P^*}{P^* + K_Z}$, $b_{21} = \delta \frac{K_p P^*}{(P^* + K_p)^2}$, $c_{32} = g_s \frac{Z^* K_Z}{(P^* + K_Z)^2}$. The corresponding characteristic equation is given by

$$\lambda^3 + A\lambda^2 + B\lambda + Ce^{-\lambda\tau_1}\lambda + e^{-\lambda\tau_2}(D\lambda + E) = 0, \quad (2.11)$$

where $A = -(a_{11} + a_{22})$, $B = a_{11}a_{22}$, $C = -a_{12}b_{21}$, $D = -a_{23}c_{32}$ and $E = a_{11}a_{23}c_{32}$. We now discuss the following cases.

Case I: $\tau_1 = \tau_2 = 0$ For non-delayed system i.e., when $\tau_1 = \tau_2 = 0$, characteristic equation (2.11) becomes

$$\lambda^3 + A\lambda^2 + B\lambda + C\lambda + D\lambda + E = 0, \quad (2.12)$$

Hence, by Routh–Hurwitz criterion, we get the stability conditions of E^*

Theorem 1 *The coexistence equilibrium point E^* of system (2.1) is asymptotically stable if $A, B + C, D > 0$ and $A(B + C) > D$.*

Case II: $\tau_1 > 0, \tau_2 = 0$ We know that $i\omega (\omega > 0)$ is a root of (2.11) if and only if ω satisfies

$$\begin{aligned} -i\omega^3 - A\omega^2 + Bi\omega + iC\omega(\cos \omega\tau_1 - i\sin \omega\tau_1) \\ + Di\omega + E = 0. \end{aligned} \quad (2.13)$$

Separating real and imaginary parts, we get

$$\begin{aligned} A\omega^2 - E &= C\omega \sin \omega\tau_1, \\ \omega^3 - (B - D)\omega &= C\omega \cos \omega\tau_1. \end{aligned} \quad (2.14)$$

These two equations give the positive values of τ and ω for which Eq. (2.11) can have purely imaginary roots. Squaring and adding these equations, we have

$$\omega^6 + p\omega^4 + q\omega^2 + r = 0, \quad (2.15)$$

where

$$\begin{aligned} p &= A^2 - 2(B + D), \\ q &= B + D - 2AE - C^2, \\ r &= E^2. \end{aligned} \quad (2.16)$$

Now we shall measure the value of τ_1 for which system (2.9) will remain stable. From Eq. (2.14) we have,

$$\tau_{1k} = \frac{1}{\omega_k} \left[\cos^{-1} \left\{ \frac{\omega_k^3 - (B - D)\omega_k}{C\omega_k} \right\} + 2k\pi \right], \quad (2.17)$$

$k = 0, 1, 2, 3, \dots$, where ω_k is the unique real positive root of Eq. (2.15). Also, differentiating Eq. (2.15) with respect to τ we get

$$\left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\omega=\omega_0, \tau=\tau_0} = \frac{U_1 V_1 + U_2 V_2}{V_1^2 + V_2^2}, \quad (2.18)$$

where

$$\begin{aligned} U_1 &= -3\omega_0^2 + (B + D) - C\tau_{10} \cos \omega_0\tau_{10}, \\ U_2 &= 2\omega_0 A + C\tau_{10} \sin \omega_0\tau_{10}, \\ V_1 &= \omega_0 C \sin \omega_0\tau_{10}, \\ V_2 &= \omega_0 C \cos \omega_0\tau_{10}. \end{aligned} \quad (2.19)$$

Hence the transversality condition is satisfied if $(U_1 V_1 + U_2 V_2) > 0$.

Theorem 2 *Assume $\tau_1 > 0, \tau_2 = 0$ and conditions of the Theorem 1 are satisfied, then the equilibrium $E^*(N^*, P^*, Z^*)$ is locally asymptotically stable for $\tau < \tau_{10}$ and unstable for $\tau > \tau_{10}$. Furthermore, the system (2.9) undergoes a Hopf bifurcation at $E^*(N^*, P^*, Z^*)$ when $\tau = \tau_{10}$, where*

$$\tau_{10} = \frac{1}{\omega_0} \left[\cos^{-1} \left\{ \frac{(AD - BC)\omega_0^2 - C\omega_0^4}{D^2 + C^2\omega_0^2} \right\} \right].$$

Case III: $\tau_1 = \bar{\tau}_1 \in (0, \tau_{10}), \tau_2 > 0$ Now we take $\tau_1 = \bar{\tau}_1 \in (0, \tau_{10})$, then we know that $i\omega (\omega > 0)$ is a root of (2.11) if and only if ω satisfies

$$\begin{aligned} -i\omega^3 - A\omega^2 + Bi\omega + iC\omega \cos \omega\bar{\tau}_1 + C\omega \sin \omega\bar{\tau}_1 \\ + Di\omega \cos \omega\tau_2 + D\omega \sin \omega\tau_2 - iE \sin \omega\tau_2 + E \cos \omega\tau_2 = 0. \end{aligned} \quad (2.20)$$

Separating real and imaginary parts, we get

$$\begin{aligned} A\omega^2 - C\omega \sin \omega\bar{\tau}_1 &= D\omega \sin \omega\tau_2 + E \cos \omega\tau_2, \\ \omega^3 - B\omega - C\omega \cos \omega\bar{\tau}_1 &= D\omega \cos \omega\tau_2 - E \sin \omega\tau_2. \end{aligned} \quad (2.21)$$

These two equations give the positive values of τ_2 and ω for which Eq. (2.11) can have purely imaginary roots. Squaring and adding these equations, we have

$$\omega^6 + p\omega^4 + q\omega^3 + r\omega^2 + s = 0, \quad (2.22)$$

where

$$\begin{aligned} p &= A^2 - 2B - 2C \cos \omega\bar{\tau}_1, \\ r &= B^2 + C^2 - D^2 + 2BC \cos \omega\bar{\tau}_1, \\ q &= -2AC\omega \sin \omega\bar{\tau}_1, \\ s &= -E^2. \end{aligned}$$

Now we shall measure the value of τ_2 for which system (2.9) will remain stable. From Eq. (2.31) we have,

$$\tilde{\tau}_{2k} = \frac{1}{\tilde{\omega}_k} \left[\cos^{-1} \left\{ \frac{(AE + BD - DC \cos \tilde{\omega}_k \bar{\tau}_1) \tilde{\omega}_k^2 + D\tilde{\omega}_k^4 - EC \tilde{\omega}_k \sin \tilde{\omega}_k \bar{\tau}_1}{E^2 + D^2 \tilde{\omega}_k^2} \right\} + 2k\pi \right],$$

$k = 0, 1, 2, 3, \dots$, where $\tilde{\omega}_k$ is the real positive root of Eq. (2.11). Also, differentiating Eq. (2.11) with respect to τ_2 we get

$$\left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\omega=\tilde{\omega}_0, \tau=\tilde{\tau}_{2_0}} > 0.$$

Hence the transversality condition is satisfied.

Theorem 3 Assume $\tau_1 = \bar{\tau}_1 \in (0, \tau_{1_0})$, $\tau > 0$ and conditions of the Theorem 1 are satisfied, then the equilibrium $E^*(N^*, P^*, Z^*)$ is locally asymptotically stable for $\tau_2 < \tilde{\tau}_{2_0}$ and unstable for $\tau_2 > \tilde{\tau}_{2_0}$. Furthermore, the system (2.9) undergoes a Hopf bifurcation at $E^*(N^*, P^*, Z^*)$ when $\tau_2 = \tilde{\tau}_{2_0}$, where

$$\tilde{\tau}_{2_0} = \frac{1}{\tilde{\omega}_0} \left[\cos^{-1} \left\{ \frac{(AE + BD - DC \cos \tilde{\omega}_0 \bar{\tau}_1) \tilde{\omega}_0^2 + D \tilde{\omega}_0^4 - EC \tilde{\omega}_0 \sin \tilde{\omega}_0 \bar{\tau}_1}{E^2 + D^2 \tilde{\omega}_0^2} \right\} \right].$$

Case IV: $\tau_1 = 0, \tau_2 > 0$

We know that $i\omega (\omega > 0)$ is a root of (2.11) if and only if ω satisfies

$$\begin{aligned} -i\omega^3 - A\omega^2 + Bi\omega + iC\omega + (Di\omega + E) \\ (\cos \omega \tau_2 - i \sin \omega \tau_2) = 0. \end{aligned}$$

Separating real and imaginary parts, we get

$$\begin{aligned} A\omega^2 = E \cos \omega \tau_2 + D\omega \sin \omega \tau_2, \\ \omega^3 - (B + C)\omega = D\omega \cos \omega \tau_2 - E \sin \omega \tau_2. \end{aligned} \quad (2.23)$$

These two equations give the positive values of τ_2 and ω for which Eq. (2.11) can have purely imaginary roots. Squaring and adding these equations, we have

$$\omega^6 + p\omega^4 + q\omega^2 + r = 0, \quad (2.24)$$

where

$$\begin{aligned} p &= A^2 - 2(B + C), \\ q &= B^2 + C^2 - D^2, \\ r &= -E^2. \end{aligned} \quad (2.25)$$

Now we shall measure the value of τ_2 for which system (2.9) will remain stable. From Eq. (2.11) we have,

$$\tau_{2_k} = \frac{1}{\tilde{\omega}_k} \left[\cos^{-1} \left\{ \frac{(EA - BD - CD) \tilde{\omega}_k^2 + D \tilde{\omega}_k^4}{E^2 + D^2 \tilde{\omega}_k^2} \right\} + 2k\pi \right], \quad (2.26)$$

$k = 0, 1, 2, 3, \dots$, where $\tilde{\omega}_k$ is the real positive root of Eq. (2.24). Also, differentiating Eq. (2.11) with respect to τ_2 we get

$$\left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\omega=\tilde{\omega}_0, \tau=\tau_{2_0}} = \frac{U_1 V_1 + U_2 V_2}{V_1^2 + V_2^2}, \quad (2.27)$$

where

$$\begin{aligned} U_1 &= -3\tilde{\omega}_0^2 + B + C + (D - E\tau_{2_0}) \cos \tilde{\omega}_0 \tau_{2_0} - D\tilde{\omega}_0 \tau_{2_0} \sin \tilde{\omega}_0 \tau_{2_0}, \\ U_2 &= -2A\tilde{\omega}_0 + (D - E\tau_{2_0}) \sin \tilde{\omega}_0 \tau_{2_0} - D\tilde{\omega}_0 \tau_{2_0} \cos \tilde{\omega}_0 \tau_{2_0}, \\ V_1 &= D\tilde{\omega}_0^2 \cos \tilde{\omega}_0 \tau_{2_0} - \tilde{\omega}_0 E \sin \tilde{\omega}_0 \tau_{2_0}, \\ V_2 &= D\tilde{\omega}_0^2 \sin \tilde{\omega}_0 \tau_{2_0} + \tilde{\omega}_0 E \cos \tilde{\omega}_0 \tau_{2_0}, \end{aligned} \quad (2.28)$$

Hence the transversality condition is satisfied if $(U_1 V_1 + U_2 V_2) > 0$.

Theorem 4 Assume $\tau_1 = 0, \tau_2 > 0$ and conditions of the Theorem 1 are satisfied, then the equilibrium $E^*(N^*, P^*, Z^*)$ is locally asymptotically stable for $\tau < \tau_{2_0}$ and unstable for $\tau > \tau_{2_0}$. Furthermore, the system (2.9) undergoes a Hopf bifurcation at $E^*(N^*, P^*, Z^*)$ when $\tau = \tau_{2_0}$, where

$$\tau_{2_0} = \frac{1}{\tilde{\omega}_0} \left[\cos^{-1} \left\{ \frac{(AD - BC) \tilde{\omega}_0^2 - C \tilde{\omega}_0^4}{D^2 + C^2 \tilde{\omega}_0^2} \right\} \right].$$

Case V: $\tau_2 = \hat{\tau}_2 \in (0, \tau_{2_0})$, $\tau_1 > 0$ Now we take $\tau_2 = \hat{\tau}_2 \in (0, \tau_{2_0})$, then we know that $i\omega (\omega > 0)$ is a root of (2.11) if and only if ω satisfies

$$\begin{aligned} -A\omega^2 + C\omega \sin \omega \tau_1 + D\omega \sin \omega \hat{\tau}_2 + E \cos \omega \hat{\tau}_2 - i\omega^3 \\ + Bi\omega + iC\omega \cos \omega \tau_1 + iD\omega \cos \omega \hat{\tau}_2 - iE \sin \omega \hat{\tau}_2 = 0. \end{aligned} \quad (2.29)$$

Separating real and imaginary parts, we get

$$\begin{aligned} -A\omega^2 + D\omega \sin \omega \hat{\tau}_2 + E \cos \omega \hat{\tau}_2 = -C\omega \sin \omega \tau_1, \\ -\omega^3 + B\omega + D\omega \cos \omega \hat{\tau}_2 - E \sin \omega \hat{\tau}_2 = -C\omega \cos \omega \tau_1 \end{aligned} \quad (2.30)$$

These two equations give the positive values of τ_2 and ω for which Eq. (2.11) can have purely imaginary roots. Squaring and adding these equations, we have

$$\omega^6 + p\omega^4 + q\omega^2 + r = 0, \quad (2.31)$$

where

$$\begin{aligned}
 p &= A^2 - 3B - 2D \cos \omega \hat{\tau}_2, \\
 r &= B^2 - C^2 + D^2 - 2AE \cos \omega \hat{\tau}_2 + 2E \sin \omega \hat{\tau}_2, \\
 q &= -2AD \sin \omega \hat{\tau}_2, \\
 s &= E^2.
 \end{aligned}$$

Now we shall measure the value of τ_1 for which system (2.9) will remain stable. From Eq. (2.30) we have,

$$\bar{\tau}_{1k} = \frac{1}{\hat{\omega}_k} \left[\cos^{-1} \left\{ \frac{\hat{\omega}_k^3 - B\hat{\omega}_k - D\hat{\omega}_k \cos \hat{\omega}_k \hat{\tau}_2 + E \sin \hat{\omega}_k \hat{\tau}_2}{C\hat{\omega}_k} \right\} + 2k\pi \right], \quad (2.32)$$

$k=0,1,2,3,\dots$, where $\hat{\omega}_k$ is the real positive root of Eq. (1.31). Also, differentiating Eq. (2.11) with respect to τ_1 we get

$$\left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\omega=\hat{\omega}_0, \tau_1=\bar{\tau}_{10}} = \frac{u_1 v_1 + u_2 v_2}{v_1^2 + v_2^2}, \quad (2.33)$$

where

$$\begin{aligned}
 u_1 &= C + \sin \hat{\omega}_0 \bar{\tau}_{10} (-2A\hat{\omega}_0 + \bar{\tau}_{10} \hat{\omega}_0^3 - B\bar{\tau}_{10} \hat{\omega}_0) \\
 &\quad + \cos \hat{\omega}_0 \bar{\tau}_{10} (B - 3\hat{\omega}_0^2 - A\bar{\tau}_{10} \hat{\omega}_0^2), \\
 u_2 &= \sin \hat{\omega}_0 \bar{\tau}_{10} (B - 3\hat{\omega}_0^2 - A\bar{\tau}_{10} \hat{\omega}_0^2) \\
 &\quad + \cos \hat{\omega}_0 \bar{\tau}_{10} (2A\hat{\omega}_0 - \bar{\tau}_{10} \hat{\omega}_0^3 + B\bar{\tau}_{10} \hat{\omega}_0), \\
 v_1 &= -A\hat{\omega}_0^3 \sin \hat{\omega}_0 \bar{\tau}_{10} + \cos \hat{\omega}_0 \bar{\tau}_{10} (B\hat{\omega}_0^2 - \hat{\omega}_0^4), \\
 v_2 &= \sin \hat{\omega}_0 \bar{\tau}_{10} (B\hat{\omega}_0^2 - \hat{\omega}_0^4) + A\hat{\omega}_0^3 \cos \hat{\omega}_0 \bar{\tau}_{10}.
 \end{aligned}$$

Hence the transversality condition is satisfied if $(U_1 V_1 + U_2 V_2) > 0$.

Theorem 5 Assume $\tau_2 = \hat{\tau}_2 \in (0, \tau_{20})$, $\tau_1 > 0$ and conditions of the Theorem 1 are satisfied, then the equilibrium $E^*(N^*, P^*, Z^*)$ is locally asymptotically stable for $\tau_1 < \bar{\tau}_{10}$ and unstable for $\tau_1 > \bar{\tau}_{10}$. Furthermore, the system (2.9) undergoes a Hopf bifurcation at $E^*(N^*, P^*, Z^*)$ when $\tau_1 = \bar{\tau}_{10}$, where

$$\bar{\tau}_{10} = \frac{1}{\hat{\omega}_0} \left[\cos^{-1} \left\{ \frac{(AD - BC)\hat{\omega}_0^2 - C\hat{\omega}_0^4}{D^2 + C^2\hat{\omega}_0^2} \right\} \right].$$

Numerical simulation

In Fig. 1, we present the geographical position of the study area which is briefly describe in “Study area” section. Now we consider the parameter set of the system as: $I_n = 1.140$, $L_r = 0.45$, $K_p = 12$, $r_{ph} = 0.1090$, $r_s = 0.049$, $M_{rp} = 0.021$, $g = 0.2041$, $s_e = 7.4$, $s_r = 7.2$, $K_z = 65$, $r_{zo} = 0.008$, $r_{fp} = 0.001$, $M_{rz} = 0.0125$, $V = 1.60$, $p_{max} = 2.62$, $T = 28.1$, $E_c = 0.2$, $d = 4$, $I_{opt} = 680$ and $I_r = 800$. Then the equilibrium population density is $E^*(2.96278, 1.7912, 5.62218)$. In Fig. 2 we show the

Fig. 2 Time series of the NPZ dynamics for different parameter conditions in Hooghly-Matla estuarine system. The model results are obtained by varying the values of temperature with fixed salinity effect. **a** Temperature (T) = 25.39 °C, salinity of downstream (S_e) = 12.17, salinity of upstream (S_r) = 6.37; **b** T = 23.38 °C, S_e = 12.17, S_r = 6.37; **c** T = 12.1 °C, S_e = 12.17, S_r = 6.37

Fig. 3 Time series of the NPZ dynamics during different conditions in Hooghly-Matla estuarine system. The model results are obtained by varying the values of salinity parameter with fixed temperature. **a** T = 25.38 °C, S_e = 12.17, S_r = 6.37; **b** Temperature (T) = 25.38 °C, salinity of downstream S_e = 12.17, salinity of upstream S_r = 4; **c** T = 25.38 °C, S_e = 12.17, S_r = 2.25

temperature effect on the dynamics of non-delayed system 2.1 by plotting the time series. Here we observe that when temperature is almost 23.38–25.39 °C, coexistence equilibrium point E^* is stable (Fig. 2a, b), i.e., nutrient (N), phytoplankton (P) and zooplankton (Z) coexist together. But if temperature decreases, then rate of phytoplankton bloom decreases and eventually due to lack of food zooplankton abundance decreases. At 12.1 °C, we see both

Fig. 4 Time series (i) and phase space (ii) diagram of system (2.1), figures show that system (2.1) is asymptotically stable. Other parameters are as in the text

Fig. 5 Time series (i, iii) and phase space (ii, iv) diagram of system (2.9) when $\tau_1 > 0, \tau_2 = 0$, i and ii show that system (2.9) is stable for $\tau_1 = 0.5 < \tau_{10} = 2.44, \tau_2 = 0$ and iii and iv show that system (2.9) is unstable for $\tau_1 = 2.7 > \tau_{10} = 2.44, \tau_2 = 0$ respectively. Other parameters are as in the text

phytoplankton and zooplankton go to zero (Fig. 2c). So, from these three figures, it is very clear that temperature dependent phytoplankton growth rate plays a very significant role to maintain the stability of coexistence equilibrium point E^* . Figure 3 (time series) shows that at higher salinity three species coexist (Fig. 3a, b) and at lowest salinity zooplankton goes to zero (Fig. 3c). So it is clear from the figure that decreasing salinity decreases the grazing rate of zooplankton and lack of low grazing rate zooplankton gradually goes to zero. Figure 4 [Time series (i) and phase space (ii)] shows that system (2.1) i.e. the non-delayed ($\tau_1 = \tau_2 = 0$) (Case I) system of system (2.9) is stable. Now if we take $\tau_1 > 0, \tau_2 = 0$ (Case II), then Fig. 5(i) (time series) and (ii) (phase space) show that system (2.9) is stable for $\tau_1 = 0.5 < \tau_{10} = 2.44, \tau_2 = 0$ and Fig. 5(iii) (time series) and (iv) (phase space) show that system (2.9) is unstable for $\tau_1 = 2.7 > \tau_{10} = 2.44, \tau_2 = 0$

respectively. If we take $\tau_1 = 0.5 = \bar{\tau}_1 \in (0, \tau_{10}), \tau_2 > 0$ (Case III), then Fig. 6(i) (time series) and (ii) (phase space) show that system (2.9) is stable for $\tau_1 = 0.5, \tau_2 = 2.1 > \bar{\tau}_{20} = 1.7$ and Fig. 6(iii) (time series) and (iv) (phase space) show that system (2.9) is unstable for $\tau_1 = 0.5, \tau_2 = 1.57 < \bar{\tau}_{20} = 1.7$ respectively. Again if we take $\tau_1 = 0, \tau_2 > 0$ (Case IV), then Fig. 7(i) (time series) and (ii) (phase space) show that system (2.9) is stable for $\tau_1 = 0, \tau_2 = 0.3 < \tau_{20} = 1.57$ and Fig. 7(iii) (time series) and (iv) (phase space) show that system (2.9) is unstable for $\tau_1 = 0, \tau_2 = 1.9 > \tau_{20} = 1.57$ respectively. Also when we take $\tau_1 > 0, \tau_2 = 0.3 = \hat{\tau}_2 \in (0, \tau_{20})$ (Case V), then Fig. 8(i) (time series) and (ii) (phase space) show that system (2.9) is stable for $\tau_1 = 1.5 < \bar{\tau}_{10}, \tau_2 = 0.3$ and Fig. 8(iii) (time series) and (iv) (phase space) show that system (2.9) is unstable for $\tau_1 = 2.5 > \bar{\tau}_{10}, \tau_2 = 0.3$ respectively.

Fig. 6 Time series (i, ii) and phase space (ii, iv) diagram of system (2.9) when $\tau_1 = 0.5 = \bar{\tau}_1 \in (0, \tau_{10})$, $\tau_2 > 0$, i and ii show that system (2.9) is stable for $\tau_1 = 0.5$, $\tau_2 = 1.57 < \bar{\tau}_{20} = 1.7$ and iii and iv show

that system (2.9) is unstable for $\tau_1 = 0.5$, $\tau_2 = 2.1 > \bar{\tau}_{20} = 1.7$ respectively. Other parameters are as in the text

Fig. 7 Time series (i, ii) and phase space (iii, iv) diagram of system (2.9) when $\tau_1 = 0$, $\tau_2 > 0$, i and ii show that system (2.9) is stable for $\tau_1 = 0$, $\tau_2 = 0.3 < \tau_{20} = 1.57$ and iii and iv show that system (2.9) is unstable for $\tau_1 = 0$, $\tau_2 = 1.9 > \tau_{20} = 1.57$ respectively. Other parameters are as in the text

Discussion

In this present work, we consider a *NPZ*-model with Holling type II functional response where photosynthesis rate of phytoplankton, salinity induced grazing rate of zooplankton and gestation delay of both phytoplankton and zooplankton population independently are considered. In the later part of the study, we show the analytical results of

the impact of delay on the stability of the system and quantitative results of the Hopf bifurcation. Delay induced deterministic system shows that there is a critical value of delay parameter below which system is stable and above which system is unstable and when the delay parameter crosses the critical value then system experiences Hopf-bifurcation. Also all the results are compared and validated by numerical examples with real data set.

Fig. 8 Time series (i, iii) and phase space (ii, iv) diagram of system (2.9) when $\tau_1 > 0, \tau_2 = 0.3 = \bar{\tau}_2 \in (0, \tau_{20})$, i and ii show that system (2.9) is stable for $\tau_1 = 1.5 < \bar{\tau}_{10}, \tau_2 = 0.3$ and iii and iv show that

system (2.9) is unstable for $\tau_1 = 2.5 > \bar{\tau}_{10}, \tau_2 = 0.3$ respectively. Other parameters are as in the text

The main purpose of this work is to observe the effect of multiple time delays along with photosynthesis rate of phytoplankton, salinity induced grazing rate of zooplankton on a NPZ-system. Results show that these biological parameters play a significant role to measure the critical value of delay parameter/parameters which is/are the key factor to maintain the stability of the system. It is quite natural that when delay parameter/parameters is/are small enough the system can persist its stability but for large delay scale, system becomes unstable. But delay parameter/parameters can not effect the equilibrium population abundance.

The model is run for the time period of 1000 days and the dynamics of NPZ is predicted by the model. The phytoplankton and zooplankton population will go extinct rather nutrient value remain uniform after certain passage of time (days) when the temperature is $< 12.1^\circ\text{C}$ and there is a constant difference between S_e and S_r in the estuarine system (Fig. 2a). Figure 2a shows that the nutrient is abundant in the system for a long time but the phytoplankton and zooplankton population eliminate after 100 and 200 days respectively. The physical conditions of the environment are not favorable for the growth of phytoplankton as a result phytoplankton population do exist in the system for a very short period of time and then eliminated from the system. Figure 2b shows a little improvement in the phytoplankton population but there is no improvement of zooplankton population in the system. Since the zooplankton population is dependent on the

phytoplankton population, the phytoplankton population grows first with gradual increase of temperature and little after the phytoplankton population. When the temperature T is fixed at 23.38°C , it shows the optimum growth for both plankton in this coastal environment. Our model shows that there is a definite role of temperature in the growth rate of zooplankton (Fig. 2c). When the temperature is raised to 23.38 and 25.38°C . the system shows a resilient condition, where the zooplankton population put itself back to its existence in this environmental condition along with phytoplankton and available nutrient(cf. Fig. 2c). The model also describes a situation where, there is a little difference in S_e and S_r (monsoon season—July to October in the tropical region) of the estuary. At 25.39°C , temperature all the components (NPZ) coexist and a stable equilibrium is attained by the system. These oscillations are continued to exist for a long time if the value of S_e approaches to S_r , when fresh water run off is more in the estuary for a long time. The above observation show that the salinity is an important factor governing the plankton dynamics in the estuary.

Model also shows that the salinity induced grazing of zooplankton (g_s) should be greater the loss due to respiration, mortality and predation (r_2), for the existence of the system. During the pre-monsoon period, the grazing pressure of zooplankton on phytoplankton is maximum; and the environmental conditions are favorable for primary production. In Hooghy-Matla estuarine system, fish population rise during monsoon. During pre-monsoon, the predation

pressure of carnivore fishes on zooplankton and herbivore fishes on phytoplankton is less in comparison to monsoon period; this situation favours the coexistence of phytoplankton and zooplankton population in the system (Mandal et al. 2012).

From the study of the system in that way, temperature dependent phytoplankton growth rate and salinity induced grazing rate of zooplankton are the key factors to describe the dynamical behavior of the system. For this reason, all the critical values of delay parameters are functions of temperature dependent phytoplankton growth rate and salinity induced grazing rate of zooplankton. Temperature and salinity effect on the system is very clearly described by Ray and Straskraba (2001), Mandal et al. (2012). In Mandal et al. (2012), authors show the temperature and salinity effect on phytoplankton growth rate and zooplankton grazing rate by real data set on the Hoogly-Matla estuary system as study area and they can plot the graph time functions of these two important system parameters. Basis on these results, we try to show the effect of gestation delays of phytoplankton and zooplankton on this system. Naturally, delay parameters become temperature and salinity dependent. So stable and unstable dynamics through Hopf bifurcation are strictly temperature and salinity dependent. It is very difficult to make proper time dependent function of temperature dependent phytoplankton growth rate and salinity induced zooplankton grazing rate from the graph time function of Mandal et al. (2012). For this reason non-autonomous system analysis is not possible, so better choice to describe such a system in numerical way. Most important motivation of this study is to describe the critical values of the delay parameters as a function of temperature and salinity intensity. So, the study is very relevant to describe that type of system and also to describe the significance of delay through the literature of ecology and mathematics.

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