Modelling effect of toxicant in a three-species food-chain system incorporating delay in toxicant uptake process by prey

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Abstract In this paper, a mathematical model is proposed and analyzed to study the effect of toxicant in a three-species food-chain system incorporating delay in toxicant uptake process by prey population. The model is formulated by using the system of non linear ordinary differential equations. In the model, it is assumed that the growth rate of prey population is affected by organismal toxicant. In this paper, we have introduced distributed delay in the environmental toxicant in the model. The distributed delay differential equations, though simple in structure, possess a rich array of solutions. The models are being analyzed by using variational matrix and Liapunov functions. The conditions for local and global stability of the equilibrium points are obtained. A region of attraction is being found for global asymptotic stability of the equilibrium points. Also, a Hopf bifurcation analysis has been performed with respect to key parameters for non-trivial equilibrium points. Furthermore, we support our analytical findings with numerical simulations.

Keywords Food-chain · Toxicant · Distributed delay · Equilibrium · Stability · Hopf-bifurcation

Mathematics Subject Classification 93A30 · 92D40 · 34D20

Introduction

It is now well known that the environmental pollution is caused by several different types of industrial discharges and chemical spills which pollute the air and contaminate the aquatic habitats (streams, rivers, lakes and oceans) with varieties of chemicals and toxicants such as arsenic, cadmium, lead, zinc, copper, iron, mercury, etc. These toxicants are responsible for the decrease in the growth rate of the exposed species and therefore, it is causing damage to terrestrial and aquatic populations (Dubey and Hussain 2006). Therefore, it is important to study the effects of toxicants on populations in order to find a theoretical threshold value of the organismal concentration of toxicant, which will predict the existence or extinction of a population or community in future.

Uncontrolled emission of toxicant into the environment is responsible for the extinction of many species and if it is not controlled then the species which are at the verge of extinction may vanish completely. Beyond effects on direct grazers, toxins may accumulate in and be transferred through marine food chains, affecting consumers at higher trophic levels, including fish, seabirds, and marine mammals (Turner and Tester 1997). According to Hladun et al. (2012) study that the Selenium (Se) is a metalloid that occurs naturally in certain alkaline soils from shale deposits of prehistoric inland seas. Agricultural water drainage dissolves Se from these naturally seleniferous soils and has caused the buildup of selenate, the predominant and bioavailable form of Se. One of the worst cases of Se pollution occurred at the Kesterson Reservoir in the San Joaquin Valley, a major drainage site for many agricultural regions of California. There is good evidence that Se accumulation can have negative effects on plant growth, insect herbivores, their predators and parasites,

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and the detritivores that feed on decaying plant and animal tissues.

In an experimental study of marine food chains (Zhang et al. 2013) the varying concentration of Hg in the microalgae species as well as the selectivity of zooplankton to the algal diet resulted in varying Hg accumulation in three zooplankton species i.e., feeding selectivity plays a key role in the transfer of MeHg along marine food chains. The destruction of zooplankton can indirectly affect fish diversity of reservoirs (Perez-Legaspi and Rico-Martinez 2001). The dynamic behaviour of the stressed ecosystem can be much more complicated than that of the unstressed system. In last few decades, many efforts have been made to study the effects of toxicant on a three-species food chain systems using mathematical models (Thomann et al. 1984; Thomann and Connolly 1984; Hallam and De Luna 1984). All these studies are related to the instantaneous effects of the toxicant on the species and/or on their biomass. The uptake of environmental toxicant by the resource-biomass may not be instantaneous. Therefore, the consideration of delay effect in such situation is very logical and close to real systems.

Often the dynamical behavior of biological systems depend on their past history and therefore it is important to incorporate continuous or distributed time delays in the model to study most of the biological processes. In general, delay differential equations exhibit more complicated dynamics on stability, periodic structure, bifurcation, and so on. The effect of the distributed delay on the stability of the model has been investigated by Shi et al. (2013). In last few decades, several investigators have proposed and analyzed mathematical models considering distributed delays in the growth dynamics of biological species (Arino et al. 2007; He and Ruan 1998; Bernard et al. 2001; Rasmussen et al. 2003; Ma and Takeuchi 1998; Gakkhar and Sahani 2009; Khare et al. 2011). Almost no studies have been conducted to investigate the modelling effect of toxicant in a three-species food-chain system incorporating delay in toxicant uptake process by prey.

Therefore in this paper we have studied the dynamical behaviour of three species food chain system under pollution stress considering the delay in toxicant uptake process by exposed prey population using mathematical model.

Mathematical model

We have incorporated delay in toxicant uptake process by exposed prey population in a three-species food-chain system. In the model, it is assumed that the growth rate of prey population is negatively affected by a organismal toxicant. In the model, we have introduced distributed delay in the

environmental toxicant. The underlying system consists of the state variables are $x(t)$, the density of the prey population; $y(t)$, the density of the intermediate predator population; $z(t)$, the density of the top predator population; $C_E(t)$, the environmental toxicant concentration; $C_0(t)$, the organism toxicant concentration; and $R_1(t)$, the distributed delay.

Taking these as state variables, we formulate the mathematical model using following system of non linear ordinary differential equations in order to study the role of distributed delay in toxicant uptake process by exposed prey population in a three-species food-chain system:

Main Model 1:

$$\frac{dx}{dt} = x \left(s_0 - s_1 C_0 - \frac{x}{k} \right) - \frac{q_1 xy}{1+x} \quad (1)$$

$$\frac{dy}{dt} = \frac{q_2 xy}{1+x} - \frac{q_3 yz}{1+y} - r_0 y - r_1 y^2 \quad (2)$$

$$\frac{dz}{dt} = \frac{q_4 yz}{1+y} - d_1 z - d_2 z^2 \quad (3)$$

$$\frac{dC_E}{dt} = q_0 - a_1 C_E - a_2 x R_1 + v x C_0 \quad (4)$$

$$\frac{dC_0}{dt} = a_2 x R_1 - b_1 C_0 - v x C_0 \quad (5)$$

$$\frac{dR_1}{dt} = \gamma_1 (C_E - R_1) \quad (6)$$

The initial conditions are

$$x(0) > 0, y(0) > 0, z(0) > 0, C_E(0) = f(0) \geq 0, C_0(0) \geq 0, R_1(0) > 0.$$

In the model, the growth equation for the prey population is logistic growth and term s_0 is the intrinsic growth rate of prey and k is the carrying capacity of environment for the prey population. r_0 and d_1 are the natural death rates of intermediate and top predator populations respectively. r_1 and d_2 are the intraspecies competition coefficients among intermediate and top predator populations respectively.

In the model $q_i u / (1 + u)$, ($i = 1, 2, 3, 4$; $u = x$ and y), account for the interactions between two different species, representing the Holling type-II functional response. This functional response is parameterized by the constant q_i . q_1 and q_3 are the predation rate of prey and intermediate predator populations respectively. q_2 and q_4 are the intermediate and top predators growth rates as result of feeding on prey and intermediate predator populations respectively. s_1 is the decreasing rate of the intrinsic growth rate associated with the uptake concentration of organismal toxicant. q_0 is the external input of toxicant into environment. The toxicant are assumed to wash out from the environment as well as from the organism at rate a_1 and b_1 , respectively. a_2 is the removal rate of toxicant from environment due to uptake by the prey species. v is the removal rate of toxicant from the prey species due to its death. It is assumed that the growth rate of uptake concentration of

organism $C_0(t)$ increases by the same amount as the depletion rate of toxicant concentration $C_E(t)$ in the environment and is assumed to be proportional to the prey population as well as the environmental concentration of the toxicant (i.e., a_2xC_E). Also it is assumed that the uptake concentration of organism may decrease with rate coefficient $v \geq 0$ due to decaying of prey population and which (i.e., $vx C_0$) may reenter into the environment (Shukla and Dubey 1997).

In the model, we have introduced distributed delay $R_1(t)$ in the term ' a_2xC_E ' which describes the uptake of environmental toxicant by the prey population (Khare et al. 2009, 2011).

$$R_1(t) = \int_{-\infty}^t \gamma_1 \exp(-\gamma_1(t-s)) C_E(s) ds$$

where $s \in (-\infty, 0]$, γ_1 is delay parameter. Now, if the effect of distributed delay is not considered in the above Model 1, then we have the following Model 2 for the effect of toxicant on three species food chain system:

Model 2: (With Toxic Effect and No Delay)

$$\frac{dx}{dt} = x \left(s_0 - s_1 C_0 - \frac{x}{k} \right) - \frac{q_1 xy}{1+x} \quad (7)$$

$$\frac{dy}{dt} = \frac{q_2 xy}{1+x} - \frac{q_3 yz}{1+y} - r_0 y - r_1 y^2 \quad (8)$$

$$\frac{dz}{dt} = \frac{q_4 yz}{1+y} - d_1 z - d_2 z^2 \quad (9)$$

$$\frac{dC_E}{dt} = q_0 - a_1 C_E - a_2 x C_E + vx C_0 \quad (10)$$

$$\frac{dC_0}{dt} = a_2 x C_E - b_1 C_0 - vx C_0 \quad (11)$$

with the initial conditions as $x(0) > 0, y(0) > 0, z(0) > 0, C_E(0) \geq 0, C_0(0) \geq 0$, where, the state variables and parameters are the same as defined for the Model 1. Now, if the effect of toxicant is not considered in the above Model 2, then we have the following Model 3 for three species food chain system:

Model 3: (Without Toxic Effect and No Delay)

$$\frac{dx}{dt} = x \left(s_0 - \frac{x}{k} \right) - \frac{q_1 xy}{1+x} \quad (12)$$

$$\frac{dy}{dt} = \frac{q_2 xy}{1+x} - \frac{q_3 yz}{1+y} - r_0 y - r_1 y^2 \quad (13)$$

$$\frac{dz}{dt} = \frac{q_4 yz}{1+y} - d_1 z - d_2 z^2 \quad (14)$$

with the initial conditions as $x(0) > 0, y(0) > 0, z(0) > 0$, where, the state variables and parameters are the same as defined for the Model 1.

Lemma 1 (Wang and Pang 2008) Suppose n is a positive integer and $f_i(t, x_1, x_2, \dots, x_n)$, ($i = 1, 2, 3, \dots, n$) are smooth functions. If $f_i|_{x_i=0, \Omega \in R_{+0}^n} \geq 0$ (where

$\Omega = (x_1, x_2, \dots, x_n)^T \in R^n$), then R_{+0}^n is an invariant domain of the following equations:

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n), (i = 1, 2, 3, \dots, n)$$

If $f_i|_{x_i=0, \Omega \in R_{+0}^n} \leq 0$ (where $\Omega = (x_1, x_2, \dots, x_n)^T \in R^n$), then R_{-0}^n is an invariant domain of the above equations.

Theorem 1

$$R_{+0}^6 := \{(x, y, z, C_E, C_0, R_1)^T | x \geq 0, y \geq 0, z \geq 0, C_E \geq 0, C_0 \geq 0, R_1 \geq 0\}$$

is an invariant domain of Eqs. (1)–(6).

Proof Denote that $\Omega := (x, y, z, C_E, C_0, R_1)^T$. For the Model 1, we notice that

$$f_1|_{x=0, \Omega \in R_{+0}^6} = \left(x \left(s_0 - s_1 C_0 - \frac{x}{k} \right) - \frac{q_1 xy}{1+x} \right) |_{x=0, \Omega \in R_{+0}^6} = 0 \geq 0,$$

$$f_2|_{y=0, \Omega \in R_{+0}^6} = \left(\frac{q_2 xy}{1+x} - \frac{q_3 yz}{1+y} - r_0 y - r_1 y^2 \right) |_{y=0, \Omega \in R_{+0}^6} = 0 \geq 0,$$

$$f_3|_{z=0, \Omega \in R_{+0}^6} = \left(\frac{q_4 yz}{1+y} - d_1 z - d_2 z^2 \right) |_{z=0, \Omega \in R_{+0}^6} = 0 \geq 0.$$

$$f_4|_{C_E=0, \Omega \in R_{+0}^6} = (q_0 - a_1 C_E - a_2 x R_1 + vx C_0) |_{C_E=0, \Omega \in R_{+0}^6} = q_0 + vx C_0 > 0,$$

$$f_5|_{C_0=0, \Omega \in R_{+0}^6} = (a_2 x R_1 - b_1 C_0 - vx C_0) |_{C_0=0, \Omega \in R_{+0}^6} = a_2 x R_1 > 0,$$

$$f_6|_{R_1=0, \Omega \in R_{+0}^6} = (\gamma_1 (C_E - R_1)) |_{R_1=0, \Omega \in R_{+0}^6} = 0 \geq 0.$$

By the Lemma 1, we get that R_{+0}^6 is an invariant domain of Eqs. (1)–(6). $\square$

Boundedness of the models

To analyze the models 1, 2 and 3, in this section we need the bounds of dependent variables involved. First we see the boundedness of Model 3, Model 2 and finally Model 1. So here, we find the region of attraction for all the models in the following lemma.

Lemma 2 The set

$$\Omega = \{(x, y, z, C_E, C_0, R_1) \in R_{+0}^6 : x(t) \leq s_0 k, x(t) + \frac{q_1}{q_2} y(t) + \frac{q_1 q_3}{q_2 q_4} z(t) \leq K_1, \\ C_E(t) + C_0(t) \leq K_2, C_E(t) + R_1(t) \leq K_5, C_E(t) \geq K_3, x(t) + C_E(t) \geq K_4\}$$

is a region of attraction for all solutions initiating in the interior of the positive region, where $K_1 = s_0 k(s_0 + 1)/\Phi_{11}$ and $\Phi_{11} = \min\{1, r_0, d_1\}$, $K_2 = (q_0 + 1)/\Phi_{12}$ and $\Phi_{12} = \min\{a_1, b_1\}$, $K_3 = q_0/\phi_{13}$ and $\phi_{13} = (a_1 + a_2 s_0 k)$, $K_4 = (q_0 - q_1 K_1)/\phi_{14}$ and $\phi_{14} = \max\{x_1 K_2, a_1 + a_2 s_0 k\}$, $K_5 = (q_0 + v K_1 K_2)/\phi_{15}$ and $\phi_{15} = \min\{a_1 - \gamma_1, \gamma_1\}$ and $a_1 > \gamma_1$.

Proof From Eq. (12) we get, $dx/dt \leq x(s_0 - \frac{x}{k})$ then by the usual comparison theorem, we get as $t \rightarrow \infty$,

$$x \leq s_0 k$$

Now, let us consider the following function:

$$W_{11}(t) = x(t) + \frac{q_1}{q_2} y(t) + \frac{q_1 q_3}{q_2 q_4} z(t)$$

by using Eqs. (12)–(14), we get $dW_{11}/dt + \Phi_{11} W_{11} \leq s_0 k(s_0 + 1)$ where $\Phi_{11} = \min\{1, r_0, d_1\}$ and then by the usual comparison theorem, we get as $t \rightarrow \infty$, $W_{11}(t) \leq s_0 k(s_0 + 1)$ and hence,

$$x(t) + \frac{q_1}{q_2} y(t) + \frac{q_1 q_3}{q_2 q_4} z(t) \leq K_1$$

where $K_1 = s_0 k(s_0 + 1)/\Phi_{11}$.

Now, let us consider the function: $W_{12}(t) = C_E(t) + C_0(t)$ by using Eqs. (10)–(11), we get $dW_{12}/dt + \Phi_{12} W_{12} \leq (q_0 + 1)$ where $\Phi_{12} = \min\{a_1, b_1\}$ and then by the usual comparison theorem, we get as $t \rightarrow \infty$, $W_{12}(t) \leq (q_0 + 1)/\phi_{12}$ and hence,

$$C_E(t) + C_0(t) \leq K_2$$

where $K_2 = (q_0 + 1)/\phi_{12}$.

Now, let us consider the function: $W_{15}(t) = C_E(t) + R_1(t)$ by using Eqs. (4) and (6), we get

$$\frac{dW_{15}}{dt} + \phi_{15} W_{15} \leq q_0 + \nu K_1 K_2$$

where $\phi_{15} = \min\{a_1 - \gamma_1, \gamma_1\}$ and $a_1 > \gamma_1$ then by the usual comparison theorem, we get as $t \rightarrow \infty$, $W_{15}(t) \leq (q_0 + \nu K_1 K_2)/\phi_{15}$ and hence,

$$C_E(t) + R_1(t) \leq K_5$$

where, $K_5 = (q_0 + \nu K_1 K_2)/\phi_{15}$.

From Eq. (10) we get, $dC_E/dt + \phi_{13} C_E \geq q_0$ where $\phi_{13} = (a_1 + a_2 s_0 k)$ then by the usual comparison theorem, we get as $t \rightarrow \infty$,

$$C_E(t) \geq K_3$$

where $K_3 = q_0/\phi_{13}$. Now, let us consider the function: $W_{14}(t) = x(t) + C_E(t)$ by using Eqs. (7) and (10), we get $dW_{14}/dt + \Phi_{14} W_{14} \geq (q_0 - q_1 K_1)$ where $\Phi_{14} = \max\{s_1 K_2 + s_0, a_1 + a_2 s_0 k\}$ and then by the usual comparison theorem, we get as $t \rightarrow \infty$, $W_{14}(t) \geq (q_0 - q_1 K_1)/\phi_{14}$ and hence,

$$x(t) + C_E(t) \geq K_4$$

where $K_4 = (q_0 - q_1 K_1)/\phi_{14}$.

Hence the solution of the models 1, 2 and 3 are bounded in Ω .

In the next section we study the dynamical behavior of three models.

Analysis of Model 1

Equilibria of Model 1

The Model 1 has four non negative equilibria in x, y, z, C_E, C_0, R_1 space namely, $E_{10}(0, 0, 0, \check{C}_E, 0, \check{R}_1)$, $E_{11}(\hat{x}, 0, 0, \hat{C}_E, \hat{C}_0, \hat{R}_1)$, $E_{12}(\bar{x}, \bar{y}, 0, \bar{C}_E, \bar{C}_0, \bar{R}_1)$ and $E_{13}(x^*, y^*, z^*, C_E^*, C_0^*, R_1^*)$. The existence of E_{10} is obvious. We prove the existence of E_{11} , E_{12} and E_{13} as follows:

- Existence of $E_{10} = (0, 0, 0, \check{C}_E, 0, \check{R}_1)$ from (4),

$$\check{C}_E = \frac{q_0}{a_1} \quad (15)$$

from (6),

$$\check{C}_E = \check{R}_1. \quad (16)$$

- Existence of $E_{11} = (\hat{x}, 0, 0, \hat{C}_E, \hat{C}_0, \hat{R}_1)$ from (1),

$$\hat{C}_0 = \frac{1}{s_1} \left(s_0 - \frac{\hat{x}}{k} \right) \quad (17)$$

$\hat{C}_0 > 0$ if $s_0 k > \hat{x}$. Addition of (4) and (5), and using (17), we get

$$\hat{C}_E = \frac{1}{a_1} \left(q_0 - \frac{b_1}{s_1} \left(s_0 - \frac{\hat{x}}{k} \right) \right) \quad (18)$$

$\hat{C}_E > 0$ if $q_0 s_1 k > b_1 (s_0 k - \hat{x})$. from (6),

$$\hat{R}_1 = \hat{C}_E \quad (19)$$

from (5),

$$A_1 \hat{x}^2 - A_2 \hat{x} - A_3 = 0 \quad (20)$$

where, $A_1 = (a_1 \nu + b_1 a_2)$, $A_2 = a_2 k (s_0 b_1 - s_1 q_0) + a_1 (\nu k s_0 - b_1) > 0$, $A_3 = a_1 b_1 s_0 k$. The Eq. (20) will have a positive root $\hat{x}$ always.

- Existence of $E_{12} = (\bar{x}, \bar{y}, 0, \bar{C}_E, \bar{C}_0, \bar{R}_1)$ Here, $\bar{x}, \bar{y}, \bar{C}_E, \bar{C}_0$ and $\bar{R}_1$ are the positive solutions of the system of algebraic equations: from (1),

$$C_0 = \frac{1}{s_1} \left[s_0 - \frac{x}{k} - \frac{q_1 y}{1+x} \right] = m_1(x, y) \quad (21)$$

from (4) and (5),

$$C_E = \frac{1}{a_1} [q_0 - b_1 m_1(x, y)] = m_2(x, y) \quad (22)$$

from (6),

$$\bar{R}_1 = \bar{C}_E \quad (23)$$

Now, considering two functions [from (1) to (4)],

$$W_1(x, y) = q_0 - (a_1 + a_2 x) m_2(x, y) + \nu x m_1(x, y) \quad (24)$$

$$W_2(x, y) = q_0 - a_1 C_E - a_2 x C_E + vx C_0 + q_2 x(s_0 - s_1 C_0 - x/k) - q_1 y(r_0 + r_1 y) \quad (25)$$

For existence of $\bar{x}$ and $\bar{y}$, the two isoclines,

$$W_1(x, y) = 0 \quad (26)$$

$$W_2(x, y) = 0 \quad (27)$$

must intersect. We note that

$$W_1(0, 0) = W_2(0, 0) = \frac{b_1 s_0}{s_1} > 0$$

Also, $W_1(x, 0) = 0$ then x will have one positive root (say) ψ_{31} , from the following quadratic equation of x ,

$$\frac{b_1 a_2}{a_1 s_1 k} x^2 - x \left[\frac{a_2}{a_1 s_1} (b_1 s_0 - q_0 s_1) - \frac{b_1}{s_1 k} \right] - \frac{b_1 s_0}{s_1} = 0$$

if $b_1 s_0 > q_0 s_1$ and $ka_2(b_1 s_0 - q_0 s_1) - b_1 a_1 > 0$. $W_1(0, y) = 0$ then $y = \frac{s_0}{q_1} = \psi_{32}$ (say). $W_2(x, 0) = 0$ then x will have one positive root (say) ψ_{33} , from the following quadratic equation of x ,

$$x^2 \left(\frac{a_2}{a_1 k} + \frac{v}{s_1} \right) - \frac{x}{a_1 s_1 k} [k(s_0(b_1 a_2 + a_1 v) - a_2 q_0 s_1) - s_1 a_1] - b_1 s_0 / s_1 = 0$$

if $k(s_0(b_1 a_2 + a_1 v) - a_2 q_0 s_1) - s_1 a_1 > 0$. $W_2(0, y) = 0$ then y will have one positive root (say) ψ_{34} , from the following quadratic equation of y ,

$$r_1 q_1 y^2 + y(r_0 q_1 + b_1 q_1 / s_1) - s_0 b_1 / s_1 = 0.$$

Thus both the isoclines intersect each other in the region

$$G_{14} = \{(x, y) : 0 < x < \psi_{33}, 0 < y < \psi_{32}\}$$

in the following two cases: (see Fig.1)

$$\text{Case (i) : } \psi_{33} > \psi_{31}, \psi_{32} > \psi_{34} \quad (28)$$

$$\text{Case (ii) : } \psi_{33} < \psi_{31}, \psi_{32} < \psi_{34} \quad (29)$$

This point of intersection will give $\bar{x}, \bar{y}$. For uniqueness of $(\bar{x}, \bar{y})$, we must have $\frac{dy}{dx} < 0$ for both the curves in the region G_{14} . For curve (26),

$$\frac{dy}{dx} = \frac{y}{1+x} - \frac{y}{x} - \frac{s_1(1+x)}{vq_1x} \left[\frac{2vx}{s_1k} + (a_1 + a_2x)m'_2(x, y) + a_2m_2(x, y) - \frac{vs_0}{s_1} \right] < 0 \quad (30)$$

and for curve (27),

$$\frac{dy}{dx} = \frac{\eta_1 + q_2(s_0 - s_1m_1(x, y) - x/k) - \eta_2 - \eta_3}{q_1[r_0 + 2r_1y]} < 0 \quad (31)$$

where, $\eta_1 = v[xm'_1(x, y) + m_1(x, y)]$, $\eta_2 = (a_1 + a_2x)m'_2(x, y) + a_2m_2(x, y)$, $\eta_3 = q_2x[s_1m'_1(x, y) + 1/k]$. In case (i), the absolute value of $\frac{dy}{dx}$ given by (30) is less than the absolute value of $\frac{dy}{dx}$ given by (31). For the case (ii), just the opposite is the condition. Knowing the values of $\bar{x}$ and $\bar{y}$; $\bar{C}_E$, $\bar{C}_0$ and $\bar{R}_1$ can be computed from the Eqs. (21) to (23).

– Existence of $E_{13}^* = (x^*, y^*, z^*, C_E^*, C_0^*, R_1^*)$

Here, $x^*, y^*, z^*, C_E^*, C_0^*$ and R_1^* are the positive solutions of the system of algebraic equations:

from (1),

$$C_0 = \frac{1}{s_1} \left[s_0 - \frac{x}{k} - \frac{q_1 y}{1+x} \right] = g_1(x, y) \quad (32)$$

from (4) and (5),

$$C_E = \frac{1}{a_1} [q_0 - b_1 g_1(x, y)] = g_2(x, y) \quad (33)$$

from (2),

$$z = \frac{1+y}{q_3} \left[\frac{q_2 x}{1+x} - r_0 - r_1 y \right] = g_3(x, y) \quad (34)$$

Fig. 1 Existence of equilibrium point E_{12} of Model 1

from (6),

$$R_1^* = C_E^* \quad (35)$$

Now, considering two functions [from (2) to (4)],

$$S_{11}(x, y) = q_0 - (a_1 + a_2x)g_2(x, y) + vxg_1(x, y) \quad (36)$$

$$S_{12}(x, y) = q_0 - (a_1 + a_2x)g_2(x, y) + vxg_1(x, y) - q_4y(r_0 + r_1y) - q_3z(d_1 + d_2z) + \frac{q_2q_4xy}{1+x} \quad (37)$$

For existence of x^* and y^* , the two isoclines,

$$S_{11}(x, y) = 0 \quad (38)$$

$$S_{12}(x, y) = 0 \quad (39)$$

must intersect. We note that

$$S_{11}(0, 0) = S_{12}(0, 0) = \frac{b_1s_0}{s_1} > 0$$

Also, $S_{11}(x, 0) = 0$ then x will have one positive root (say) ψ_{41} , from the following quadratic equation of x ,

$$x^2(v + b_1a_2/a_1)/s_1k - x[(b_1s_0 - s_1q_0)a_2/s_1a_1 + (vs_0k - b_1)/s_1k] - s_0b_1/s_1 = 0$$

if $b_1s_0 > s_1q_0$ and $vs_0k > b_1$. $S_{11}(0, y) = 0$ then $y = \frac{s_0}{q_1} = \psi_{42}$ (say).

$S_{12}(x, 0) = 0$ then x will have one positive root (say) ψ_{43} , from the following quadratic equation of x ,

$$x^3(v + \frac{b_1a_2}{a_1})\frac{q_3}{s_1k} - x^2[(b_1s_0 - q_0s_1)\frac{a_2q_3}{a_1s_1} + (vs_0k - b_1)\frac{q_3}{s_1k} - (b_1a_2 + va_1)\frac{q_3}{ks_1a_1}] - x[(v + b_1)\frac{s_0q_3}{s_1} + (b_1s_0 - q_0s_1)\frac{a_2q_3}{a_1s_1} - (q_2 - r_0 + \frac{b_1q_3}{s_1k})] - (r_0 + \frac{b_1s_0q_0}{s_1}) = 0.$$

if $(b_1s_0 - q_0s_1)\frac{a_2q_3}{a_1s_1} + (vs_0k - b_1)\frac{q_3}{s_1k} - (b_1a_2 + va_1)\frac{q_3}{ks_1a_1} > 0$

and $[(v + b_1)\frac{s_0q_3}{s_1} + (b_1s_0 - q_0s_1)\frac{a_2q_3}{a_1s_1} - (q_2 - r_0 + \frac{b_1q_3}{s_1k})] > 0$.

$S_{12}(0, y) = 0$ then y will have one positive root (say) ψ_{44} , from the following fourth degree equation of y ,

$$G_1y^4 + G_2y^3 - G_3y^2 + G_4y - G_5 = 0$$

$G_1 = d_2r_1^2/q_3 > 0$, $G_2 = 2r_1d_2(r_1 + r_0)/q_3$, $G_3 = d_1r_1 - [2r_0r_1 + (r_1 + r_0)^2d_2/q_3] - r_1q_4$, $G_4 = b_1q_1/s_1 + r_0q_4 - d_1(r_1 + r_0) + [2r_0(r_0 + r_1)]d_2/q_3 > 0$ and $G_5 = b_1s_0/s_1 + d_1r_0 - d_2r_0^2/q_3 > 0$

if $d_1r_1 - [2r_0r_1 + (r_1 + r_0)^2d_2/q_3] - r_1q_4 > 0$, $b_1q_1/s_1 + r_0q_4 - d_1(r_1 + r_0) + [2r_0(r_0 + r_1)]d_2/q_3 > 0$ and

$$b_1s_0/s_1 + d_1r_0 - d_2r_0^2/q_3 > 0.$$

Thus both the isoclines intersect each other in the region

$$G_{15} = \{(x, y) : 0 < x < \psi_{43}, 0 < y < \psi_{42}\}$$

in the following two cases: (see Fig. 2)

$$\text{Case (i)} : \psi_{43} > \psi_{41}, \psi_{42} > \psi_{44} \quad (40)$$

$$\text{Case (ii)} : \psi_{43} < \psi_{41}, \psi_{42} < \psi_{44} \quad (41)$$

This point of intersection will give x^*, y^* . For uniqueness of (x^*, y^*) , we must have $\frac{dy}{dx} < 0$ for both the curves in the region G_{15} .

For curve (38),

$$\frac{dy}{dx} = \frac{y}{1+x} - \frac{y}{x} - \frac{s_1(1+x)}{vq_1x} \left[\frac{2vx}{s_1k} + (a_1 + a_2x)g_2'(x, y) + a_2g_2(x, y) - \frac{vs_0}{s_1} \right] < 0 \quad (42)$$

and for curve (39),

$$\frac{dy}{dx} = \frac{\eta_{11} - \eta_{12} - q_3g_3'(x, y)[d_1 + 2d_2g_3(x, y)] + \frac{q_2q_4y}{(1+x)^2}}{\frac{q_2q_4x}{1+x} - q_4(r_0 + 2r_1y)} < 0 \quad (43)$$

Fig. 2 Existence of equilibrium point E_{13}^* of Model 1

where, $\eta_{11} = v[xg'_1(x, y) + g_1(x, y)]$, $\eta_{12} = (a_1 + a_2x)g'_2(x, y) + a_2g_2(x, y)$. In case (i), the absolute value of $\frac{dy}{dx}$ given by (42) is less than the absolute value of $\frac{dy}{dx}$ given by (43). For the case (ii), just the opposite is the condition.

Knowing the value of x^* and y^* ; z^* , C_E^* , C_0^* and R_1^* can be computed from the Eqs. (32) to (35).

Dynamical behaviour of the Model 1

Local stability

The stability behavior of E_{10} can be studied by computing variational matrix, and E_{11} , E_{12} and E_{13}^* can be studied by computing Liapunov's direct method.

The general variational matrix corresponding to the Model 1 is

$$J(x, y, z, C_E, C_0, R_1) = \begin{bmatrix} c_{11} & -c_{12} & 0 & 0 & -c_{15} & 0 \\ c_{21} & c_{22} & -c_{23} & 0 & 0 & 0 \\ 0 & c_{32} & c_{33} & 0 & 0 & 0 \\ -c_{41} & 0 & 0 & -c_{44} & c_{45} & -c_{46} \\ c_{51} & 0 & 0 & 0 & -c_{55} & c_{56} \\ 0 & 0 & 0 & c_{64} & 0 & -c_{66} \end{bmatrix}$$

where,

$$\begin{aligned} c_{11} &= x\left(\frac{yq_1}{(1+x)^2} - \frac{1}{k}\right) + (s_0 - s_1C_0 - \frac{x}{k} - \frac{q_1y}{1+x}), \\ c_{12} &= \frac{q_1x}{1+x}, c_{15} = s_1x, \\ c_{21} &= \frac{q_2y}{(1+x)^2}, c_{22} = y\left(\frac{q_3z}{(1+y)^2} - r_1\right) \\ &+ \left(\frac{q_2x}{1+x} - \frac{q_3z}{1+y} - r_0 - r_1y\right), c_{23} = \frac{q_3y}{1+y}, \\ c_{32} &= \frac{q_4z}{(1+y)^2}, c_{33} = \frac{q_4y}{1+y} - d_1 - 2d_2z, \\ c_{41} &= a_2R_1 - vC_0, c_{44} = a_1, c_{45} = vx, \\ c_{46} &= a_2x, c_{51} = a_2R_1 - vC_0, c_{55} = b_1 + vx, \\ c_{56} &= a_2x, c_{64} = \gamma_1, c_{66} = \gamma_1 \end{aligned}$$

About E_{10} , the eigenvalues of the characteristic equation are s_0 , $-r_0$, $-d_1$, $-a_1$, $-b_1$ and $-\gamma_1$, which shows that E_{10} is unstable.

About E_{11} , two of the eigenvalues of the characteristic equations are $q_2\hat{x}/(1+\hat{x}) - r_0$ and $-d_1$ and the other four eigenvalues are given by the roots of the following equation is

$$\lambda^4 + \lambda^3J_1 + \lambda^2J_2 + \lambda J_3 + J_4 = 0 \quad (44)$$

where, $J_1 = \gamma_1 + a_1 + (b_1 + v\hat{x}) + \hat{x}/k$, $J_2 = (b_1 + v\hat{x})(a_1 + \gamma_1 + \hat{x}/k) + (a_1 + \gamma_1)\hat{x}/k + (a_2\hat{R}_1 - v\hat{C}_0)\frac{q_1\hat{x}}{1+\hat{x}} - \gamma_1(a_2\hat{x} - a_1) > 0$, $J_3 = (a_1 + \gamma_1)(b_1 + v\hat{x})\frac{\hat{x}}{k} - \gamma_1(b_1 + v\hat{x})(a_2\hat{x} - a_1) + (a_2\hat{R}_1 - v\hat{C}_0)[(a_1 + \gamma_1)\frac{q_1\hat{x}}{1+\hat{x}} - a_2\hat{x}\gamma_1] - \gamma_1(a_2\hat{x}$

$$-a_1)\frac{\hat{x}}{k}, J_4 = \gamma_1(a_2\hat{R}_1 - v\hat{C}_0)\left[\frac{q_1q_1\hat{x}}{1+\hat{x}} - (2(\frac{q_1\hat{x}}{1+\hat{x}}) + \frac{\hat{x}}{k})a_2\hat{x}\right] - \gamma_1(b_1 + v\hat{x})(a_2\hat{x} - a_1)\frac{\hat{x}}{k}.$$

According to Routh-Hurwitz criteria E_{11} is locally asymptotically stable if

$$\frac{q_2}{r_0} < \frac{1}{\hat{x}} + 1 \Rightarrow \frac{q_2}{r_0} < 1$$

$$J_i > 0 \quad (i = 1, 2, 3, 4),$$

$$\frac{\hat{C}_E}{\hat{C}_0} < \frac{v}{a_2}, \hat{x} < \frac{a_1}{a_2}, \hat{x} < \frac{a_1}{2a_2}, a_2\hat{x}(1 + \hat{x}) < q_1k(a_1 - 2a_2\hat{x})$$

and $J_1J_2J_3 > J_3^2 + J_1^2J_4$, i.e.,

$$\begin{aligned} &(\gamma_1 + a_1 + (b_1 + v\hat{x}) + \hat{x}/k)[(b_1 + v\hat{x})(a_1 + \gamma_1 + \hat{x}/k) + (a_1 + \gamma_1)\hat{x}/k + (a_2\hat{R}_1 - v\hat{C}_0)\frac{q_1\hat{x}}{1+\hat{x}} - \gamma_1(a_2\hat{x} - a_1)][(a_1 + \gamma_1)(b_1 + v\hat{x})\frac{\hat{x}}{k} - \gamma_1(b_1 + v\hat{x})(a_2\hat{x} - a_1) + (a_2\hat{R}_1 - v\hat{C}_0)[(a_1 + \gamma_1)\frac{q_1\hat{x}}{1+\hat{x}} - a_2\hat{x}\gamma_1] - \gamma_1(a_2\hat{x} - a_1)\frac{\hat{x}}{k}] > [(a_1 + \gamma_1)(b_1 + v\hat{x})\frac{\hat{x}}{k} - \gamma_1(b_1 + v\hat{x})(a_2\hat{x} - a_1) + (a_2\hat{R}_1 - v\hat{C}_0)[(a_1 + \gamma_1)\frac{q_1\hat{x}}{1+\hat{x}} - a_2\hat{x}\gamma_1] - \gamma_1(a_2\hat{x} - a_1)\frac{\hat{x}}{k}]^2 + (\gamma_1 + a_1 + (b_1 + v\hat{x}) + \hat{x}/k)^2 [\gamma_1(a_2\hat{R}_1 - v\hat{C}_0)\frac{q_1\hat{x}}{1+\hat{x}} - (2(\frac{q_1\hat{x}}{1+\hat{x}}) + \frac{\hat{x}}{k})a_2\hat{x}] - \gamma_1(b_1 + v\hat{x})(a_2\hat{x} - a_1)\frac{\hat{x}}{k}]. \end{aligned}$$

Remark 1 From the stability conditions of E_{11} , the following may be noted:

- the ratio of the growth rate of intermediate predator and its death rate is less than one,
- the ratio of the concentration of environmental toxicant and the concentration of organismal toxicant is less than the ratio of the removal rate of toxicant from the prey species and the toxicant uptake by the prey species,
- the equilibria of prey is less than the ratio of toxicant wash out from the environment and removal rate of toxicant from the environment due to uptake of by the prey species, to the ratio of toxicant wash out from the environment and the twice of removal rate of toxicant from the environment due to uptake of by the prey species, are satisfied then only prey population will survive in the presence of toxicant and when the distributed delay is considered.

Theorem 2 If the following inequalities hold

$$q_1k\bar{y} < (1 + \bar{x})^2 \quad (45)$$

$$q_4\bar{y} < d_1(1 + \bar{x}) \quad (46)$$

$$9[s_1\bar{x} + T_4(v\bar{C}_0 - a_2\bar{R}_1)]^2 < \bar{x}T_4(b_1 + v\bar{x})\left(\frac{1}{k} - \frac{q_1\bar{y}}{(1 + \bar{x})^2}\right) \quad (47)$$

$$6T_4(a_2\bar{x})^2 < \gamma_1T_5(b_1 + v\bar{x}) \quad (48)$$

where,

$$T_1 = \frac{q_1 \bar{x}(1 + \bar{x})}{q_2 \bar{y}} > 0 \quad (49)$$

$$T_2 > \frac{2\bar{x}q_1(1 + \bar{x})(\frac{q_3}{1+\bar{y}})^2}{r_1 q_2 (\frac{q_4 \bar{y}}{1+\bar{y}} - d_1)} \quad (50)$$

$$T_3 < \frac{a_1 \bar{x}}{9(v\bar{C}_0 - a_2 \bar{R}_1)^2} \left(\frac{1}{k} - \frac{q_1 \bar{y}}{(1 + \bar{x})^2} \right) \quad (51)$$

$$T_4 > \frac{v^2 \bar{x}^3}{(b_1 + v\bar{x})(v\bar{C}_0 - a_2 \bar{R}_1)^2} \left(\frac{1}{k} - \frac{q_1 \bar{y}}{(1 + \bar{x})^2} \right) \quad (52)$$

$$T_5 < \frac{a_1^2 \bar{x}}{54\gamma_1(v\bar{C}_0 - a_2 \bar{R}_1)^2} \left(\frac{1}{k} - \frac{q_1 \bar{y}}{(1 + \bar{x})^2} \right) \quad (53)$$

then the equilibrium E_{12} is locally asymptotically stable.

Proof We first linearize the system about the equilibrium E_{12} by using the following transformations

$$x = \bar{x} + n_1; \quad y = \bar{y} + n_2; \quad z = \bar{z} + n_3; \quad C_E = \bar{C}_E + n_4; \\ C_0 = \bar{C}_0 + n_5; \quad R_1 = \bar{R}_1 + n_6;$$

where n_1, n_2, n_3, n_4, n_5 and n_6 are small perturbations around E_{12} . Then we get the following linearized the system,

$$\frac{dn_1}{dt} = -n_1 \bar{x} \left(\frac{1}{k} - \frac{q_1 \bar{y}}{(1 + \bar{x})^2} \right) - n_2 \left(\frac{q_1 \bar{x}}{1 + \bar{x}} \right) - n_5 s_1 \bar{x}$$

$$\frac{dn_2}{dt} = n_1 \left(\frac{q_2 \bar{y}}{(1 + \bar{y})^2} \right) - n_2 r_1 \bar{y} - n_3 \left(\frac{q_3 \bar{y}}{1 + \bar{y}} \right)$$

$$\frac{dn_3}{dt} = -n_3 (d_1 - \frac{q_4 \bar{y}}{1 + \bar{y}})$$

$$\frac{dn_4}{dt} = n_1 (v\bar{C}_0 - a_2 \bar{R}_1) - a_1 n_4 + v\bar{x} n_5 - a_2 \bar{x}^* n_6$$

$$\frac{dn_5}{dt} = -n_1 (v\bar{C}_0 - a_2 \bar{R}_1) - n_5 (b_1 + v\bar{x}) + n_6 a_2 \bar{x}$$

$$\frac{dn_6}{dt} = n_4 \gamma_1 - n_6 \gamma_1$$

Now consider the following positive definite function

$$V_{32} = \frac{1}{2} n_1^2 + T_1 \frac{1}{2} n_2^2 + T_2 \frac{1}{2} n_3^2 + T_3 \frac{1}{2} n_4^2 + T_4 \frac{1}{2} n_5^2 + T_5 \frac{1}{2} n_6^2 \quad (54)$$

$$\frac{dV_{32}}{dt} = n_1 \frac{dn_1}{dt} + T_1 n_2 \frac{dn_2}{dt} + T_2 n_3 \frac{dn_3}{dt} + T_3 n_4 \frac{dn_4}{dt} \\ + T_4 n_5 \frac{dn_5}{dt} + T_5 n_6 \frac{dn_6}{dt} \quad (55)$$

$$\frac{dV_{32}}{dt} = -n_1^2 \bar{x} \left(\frac{1}{k} - \frac{q_1 \bar{y}}{(1 + \bar{x})^2} \right) - n_2^2 r_1 T_1 \bar{y} \\ - n_3^2 T_2 (d_1 - \frac{q_4 \bar{y}}{1 + \bar{y}}) - n_4^2 T_3 a_1 \\ - n_5^2 T_4 (b_1 + v\bar{x}) - n_6^2 \gamma_1 T_5 - n_1 n_2 \left(\frac{q_1 \bar{x}}{1 + \bar{x}} - \frac{T_1 q_2 \bar{y}}{(1 + \bar{x})^2} \right) \\ - n_1 n_4 T_3 (v\bar{C}_0 - a_2 \bar{R}_1) \\ - n_1 n_5 (s_1 \bar{x} + T_4 (v\bar{C}_0 - a_2 \bar{R}_1)) - n_2 n_3 \frac{q_3 \bar{y} T_1}{1 + \bar{y}} + n_4 n_5 v\bar{x} T_3 \\ + n_4 n_6 \gamma_1 T_5 + n_5 n_6 a_2 \bar{x} T_4$$

Now using the sylvester's criterion in the quadratic forms

$$\frac{dV_{32}}{dt} \leq - [((t_{11}/2)n_1^2 + t_{12}n_1n_2 + (t_{22}/2)n_2^2) \\ + ((t_{11}/2)n_1^2 - t_{14}n_1n_4 + (t_{44}/2)n_4^2) \\ + ((t_{11}/2)n_1^2 + t_{15}n_1n_5 + (t_{55}/2)n_5^2) \\ + ((t_{22}/2)n_2^2 + t_{23}n_2n_3 + (t_{33}/2)n_3^2) \\ + ((t_{44}/2)n_2^2 - t_{45}n_4n_5 + (t_{55}/2)n_5^2) \\ + ((t_{44}/2)n_2^2 - t_{46}n_4n_6 + (t_{66}/2)n_6^2) \\ + ((t_{55}/2)n_2^2 - t_{56}n_2n_6 + (t_{66}/2)n_6^2)]$$

where,

$$t_{11} = \left(\frac{1}{k} - \frac{q_1 \bar{y}}{(1 + \bar{x})^2} \right) \bar{x}/3, t_{12} = \frac{q_1 \bar{x}}{1 + \bar{x}} - \frac{T_1 q_2 \bar{y}}{(1 + \bar{x})^2},$$

$$t_{14} = T_3 (v\bar{C}_0 - a_2 \bar{R}_1),$$

$$t_{15} = s_1 \bar{x} + T_4 (v\bar{C}_0 - a_2 \bar{R}_1), t_{22} = r_1 T_1 \bar{y}/2, t_{23} = \frac{q_3 \bar{y} T_1}{1 + \bar{y}},$$

$$t_{33} = T_2 (d_1 - \frac{q_4 \bar{y}}{1 + \bar{y}}),$$

$$t_{44} = a_1 T_3/3, t_{45} = v\bar{x} T_3, t_{46} = \gamma_1 T_5, t_{55} = (b_1 + v\bar{x}) T_4/3,$$

$$t_{56} = a_2 \bar{x} T_4,$$

$$t_{66} = \gamma_1 T_5/2.$$

Sufficient conditions for dV_{32}/dt to be negative definite are that the following inequalities hold:

$$t_{11} > 0 \quad (56)$$

$$t_{33} > 0 \quad (57)$$

$$t_{11} t_{22} > t_{12}^2 \quad (58)$$

$$t_{11} t_{44} > t_{14}^2 \quad (59)$$

$$t_{11} t_{55} > t_{15}^2 \quad (60)$$

$$t_{22} t_{33} > t_{23}^2 \quad (61)$$

$$t_{44}t_{55} > t_{45}^2 \quad (62)$$

$$t_{44}t_{66} > t_{46}^2 \quad (63)$$

$$t_{55}t_{66} > t_{56}^2 \quad (64)$$

We note that the third, forth, sixth, seventh and eighth inequalities, i.e., $t_{11}t_{22} > t_{12}^2$, $t_{11}t_{44} > t_{14}^2$, $t_{22}t_{33} > t_{23}^2$, $t_{44}t_{55} > t_{45}^2$ and $t_{44}t_{66} > t_{46}^2$ are satisfied due to the proper choice of T_1, T_2, T_3, T_4 and T_5 and other inequalities, (45) $\Rightarrow$ (56), (46) $\Rightarrow$ (57), (47) $\Rightarrow$ (60) and (48) $\Rightarrow$ (64). Hence V_{32} is a Liapunov function with respect to E_{12}^- , proving the theorem.

Theorem 3 *If the following inequalities hold*

$$q_1ky^* < (1+x^*)^2 \quad (65)$$

$$q_3z^* < r_1(1+y^*)^2 \quad (66)$$

$$9(s_1x^* + L_4(vC_0^* - a_2R_1^*))^2 < x^*L_4(b_1 + vx^*) \left(\frac{1}{k} - \frac{q_1y^*}{(1+x^*)^2} \right) \quad (67)$$

$$9(vx^*L_5)^2 < a_1L_3L_4(b_1 + vx^*) \quad (68)$$

where,

$$L_1 = \frac{q_1x^*(1+x^*)}{q_2y^*} > 0$$

$$L_2 = \frac{q_1q_3x^*}{q_2q_4z^*}(1+x^*)(1+y^*) > 0$$

$$L_3 < \frac{x^*a_1}{9(vC_0^* - a_2R_1^*)} \left(\frac{1}{k} - \frac{q_1y^*}{(1+x^*)^2} \right)$$

$$L_4 < \frac{\gamma_1L_5(b_1 + vx^*)}{6a_2^2x^{*2}} > 0$$

$$L_5 > \frac{a_1a_2x^{*2}}{9\gamma_1(vC_0^* - a_2R_1^*)} \left(\frac{1}{k} - \frac{q_1y^*}{(1+x^*)^2} \right)$$

then the positive equilibrium E_{13}^* is locally asymptotically stable.

Proof We first linearize the system about the equilibrium E_{13}^* by using the following transformations

$$x = x^* + n_1; \quad y = y^* + n_2; \quad z = z^* + n_3; \quad C_E = C_E^* + n_4; \\ C_0 = C_0^* + n_5; \quad R_1 = R_1^* + n_6;$$

where n_1, n_2, n_3, n_4, n_5 and n_6 are small perturbations around E_{13}^* . Then we get the following linearized the system,

$$\begin{aligned} \frac{dn_1}{dt} &= -n_1x^* \left(\frac{1}{k} - \frac{q_1y^*}{(1+x^*)^2} \right) - n_2 \frac{q_1x^*}{1+x^*} - n_5s_1x^* \\ \frac{dn_2}{dt} &= n_1 \frac{q_2y^*}{(1+x^*)^2} - n_2y^*(r_1 - \frac{q_3z^*}{(1+y^*)^2}) - n_3 \frac{q_3y^*}{1+y^*} \\ \frac{dn_3}{dt} &= n_2 \frac{q_4z^*}{(1+y^*)^2} - n_3d_2z^* \\ \frac{dn_4}{dt} &= n_1(vC_0^* - a_2R_1^*) - a_1n_4 + vx^*n_5 - a_2x^*n_6 \\ \frac{dn_5}{dt} &= -n_1(vC_0^* - a_2R_1^*) - n_5(b_1 + vx^*) + n_6a_2x^* \\ \frac{dn_6}{dt} &= n_4\gamma_1 - n_6\gamma_1 \end{aligned}$$

Now consider the following positive definite function

$$\begin{aligned} V_{33} &= \frac{1}{2}n_1^2 + L_1\frac{1}{2}n_2^2 + L_2\frac{1}{2}n_3^2 + L_3\frac{1}{2}n_4^2 + L_4\frac{1}{2}n_5^2 + L_5\frac{1}{2}n_6^2 \\ \frac{dV_{33}}{dt} &= n_1\frac{dn_1}{dt} + L_1n_2\frac{dn_2}{dt} + L_2n_3\frac{dn_3}{dt} + L_3n_4\frac{dn_4}{dt} \\ &\quad + L_4n_5\frac{dn_5}{dt} + L_5n_6\frac{dn_6}{dt} \\ \frac{dV_{33}}{dt} &= -n_1^2x^* \left(\frac{1}{k} - \frac{q_1y^*}{(1+x^*)^2} \right) \\ &\quad - n_2^2L_1y^*(r_1 - \frac{q_3z^*}{(1+y^*)^2}) - n_3^2L_2d_2z^* \\ &\quad - n_4^2L_3a_1 - n_5^2L_4(b_1 + vx^*) - n_6^2\gamma_1L_5 \\ &\quad - n_1n_2 \left(\frac{q_1x^*}{1+x^*} - \frac{L_1q_2y^*}{(1+x^*)^2} \right) \\ &\quad + n_1n_4L_3(vC_0^* - a_2R_1^*) - n_1n_5(s_1x^* + L_4(vC_0^* - a_2R_1^*)) \\ &\quad - n_2n_3 \left(\frac{L_1q_3y^*}{1+y^*} - \frac{L_2q_4z^*}{(1+y^*)^2} \right) + n_4n_5L_3vx^* \\ &\quad - n_4n_6(a_2x^*L_3 - L_5\gamma_1) + n_5n_6a_2x^*L_4 \end{aligned}$$

Now using the sylvester's criterion in the quadratic forms

$$\begin{aligned} \frac{dV_{33}}{dt} &\leq -[(e_{11}/2)n_1^2 + e_{12}n_1n_2 + (e_{22}/2)n_2^2 \\ &\quad + ((e_{11}/2)n_1^2 - e_{14}n_1n_4 + (e_{44}/2)n_4^2) \\ &\quad + ((e_{11}/2)n_1^2 + e_{15}n_1n_5 + (e_{55}/2)n_5^2) \\ &\quad + ((e_{22}/2)n_2^2 + e_{23}n_2n_3 + (e_{33}/2)n_3^2) \\ &\quad + ((e_{44}/2)n_4^2 - e_{45}n_4n_5 + (e_{55}/2)n_5^2) \\ &\quad + ((e_{44}/2)n_4^2 + e_{46}n_4n_6 + (e_{66}/2)n_6^2) \\ &\quad + ((e_{55}/2)n_5^2 - e_{56}n_5n_6 + (e_{66}/2)n_6^2)] \end{aligned}$$

where,

$$\begin{aligned} e_{11} &= \left(\frac{1}{k} - \frac{q_1 y^*}{(1+x^*)^2} \right) x^*/3, e_{12} \\ &= \left(\frac{q_1 x^*}{1+x^*} - \frac{L_1 q_2 y^*}{(1+x^*)^2} \right), e_{14} = L_3 (vC_0^* - a_2 R_1^*), \\ e_{15} &= (s_1 x^* + L_4 (vC_0^* - a_2 R_1^*)), e_{22} = L_1 (r_1 - \frac{q_3 z^*}{(1+y^*)^2}) y^*/2, \\ e_{23} &= \frac{L_1 q_3 y^*}{1+y^*} - \frac{L_2 q_4 z^*}{(1+y^*)^2}, e_{33} = L_2 d_2 z^*, e_{44} = a_1 L_3/3, \\ e_{45} &= L_3 v x^*, e_{46} = a_2 x^* L_3 - \gamma_1 L_5, e_{55} = L_4 (b_1 + v x^*)/3, \\ e_{56} &= a_2 x^* L_4, e_{66} = \gamma_1 L_5/2. \end{aligned}$$

Sufficient conditions for dV_{33}/dt to be negative definite are that the following inequalities hold:

$$e_{11} > 0 \quad (69)$$

$$e_{22} > 0 \quad (70)$$

$$e_{11}e_{22} > e_{12}^2 \quad (71)$$

$$e_{11}e_{44} > e_{14}^2 \quad (72)$$

$$e_{11}e_{55} > e_{15}^2 \quad (73)$$

$$e_{22}e_{33} > e_{23}^2 \quad (74)$$

$$e_{44}e_{55} > e_{45}^2 \quad (75)$$

$$e_{44}e_{66} > e_{46}^2 \quad (76)$$

$$e_{55}e_{66} > e_{56}^2 \quad (77)$$

We note that the third, fourth, sixth, eighth and ninth inequalities, i.e., $e_{11}e_{22} > e_{12}^2$, $e_{11}e_{44} > e_{14}^2$, $e_{22}e_{33} > e_{23}^2$, $e_{44}e_{66} > e_{46}^2$ and $e_{55}e_{66} > e_{56}^2$ are satisfied due to the proper choice of L_1 , L_2 , L_3 , L_4 and L_5 and other inequalities, (65) $\Rightarrow$ (69), (66) $\Rightarrow$ (70), (67) $\Rightarrow$ (73) and (68) $\Rightarrow$ (75). Hence V_{33} is a Liapunov function with respect to E_{13}^* , proving the theorem.

Global stability

Theorem 4 E_{13}^* is globally asymptotically stable if $q_1^* > q_1$, $q_3^* > q_3$, $a_1 > a_1^*$, $b_1 > b_1^*$, $v < v^*$, $a_2^* > a_2$ and $\gamma_1 > \gamma_1^*$, where, q_1^* , q_3^* , a_1^* , b_1^* , v^* , a_2^* and γ_1^* are given in proof.

Proof We first linearize the system about the equilibrium E_{13}^* by using the following transformations

$$\begin{aligned} x &= x^* + n_1; \quad y = y^* + n_2; \quad z = z^* + n_3; \quad C_E = C_E^* + n_4; \\ C_0 &= C_0^* + n_5; \quad R_1 = R_1^* + n_6; \end{aligned}$$

where n_1 , n_2 , n_3 , n_4 , n_5 and n_6 are small perturbations around E_{13}^* . Then we get the following linearized the system,

$$\begin{aligned} \frac{dn_1}{dt} &= (x^* + n_1) \left[-n_1 \left(\frac{1}{k} - \frac{q_1 y^*}{\pi_{31}} \right) - n_2 \frac{q_1 (1+x^*)}{\pi_{31}} - n_5 s_1 \right] \\ \frac{dn_2}{dt} &= (y^* + n_2) \left[\frac{q_2 n_1}{\pi_{31}} - n_2 \left(r_1 - \frac{q_3 z^*}{\pi_{32}} \right) - n_3 \frac{q_3 (1+y^*)}{\pi_{32}} \right] \\ \frac{dn_3}{dt} &= (z^* + n_3) \left[n_2 \frac{q_4}{\pi_{32}} - n_3 d_2 \right] \\ \frac{dn_4}{dt} &= n_1 (vC_0^* - a_2 R_1^*) - a_1 n_4 + v x n_5 - a_2 x n_6 \\ \frac{dn_5}{dt} &= -n_1 (vC_0^* - a_2 R_1^*) - n_5 (b_1 + v x) + n_6 a_2 x \\ \frac{dn_6}{dt} &= n_4 \gamma_1 - n_6 \gamma_1 \end{aligned}$$

Where, $\pi_{31} = (1+x)(1+x^*)$, $\pi_{32} = (1+y)(1+y^*)$. Now, we consider the following positive definite function:

$$\begin{aligned} V_{34} &= \left\{ n_1 - x^* \log \left(1 + \frac{n_1}{x^*} \right) \right\} + F_1 \left\{ n_2 - y^* \log \left(1 + \frac{n_2}{y^*} \right) \right\} \\ &\quad + F_2 \left\{ n_3 - z^* \log \left(1 + \frac{n_3}{z^*} \right) \right\} + \frac{F_3}{2} n_4^2 + \frac{F_4}{2} n_5^2 + \frac{F_5}{2} n_6^2 \\ \frac{dV_{34}}{dt} &= \frac{n_1}{x^* + n_1} \frac{dn_1}{dt} + \frac{n_2 F_1}{y^* + n_2} \frac{dn_2}{dt} + \frac{n_3 F_2}{z^* + n_3} \frac{dn_3}{dt} + F_3 n_4 \frac{dn_4}{dt} \\ &\quad + F_4 n_5 \frac{dn_5}{dt} + F_5 n_6 \frac{dn_6}{dt} \\ \frac{dV_{34}}{dt} &= -n_1^2 x^* \left(\frac{1}{k} - \frac{q_1 y^*}{\pi_{31}} \right) - n_2^2 F_1 \left(r_1 - \frac{q_3 z^*}{\pi_{32}} \right) - n_3^2 F_2 d_2 \\ &\quad - n_4^2 a_1 - n_5^2 (b_1 + v x) - n_6^2 \gamma_1 - n_1 n_2 (q_1 (1+x^*) \\ &\quad - F_1 q_2) / \pi_{31} + n_1 n_4 (vC_0^* - a_2 R_1^*) \\ &\quad - n_1 n_5 (s_1 + (vC_0^* - a_2 R_1^*)) - n_2 n_3 (F_1 q_3 (1+y^*) \\ &\quad - F_2 q_4) / \pi_{32} + n_4 n_5 v x - n_4 n_6 (a_2 x - \gamma_1) + n_5 n_6 a_2 x \end{aligned} \quad (78)$$

Now, choosing $F_1 = \frac{q_1}{q_2} (1+x^*) > 0$ and $F_2 = \frac{q_1 q_3}{q_2 q_4} (1+x^*)(1+y^*) > 0$ from above Eq. (78). Now using the sylvester's criterion in the quadratic forms,

$$\begin{aligned} \frac{dV_{34}}{dt} &\leq -[(f_{11}/2)n_1^2 + f_{14}n_1n_4 + (f_{44}/2)n_4^2] + ((f_{11}/2)n_1^2 \\ &\quad + f_{15}n_1n_5 + (f_{55}/2)n_5^2) + ((f_{44}/2)n_4^2 - f_{45}n_4n_5 \\ &\quad + (f_{55}/2)n_5^2) + ((f_{44}/2)n_4^2 + f_{46}n_4n_6 + (f_{66}/2)n_6^2) \\ &\quad + ((f_{55}/2)n_5^2 - f_{56}n_5n_6 + (f_{66}/2)n_6^2) \end{aligned}$$

where, $f_{11} = (\frac{1}{k} - \frac{q_1 y^*}{\pi_{31}})/2$, $f_{14} = vC_0^* - a_2 R_1^*$, $f_{15} = s_1 + (vC_0^* - a_2 R_1^*)$, $f_{44} = a_1/3$, $f_{45} = vx$, $f_{46} = a_2 x - \gamma_1$, $f_{55} = (b_1 + vx)/3$, $f_{56} = a_2 x$, $f_{66} = \gamma_1/2$.

Sufficient conditions for dV_{34}/dt to be negative definite are that the following inequalities hold:

$$(i)f_{11} > 0, \quad (ii)f_{22} > 0, \quad (iii)f_{11}f_{44} > f_{14}^2, \quad (iv)f_{11}f_{55} > f_{15}^2, \\ (v)f_{44}f_{55} > f_{45}^2, \quad (vi)f_{44}f_{66} > f_{46}^2, \quad (vii)f_{55}f_{66} > f_{56}^2.$$

Clearly, by Liapunov's direct method E_{13}^* is globally asymptotically stable. Further, the above conditions (i)–(vii), obtained by using Sylvester's criteria, can be rewritten respectively in the following form:

$$(1 + x_{\max})(1 + x^*) > q_1 k y^* \quad (79)$$

$$(1 + y_{\max})(1 + y^*) r_1 > q_3 z^* \quad (80)$$

$$a_1 \left(\frac{1}{k} - \frac{q_1 y^*}{(1 + x_{\max})(1 + x^*)} \right) > 6(vC_0^* - a_2 R_1^*)^2 \quad (81)$$

$$(b_1 + vx_{\max}) \left(\frac{1}{k} - \frac{q_1 y^*}{(1 + x_{\max})(1 + x^*)} \right) > 6(s_1 + (vC_0^* - a_2 R_1^*))^2 \quad (82)$$

$$a_1(b_1 + vx_{\max}) > 6v^2 x_{\min}^2 \quad (83)$$

$$a_1 \gamma_1 > 6(a_2 x_{\min} - \gamma_1)^2 \quad (84)$$

$$\gamma_1(b_1 + vx_{\max}) > 6a_2^2 x_{\min}^2 \quad (85)$$

where $x_{\max}$, $y_{\max}$ and $x_{\min}$ can be obtained from region of attraction Ω . Here, we have taken above inequalities in such a way that right hand side becomes minimum and the left hand side becomes maximum. This particular case represents the weakest situation for the inequalities to be held and, moreover, if inequalities are dissatisfied in this situation then the paths starting in Ω will not go to interior equilibrium point. Thus, under this particular situation we will break the above inequalities (79)–(85) to obtain the threshold conditions depending upon toxicant and predation related parameters.

Now, by some algebraic manipulation it is clear that inequality (79), holds true when $q_1^* > q_1$, where

$$q_1^* = \frac{(1 + x_{\max})(1 + x^*)}{k y^*}$$

Now, by some algebraic manipulation it is clear that inequality (80), holds true when $q_3^* > q_3$, where

$$q_3^* = \frac{r_1}{z^*} (1 + y_{\max})(1 + y^*)$$

Now, by some algebraic manipulation it is clear that inequality (81), holds true when $a_1^* > a_1$, where

$$a_1^* = \frac{6(vC_0^* - a_2 R_1^*)^2}{\left(\frac{1}{k} - \frac{q_1 y^*}{(1 + x_{\max})(1 + x^*)} \right)}$$

Now, by some algebraic manipulation it is clear that inequality (82), holds true when $b_1 > b_1^*$, where

$$b_1^* = \frac{6(s_1 + (vC_0^* - a_2 R_1^*))^2}{\left(\frac{1}{k} - \frac{q_1 y^*}{(1 + x_{\max})(1 + x^*)} \right)} - vx_{\max}$$

Now, by some algebraic manipulation it is clear that inequality (83), we obtain the following result:

$$F_{11}(v) = \left(\frac{6x_{\min}^2}{a_1} \right) v^2 - vx_{\max}v - b_1 < 0$$

by theory of equations there exists a positive root of $F_{11}(v) = 0$, say it is $v = v^*$. Then, clearly the above inequality holds when $v < v^*$.

Now, by some algebraic manipulation it is clear that inequality (84), holds true when $a_2^* > a_2$, where

$$a_2^* = \frac{1}{x_{\min}} \left(\sqrt{\frac{a_1 \gamma_1}{6}} + \gamma_1 \right)$$

Now, by some algebraic manipulation it is clear that inequality (85), holds true when $\gamma_1 > \gamma_1^*$, where

$$\gamma_1^* = \frac{6a_2^2 x_{\min}^2}{b_1 + vx_{\max}}$$

Its clear that E_{13}^* is globally asymptotically stable if $q_1^* > q_1$, $q_3^* > q_3$, $a_1 > a_1^*$, $b_1 > b_1^*$, $v < v^*$, $a_2^* > a_2$ and $\gamma_1 > \gamma_1^*$. This completes the proof of the theorem.

Analysis of Model 2

Equilibria of Model 2

The Model 2 has four non negative equilibria in x, y, z, C_E, C_0 space namely, $\check{E}_{20} = (0, 0, 0, \check{C}_E, 0)$, $\hat{E}_{21} = (\hat{x}, 0, 0, \hat{C}_E, \hat{C}_0)$, $\bar{E}_{22} = (\bar{x}, \bar{y}, 0, \bar{C}_E, \bar{C}_0)$ and $E_{23}^{**} = (x^{**}, y^{**}, z^{**}, C_E^{**}, C_0^{**})$. The existence of E_{20} is obvious.

Here we observe that, the equilibria of Model 2 and Model 1 have the same non-negative conditions.

Dynamical behaviour of the Model 2

Local stability

The stability behavior of E_{20} can be studied by computing variational matrix, and $\hat{E}_{21}$, $\bar{E}_{22}$ and E_{23}^{**} can be studied by computing Liapunov's direct method.

The general variational matrix corresponding to the Model 2 is

$$J(x, y, z, C_E, C_0) = \begin{bmatrix} n_{11} & -n_{12} & 0 & 0 & -n_{15} \\ n_{21} & n_{22} & -n_{23} & 0 & 0 \\ 0 & n_{32} & n_{33} & 0 & 0 \\ n_{41} & 0 & 0 & -n_{44} & n_{45} \\ -n_{51} & 0 & 0 & n_{54} & -n_{55} \end{bmatrix}$$

where,

$$\begin{aligned} n_{11} &= x \left(\frac{yq_1}{(1+x)^2} - \frac{1}{k} \right) + (s_0 - s_1 C_0 - \frac{x}{k} - \frac{q_1 y}{1+x}), \\ n_{12} &= \frac{q_1 x}{1+x}, n_{15} = s_1 x, \\ n_{21} &= \frac{q_2 y}{(1+x)^2}, n_{22} = y \left(\frac{q_3 z}{(1+y)^2} - r_1 \right) \\ &+ \left(\frac{q_2 x}{1+x} - \frac{q_3 z}{1+y} - r_0 - r_1 y \right), n_{23} = \frac{q_3 y}{1+y}, \\ n_{32} &= \frac{q_4 z}{(1+y)^2}, n_{33} = \frac{q_4 y}{1+y} - d_1 - 2d_2 z, \\ n_{41} &= v C_0 - a_2 C_E, n_{44} = a_1 + a_2 x, \\ n_{45} &= vx, n_{51} = v C_0, n_{54} = a_2 x, n_{55} = b_1 + vx. \end{aligned}$$

About E_{20} , the eigenvalues of the characteristic equation are $s_0, -r_0, -d_1, -a_1$ and $-b_1$, which shows that E_{20} is unstable.

About E_{21} , two eigenvalues of the characteristic equation are, $-d_1, \frac{q_2 \hat{x}}{1+\hat{x}} - r_0$ and the other three eigenvalues are given by the roots of the following equation

$$\lambda^3 + \lambda^2 M_1 + \lambda M_2 + M_3 = 0 \quad (86)$$

$$M_1 = \hat{x}/k + (a_1 + b_1) + \hat{x}(v + a_2),$$

$$M_2 = [(a_1 + b_1) + \hat{x}(v + a_2)] \hat{x}/k + a_1 b_1 + \hat{x} (b_1 a_2 + a_1 v) - v s_1 \hat{x}^2,$$

$$M_3 = [(a_1 + b_1) + \hat{x}(v + a_2)] \hat{x}/k - s_1 \hat{x}^2 [v(a_1 + a_2 \hat{x}) + \hat{C}_E a_2^2 + v a_2 \hat{C}_0].$$

According to Routh-Hurwitz criteria E_{21} is locally asymptotically stable if

$$\frac{q_2}{r_0} < \frac{1}{\hat{x}} + 1 \Rightarrow \frac{q_2}{r_0} < 1$$

$$M_2 > 0, \text{ i.e., } [(a_1 + b_1) + \hat{x}(v + a_2)] \hat{x}/k + a_1 b_1 + \hat{x}(b_1 a_2 + a_1 v) > v s_1 \hat{x}^2,$$

$$M_3 > 0, \text{ i.e., } [(a_1 + b_1) + \hat{x}(v + a_2)] \hat{x}/k > s_1 \hat{x}^2 [v(a_1 + a_2 \hat{x}) + \hat{C}_E a_2^2 + v a_2 \hat{C}_0] \text{ and } M_1 M_2 - M_3 > 0, \text{ i.e., } [(a_1 + b_1) + \hat{x}(v + a_2) + \frac{\hat{x}}{k} [b_1(a_1 + \hat{x} a_2) + v \hat{x}(a_1 - s_2 \hat{x})] + s_1 \hat{x}^2 [v(a_1 + a_2 \hat{x}) + \hat{C}_E a_2^2 + v a_2 \hat{C}_0]] \frac{\hat{x}}{k} [(a_1 + b_1) + \hat{x}(v + a_2)] \frac{\hat{x}}{k} + (a_1 + b_1) + \hat{x}(v + a_2) - 1] > 0,$$

$$M_1 M_2 - M_3 > 0 \text{ if}$$

$$\frac{1}{v + a_2} < \hat{x} < \frac{a_1}{s_1}.$$

Remark 2 From the stability conditions of E_{21} , the following may be noted:

- (i) the ratio of the growth rate of intermediate predator and its death rate is less than one,
- (ii) the equilibria of prey is lies between the residence time of organismal concentration in the population and the ratio of decay rate of environment toxicant concentration and the prey population due to toxicant are satisfied then only prey population will survive in the presence of toxicant.

Remark 3

$$\frac{\partial \hat{x}}{\partial \hat{C}_0} = -s_1 k < 0 \quad (87)$$

It may be noted from Eq. (87) that the prey population will decrease with respect to increasing organismal toxicant concentration.

About E_{22} , one of the eigenvalues of the characteristic equation is $q_4 \bar{y}/(1 + \bar{y}) - d_1$ and the other four eigenvalues are given by the roots of the following equation is

$$\lambda^4 + \lambda^3 U_1 + \lambda^2 U_2 + \lambda U_3 + U_4 = 0 \quad (88)$$

$$\text{where, } U_1 = p_{15} + p_{14} + r_{11} \bar{y} - \bar{x} p_{11}$$

$$U_2 = p_{12} p_{13} - \bar{x} p_{11} (r_{11} \bar{y} + p_{14}) + r_{11} \bar{y} p_{14} - s_1 \bar{x} v \bar{C}_0 - v a_2 \bar{x}^2 + (r_{11} \bar{y} + p_{14} - \bar{x} p_{11}) p_{15}$$

$$U_3 = (p_{12} p_{13} - \bar{x} p_{11} r_{11} \bar{y}) p_{14} - s_1 \bar{x} v \bar{C}_0 (r_{11} \bar{y} + p_{14}) + s_1 \bar{x} (v \bar{C}_0 - a_2 \bar{C}_E) a_2 \bar{x} - v a_2 \bar{x}^2 (r_{11} \bar{y} - \bar{x} p_{11}) + (p_{12} p_{13} - \bar{x} p_{11} r_{11} \bar{y}) p_{15} + (r_{11} \bar{y} - \bar{x} p_{11}) p_{14} p_{15}$$

$$U_4 = s_1 \bar{x} r_{11} \bar{y} ((v \bar{C}_0 - a_2 \bar{C}_E) a_2 \bar{x} - p_{14} v \bar{C}_0) + (p_{12} p_{13} - \bar{x} p_{11} r_{11} \bar{y}) (p_{14} p_{15} - v a_2 \bar{x}^2)$$

$$p_{11} = \frac{q_1 \bar{y}}{(1+\bar{x})^2} - \frac{1}{k}, p_{12}$$

$$= \frac{q_1 \bar{x}}{1+\bar{x}}, p_{13} = \frac{q_2 \bar{y}}{(1+\bar{x})^2}, p_{14} = a_1 + a_2 \bar{x}, p_{15} = b_1 + v \bar{x}.$$

From the Routh-Hurwitz's criteria it is found that E_{22} is locally asymptotically stable if the following conditions hold good.

$$q_4 \bar{y} < d_1 (1 + \bar{y}), \quad (89)$$

$$U_i > 0, i = 1, 2, 3, 4, U_1 U_2 > U_3 \text{ i.e.,}$$

$$\begin{aligned} &((b_1 + v \bar{x})(1 - \bar{x}) + (r_{11} \bar{y} - \bar{x} p_{11})) \frac{\bar{x} \bar{y} q_1 q_2}{(1+\bar{x})^3} + (a_1 + a_2 \bar{x} + r_{11} \bar{y}) [(a_1 + a_2 \bar{x})(1 + r_{11} \bar{y}) + (b_1 + v \bar{x}) + r_{11} \bar{y} - \bar{x} p_{11}] - \bar{x} (a_1 + a_2 \bar{x} + r_{11} \bar{y}) p_{11} [a_1 + b_1 + \bar{x}(a_2 + v) + r_{11} \bar{y} - \bar{x} p_{11}] + \bar{x} v s_1 \bar{C}_0 \\ &(\bar{x} p_{11} - b_1) - (a_1 + b_1 + \bar{x}(a_2 + v)) - v a_2 \bar{x}^2 - s_1 \bar{x}^2 (v \bar{C}_0 (a_2 + v) - a_2^2 \bar{C}_E) - \bar{x} (b_1 + v \bar{x}) (b_1 + v \bar{x} + r_{11} \bar{y}) p_{11} + \bar{x}^2 (b_1 + v \bar{x}) p_{11} (r_{11} \bar{y} + p_{11}) > 0, \text{ and } U_1 U_2 U_3 > (U_3^2 + U_1^2 U_4), \text{ i.e.,} \\ &(p_{14} + p_{15} + r_{11} \bar{y} - \bar{x} p_{11}) [p_{12} p_{13} - \bar{x} p_{11} (r_{11} \bar{y} + p_{14}) + r_{11} \bar{y} p_{14}] \end{aligned}$$

$$\begin{aligned}
 & -\bar{x}v(s_1\bar{C}_0 + a_2\bar{x}) + p_{15}(r_1\bar{y} + p_{14} - \bar{x}p_{11})[(p_{12}p_{13} - \bar{x}p_{11} \\
 & r_1\bar{y}) (p_{14} + \bar{x}p_{15}) + (r_1\bar{y} - \bar{x}p_{11})(p_{14}p_{15} - va_2\bar{x}^2) - s_1\bar{x}v\bar{C}_0 \\
 & (r_1\bar{y} + p_{14}) + s_1a_2\bar{x}^2 (v\bar{C}_0 - a_2\bar{C}_E)] > [(p_{12}p_{13} - \bar{x}p_{11}r_1\bar{y}) \\
 & (p_{14} + \bar{x}p_{15}) + (r_1\bar{y} - \bar{x}p_{11})(p_{14}p_{15} - va_2\bar{x}^2) - s_1\bar{x}v\bar{C}_0(r_1\bar{y} \\
 & + p_{14}) + s_1a_2\bar{x}^2(v\bar{C}_0 - a_2\bar{C}_E)]^2 + (p_{14} + p_{15} + r_1\bar{y} - \bar{x}p_{11})^2 \\
 & [\bar{x}y r_1 s_1 ((v\bar{C}_0 - a_2\bar{C}_E)a_2\bar{x} \\
 & - v\bar{C}_0 p_{14}) + (p_{12}p_{13} - \bar{x}y r_1 p_{11})(p_{14}p_{15} - va_2\bar{x}^2)].
 \end{aligned}$$

The characteristic equation of E_{23}^* is

$$\lambda^5 + \lambda^4 Q_1 + \lambda^3 Q_2 + \lambda^2 Q_3 + \lambda Q_4 + Q_5 = 0 \quad (90)$$

where,

$$\begin{aligned}
 Q_1 &= (d_2 z^{**} + q_{17} + q_{18}) - (x^{**} q_{11} + y^{**} q_{14}) \\
 Q_2 &= q_{12} q_{13} + x^{**} q_{11} y^{**} q_{14} + q_{15} q_{16} - (x^{**} q_{11} + y^{**} q_{14}) d_2 z^{**} \\
 &\quad - (x^{**} q_{11} + y^{**} q_{14} - d_2 z^{**}) q_{17} - s_1 v x^{**} C_0^{**} \\
 &\quad - va_2 (x^{**})^2 + (d_2 z^{**} + q_{17} - x^{**} q_{11} - y^{**} q_{14}) q_{18} \\
 Q_3 &= (q_{12} q_{13} + x^{**} q_{11} y^{**} q_{14}) d_2 z^{**} - x^{**} q_{11} q_{15} q_{16} \\
 &\quad + (q_{12} q_{13} + x^{**} q_{11} y^{**} q_{14} + q_{15} q_{16} \\
 &\quad - x^{**} q_{11} d_2 z^{**} - y^{**} q_{14} d_2 z^{**}) q_{17} + (y^{**} q_{14} - d_2 z^{**} \\
 &\quad - q_{17}) s_1 x^{**} v C_0^{**} + (s_1 x^{**} (v C_0^{**} - a_2 C_E^{**}) + q_{11} v (x^{**})^2 \\
 &\quad + y^{**} q_{14} v x^{**} - d_2 z^{**} v x^{**}) a_2 x^{**} + (q_{12} q_{13} + x^{**} q_{11} y^{**} q_{14} \\
 &\quad + q_{15} q_{16} - x^{**} q_{11} d_2 z^{**} - y^{**} q_{14} d_2 z^{**} - x^{**} q_{11} q_{17} \\
 &\quad - y^{**} q_{14} q_{17} + d_2 z^{**} q_{17}) q_{18} \\
 Q_4 &= q_{12} q_{13} d_2 z^{**} q_{17} + x^{**} q_{11} y^{**} q_{14} d_2 z^{**} q_{17} - x^{**} q_{11} q_{15} q_{16} q_{17} \\
 &\quad + (y^{**} q_{14} d_2 z^{**} - q_{15} q_{16}) s_1 x^{**} v C_0^{**} \\
 &\quad - (d_2 z^{**} - y^{**} q_{14}) s_1 x^{**} q_{17} v C_0^{**} + (d_2 z^{**} - y^{**} q_{14}) \\
 &\quad (v C_0^{**} - a_2 C_E^{**}) s_1 a_2 (x^{**})^2 + (x^{**} q_{11} d_2 z^{**} + y^{**} q_{14} d_2 z^{**} \\
 &\quad - q_{12} q_{13} - x^{**} q_{11} y^{**} q_{14} - q_{15} q_{16}) v a_2 x^{**2} \\
 &\quad - x^{**} q_{11} q_{15} q_{16} q_{18} + (q_{12} q_{13} + x^{**} q_{11} y^{**} q_{14}) d_2 z^{**} q_{18} \\
 &\quad + (q_{12} q_{13} + x^{**} q_{11} y^{**} q_{14} + q_{15} q_{16} - x^{**} q_{11} d_2 z^{**} \\
 &\quad - y^{**} q_{14} d_2 z^{**}) q_{17} q_{18} \\
 Q_5 &= s_1 x^{**} (y^{**} q_{14} d_2 z^{**} - q_{15} q_{16}) (a_1 v C_0^{**} + a_2^2 C_E^{**} x^{**}) \\
 &\quad + (x^{**} q_{11} q_{15} q_{16} - q_{12} q_{13} d_2 z^{**} - x^{**} q_{11} y^{**} q_{14} d_2 z^{**}) v x^{**} a_2 x^{**} \\
 &\quad - x^{**} q_{11} q_{15} q_{16} q_{17} q_{18} + (q_{12} q_{13} + x^{**} q_{11} y^{**} q_{14}) d_2 z^{**} q_{17} q_{18}
 \end{aligned}$$

and,

$$\begin{aligned}
 q_{11} &= \frac{q_1 y^{**}}{(1+x^{**})^2} - \frac{1}{k}, q_{12} = \frac{q_1 x^{**}}{1+x^{**}}, q_{13} = \frac{q_2 y^{**}}{(1+x^{**})^2}, q_{14} = \frac{q_3 z^{**}}{(1+y^{**})^2} \\
 -r_1, \quad q_{15} &= \frac{q_3 y^{**}}{1+y^{**}}, q_{16} = \frac{q_4 z^{**}}{(1+y^{**})^2}, q_{17} = a_1 + a_2 x^{**}, \quad q_{18} = \\
 & b_1 + v x^{**}.
 \end{aligned}$$

According to Routh-Hurwitz's criteria for the equilibrium point E_{23}^* to be locally asymptotically stable, following conditions must be satisfied

$$Q_i > 0, \quad i = 1, 2, 3, 4, 5, \quad Q_1 Q_2 > Q_3,$$

$Q_1 Q_2 Q_3 > (Q_3^2 + Q_1^2 Q_4)$ and $(Q_3 Q_4 - Q_2 Q_5) (Q_1 Q_2 - Q_3) > (Q_1 Q_4 - Q_5)^2$. It is difficult to interpret the results in ecological terms from these complicated expressions, however, numerical examples are taken and graphs are plotted to illustrate the dynamical behaviour of the system about equilibrium E_{23}^* .

Hopf-bifurcation

Again, in the similar way the equilibrium population densities are functions of s_1 and the coefficients of the characteristic Eq. (90) are functions of the parameter s_1 . Now we can use the notation $Q_i = Q_i(s_1)$ for $i = 1, 2, 3, 4, 5$. Now noting that the quantities Q_i s are smooth functions of the parameter s_1 . As we have explained the definition of Hopf-bifurcation in previous section. Without knowing eigenvalues, Liu (1994) proved that (referring the result to the current case): if $Q_i(s_1)$,

$$\Delta_1(s_1) = Q_1(s_1)Q_2(s_1) - Q_3(s_1),$$

$$\Delta_2(s_1) = Q_1(s_1)Q_2(s_1)Q_3(s_1) - (Q_3^2(s_1) + Q_1^2(s_1)Q_4(s_1)),$$

$\Delta_3(s_1) = [Q_3(s_1)Q_4(s_1) - Q_2(s_1)Q_5(s_1)][Q_1(s_1)Q_2(s_1) - Q_3(s_1)] - [Q_1(s_1)Q_4(s_1) - Q_5(s_1)]^2$ are smooth functions of the parameter ' s_1 ' in an open interval containing $s_1^{**} \in \mathcal{R}^+$ such that following conditions hold: (iii_{*}) $Q_1(s_1^{**}) > 0$, $\Delta_1(s_1^{**}) > 0$, $\Delta_2(s_1^{**}) > 0$ and $\Delta_3(s_1^{**}) = 0$; (iv_{*}) $d\Delta_3(s_1)/ds_1|_{s_1=s_1^{**}} \neq 0$ then (iii_{*}) and (iv_{*}) are equivalent to conditions (i) and (ii) for the occurrence of a simple Hopf-bifurcation at $s_1 = s_1^{**}$. Hence, in the similar way, we can propose the following theorem:

Theorem 5 If a critical value s_1^{**} of parameter s_1 be found such that $Q_i(s_1^{**}) > 0$, $\Delta_1(s_1^{**}) > 0$, $\Delta_2(s_1^{**}) > 0$, $\Delta_3(s_1^{**}) = 0$ and further $\Delta_3' \neq 0$ (where primes denotes differentiation with respect to s_1) then the system (7)–(11) undergoes Hopf-bifurcation around E_{23}^* . (for proof see Appendix 1).

Global stability

Theorem 6 If the following inequalities hold in the region Ω

$$k q_1 y^{**} < \pi_{21} \quad (91)$$

$$v x^{**} < a_1 + a_2 x^{**} \quad (92)$$

$$2[H_1 q_3 (1 + y^{**}) + q_4 H_2]^2 < H_1 H_2 d_2 \pi_{22}^2 \left(r_1 + \frac{q_3 z^{**}}{\pi_{22}} \right) \quad (93)$$

$$4a_2 x^{**} > H_3 (b_1 + v x^{**}) (a_1 + (a_2 - v) x^{**}) \quad (94)$$

where,

$$H_1 = \frac{q_1 (1 + x^{**})}{q_2} > 0 \quad (95)$$

$$H_2 < \left(\frac{1}{k} - \frac{q_1 y^{**}}{\pi_{21}} \right) \frac{a_1 + (a_2 - v) x^{**}}{6(a_2 C_E - v C_0)^2} \quad (96)$$

$$H_3 = \frac{s_1}{a_2 C_E - v C_0} > 0 \quad (97)$$

where, $\pi_{21} = (1 + s_0 k)(1 + x^{**})$, $\pi_{22} = (1 + K_1)(1 + y^{**})$, then the positive equilibrium E_{23}^* is globally asymptotically

stable with respect to all solutions initiating in the interior of positive region Ω . (for proof see Appendix 2).

Analysis of Model 3

Equilibria of Model 3

The Model 3 has following four non-negative equilibria in x, y, z space namely, $E_{30} = (0, 0, 0)$, $E_{31} = (\ddot{x}, 0, 0)$, $E_{32} = (\ddot{x}, \ddot{y}, 0)$ and $E_{33} = (\ddot{x}, \ddot{y}, \ddot{z})$. The existence of E_{30} is obvious. We prove the existence of E_{31} , E_{32} and E_{33} as follows:

1. Existence of $E_{31} = (\ddot{x}, 0, 0)$, where,

$$\ddot{x} = s_0 k \quad (98)$$

2. Existence of $E_{32} = (\ddot{x}, \ddot{y}, 0)$, where,

$$\ddot{y} = \left(s_0 - \frac{\ddot{x}}{k} \right) \left(\frac{1 + \ddot{x}}{q_1} \right) > 0 \quad (99)$$

$\ddot{y} > 0$ if $s_0 k > \ddot{x}$. Using (99) in (13), we get

$$C_1 \ddot{x}^3 + C_2 \ddot{x}^2 + C_3 \ddot{x} - C_4 = 0 \quad (100)$$

where, $C_1 = r_1$, $C_2 = r_1(2 - s_0 k) > 0$, $C_3 = q_1 k(q_2 - r_0) - r_1(2s_0 r_1 - 1) > 0$ and $C_4 = k(s_0 r_1 + r_0 q_1)$. The Eq. (100) will have a positive root $\ddot{x}$ always.

3. Existence of $E_{33} = (\ddot{x}, \ddot{y}, \ddot{z})$

Here, $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ are the positive solutions of the system of algebraic equations:

from (14),

$$z = \left(\frac{q_4 y}{1 + y} - d_1 \right) \frac{1}{d_2} = S_1(y) \quad (101)$$

Now, considering two functions [from (12) and (13)],

$$K_{11}(x, y) = s_0 k(1 + x) - x(1 + x) - q_1 k y \quad (102)$$

$$K_{12}(x, y) = s_0 - \frac{x}{k} + \left(\frac{q_2 x - q_1 y}{1 + x} \right) - \frac{q_3 z}{1 + y} - (r_0 + r_1 y) \quad (103)$$

For existence of $\ddot{x}$ and $\ddot{y}$, the two isoclines,

$$K_{11}(x, y) = 0 \quad (104)$$

$$K_{12}(x, y) = 0 \quad (105)$$

must intersect. We note that

$$K_{11}(0, 0) = s_0 k > 0$$

$$K_{12}(0, 0) = s_0 - r_0 > 0$$

$$K_{12}(0, 0) > 0 \text{ if } s_0 > r_0.$$

Also, $K_{11}(x, 0) = 0$ then x will have one positive root (say) ψ_1 , from the following quadratic equation of x ,

$$x^2 - x(s_0 k - 1) - s_0 k = 0$$

if $s_0 k > 1$.

$$K_{11}(0, y) = 0 \text{ then } y = \frac{s_0}{q_1} = \psi_2 \text{ (say).}$$

$K_{12}(x, 0) = 0$ then $x =$ will have one positive root (say) ψ_3 , from the following quadratic equation of x ,

$$d_2 x^2 - x[kd_1 q_3 + kd_2(s_0 - r_0) + d_2(kq_2 - 1)] - k[d_1 q_3 + d_2(s_0 - r_0)] = 0$$

if $s_0 > r_0$ and $kq_2 > 1$.

$K_{12}(0, y) = 0$ then y will have one positive root (say) ψ_4 , from the following cubic equation of y ,

$$N_1 y^3 + N_2 y^2 - N_3 y - N_4 = 0$$

$N_1 = (r_1 + q_1)d_2$, $N_2 = d_2[2(q_1 + r_1) - (s_0 - r_0)] > 0$, $N_3 = 2d_2(s_0 - r_0) - [(r_1 + q_1)d_2 + q_3(q_4 - d_1)] > 0$ and $N_4 = d_1 q_3 + (s_0 - r_0)d_2 > 0$. Thus both the isoclines intersect each other in the region

$$G_{11} = \{(x, y) : 0 < x < \psi_3, 0 < y < \psi_2\}$$

in the following two cases: (see Fig. 3)

$$\text{Case (i) : } \psi_3 > \psi_1, \psi_2 > \psi_4 \quad (106)$$

$$\text{Case (ii) : } \psi_3 < \psi_1, \psi_2 < \psi_4 \quad (107)$$

This point of intersection will give $\ddot{x}, \ddot{y}$. For uniqueness of $(\ddot{x}, \ddot{y})$, we must have $\frac{dy}{dx} < 0$ for both the curves in the region G_{11} .

For curve (104),

$$\frac{dy}{dx} = \frac{s_0 k - (2x + 1)}{q_1 k} < 0 \quad (108)$$

and for curve (105),

$$\frac{dy}{dx} = \frac{\frac{1}{(1+x)^2} (q_1 y + q_2) - \frac{1}{k} - \frac{S'_1(y) q_3}{1+y}}{\left(r_1 + \frac{q_1}{1+x} - \frac{S_1(y) q_3}{(1+y)^2} \right)} < 0 \quad (109)$$

In case (i), the absolute value of $\frac{dy}{dx}$ given by (108) is less than the absolute value of $\frac{dy}{dx}$ given by (109). For the case (ii), just the opposite is the condition.

Knowing the value of $\ddot{x}$ and $\ddot{y}$; $\ddot{z}$ can be computed from the Eq. (101).

Dynamical behaviour of the Model 3

Local stability

The general variational matrix corresponding to the Model 3 is

Fig. 3 Existence of equilibrium point E_{33} of Model 3

$$J(x, y, z) = \begin{bmatrix} b_{11} & -b_{12} & 0 \\ b_{21} & b_{22} & -b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix}$$

where,

$$b_{11} = s_0 - \frac{2x}{k} - \frac{q_1 y}{(1+x)^2}, \quad b_{12} = \frac{q_1 x}{(1+x)}, \quad b_{21} = \frac{q_2 y}{(1+x)^2}, \quad b_{22} = \frac{q_2 x}{(1+x)} - \frac{q_3 z}{(1+y)^2} - r_0 - 2r_1 y, \quad b_{23} = \frac{q_3 y}{(1+y)}, \quad b_{32} = \frac{q_4 z}{(1+y)^2}, \quad b_{33} = \frac{q_4 y}{1+y} - d_1 - 2d_2 z.$$

- About E_{30} , the eigenvalues of the characteristic equation are $s_0, -r_0$ and $-d_1$, which shows that E_{30} is unstable.
- About E_{31} , the eigen values of the characteristic equation are $-s_0, \frac{q_2 \tilde{x}}{1+\tilde{x}} - r_0$ and $-d_1$, which shows E_{31} is locally asymptotically stable if

$$q_2 \tilde{x} < r_0(1 + \tilde{x}) \quad (110)$$

which implies that,

$$q_2 - r_0 < \frac{r_0}{s_0 k} \quad (111)$$

holds good.

Remark 4 From (111), it may be noted that if the difference of the growth rate of intermediate predator and its natural death rate is less than its natural death rate is divided by the product of intrinsic growth rate and carrying capacity of the prey then the prey population will only survive and both the predators will tend to extinction.

- About E_{32} , the eigen values of the characteristic equation are $\frac{q_1 \tilde{y}}{(1+\tilde{x})^2} - \frac{1}{k}, -r_1 \tilde{y}$ and $\frac{q_4 \tilde{y}}{(1+\tilde{y})} - d_1$, which shows E_{32} is locally asymptotically stable if

$$q_1 \tilde{y} k < (1 + \tilde{x})^2 \quad (112)$$

which implies that,

$$\tilde{x} < s_0 k < (1 + 2\tilde{x}) \quad (113)$$

and

$$0 < \tilde{y} < \frac{d_1}{q_4 - d_1} \quad (114)$$

if $d_1 < q_4$, holds good.

Remark 5 From (113) and (114), following may be noted:

- if the equilibria of prey is less than the product of intrinsic growth rate and carrying capacity of prey, which is again less than the sum of unity and twice of equilibria of prey,
- if the equilibria of intermediate predator is less than the ratio of natural death rate of top predator and the natural death rate of top predator is subtracted from the growth rate of top predator, then prey and intermediate predator population will survive and top predator population will tend to extinction.

- About E_{33} , the characteristic equation is given by

$$\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0 \quad (115)$$

where,

$$B_1 = d_2 \tilde{z} + \tilde{x} \left(\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2} \right) + \tilde{y} \left(r_1 - \frac{q_3 \tilde{z}}{(1+\tilde{y})^2} \right), \quad B_2 = \frac{q_1 q_2 \tilde{x} \tilde{y}}{(1+\tilde{x})^3} + \frac{q_3 q_4 \tilde{y} \tilde{z}}{(1+\tilde{y})^3} + \tilde{y} \tilde{z} d_2 \left(r_1 - \frac{q_3 \tilde{z}}{(1+\tilde{y})^2} \right) + \tilde{x} \left(\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2} \right) [\tilde{z} d_2 + \tilde{y} \left(r_1 - \frac{q_3 \tilde{z}}{(1+\tilde{y})^2} \right)], \quad B_3 = \frac{\tilde{x} \tilde{y} \tilde{z} q_1 q_2 d_2}{(1+\tilde{x})^3} + \tilde{x} \tilde{y} \tilde{z} \left(\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2} \right) [d_2 \left(r_1 - \frac{q_3 \tilde{z}}{(1+\tilde{y})^2} \right) + \frac{q_3 q_4}{(1+\tilde{y})^3}].$$

According to Routh-Hurwitz criteria E_{33} is locally asymptotically stable if $B_1 > 0, B_2 > 0, B_3 > 0$ and $B_1 B_2 > B_3$, i.e.,

$$\theta_3 (\tilde{x} \theta_1 + \tilde{y} \theta_2) + (\tilde{y} \theta_2 + \tilde{z} d_2) [\tilde{x} \theta_1 (\tilde{x} \theta_1 + \tilde{y} \theta_2 + d_2 \tilde{z}) + (\theta_4 + \tilde{y} \tilde{z} d_2 \theta_2)] > 0, \text{ and}$$

$$q_1 \tilde{y} k < (1 + \tilde{x})^2, \quad (116)$$

$$q_3 \tilde{z} < r_1 (1 + \tilde{y})^2. \quad (117)$$

Where, $\theta_1 = \frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2} > 0$, $\theta_2 = r_1 - \frac{q_3 \tilde{z}}{(1+\tilde{y})^2} > 0$, $\theta_3 = \frac{\tilde{y} \tilde{z} q_1 q_2}{(1+\tilde{x})^3}$ and $\theta_4 = \frac{\tilde{y} \tilde{z} q_3 q_4}{(1+\tilde{y})^3}$.

Remark 6 Following may be noted from the above conditions (116–117):

- (i) if the product of predation rate of prey and the equilibria of intermediate predator and the carrying capacity of prey is less than the square of the sum of unity and the equilibria of prey,
- (ii) if the product of predation rate of intermediate predator and the equilibria of top predator less than the product of intraspecies competition coefficient among intermediate predator and the square of the sum of unity and the equilibria of intermediate predator then all the populations will survive.

Hopf-bifurcation

We are now in a position to make an attempt to find out the conditions under which the system undergoes Hopf-bifurcation. For this purpose, we choose the parameter q_4 as bifurcation parameter as it plays a crucial role in Holling type II functional response which describes the predation of intermediate consumer. We shall now apply the Liu's criteria, Liu (1994) to obtain the conditions for small amplitude periodic solution arising from Hopf-bifurcation.

As the equilibrium population densities are functions of q_4 , the coefficients of the characteristic Eq. (115) are functions of the parameter q_4 and hence we can use the notation $B_i = B_i(q_4)$ for $i = 1, 2, 3$. Noting that the quantities B_i s are smooth functions of the parameter q_4 , we first state in our case, the definition of a simple Hopf-Bifurcation.

If a critical value $\tilde{q}_4$ of parameter q_4 be found such that (i) a simple pair of complex conjugate eigenvalues of characteristic equation exists, say, $\lambda_1(q_4) = u(q_4) + iv(q_4)$, $\lambda_2(q_4) = u(q_4) - iv(q_4) = \overline{\lambda_1}(q_4)$. These eigen values will become purely imaginary at $q_4 = \tilde{q}_4$, i.e. $\lambda_1(\tilde{q}_4) = iv_0$, $\lambda_2(\tilde{q}_4) = -iv_0$, with $v(\tilde{q}_4) = v_0 > 0$, and the other eigenvalue remains real and negative; and (ii) the transversality condition, $dRe\lambda_i(\tilde{q}_4)/dq_4|_{q_4=\tilde{q}_4} = du(q_4)/dq_4|_{q_4=\tilde{q}_4} \neq 0$ is satisfied. Then we find at $q_4 = \tilde{q}_4$, a simple Hopf-bifurcation. Without knowing eigenvalues, Liu (1994) proved that (referring the result to the current case): if $B_1(q_4), B_3(q_4), \Delta(q_4) = B_1(q_4)B_2(q_4) - B_3(q_4)$ are smooth functions of the parameter ' q_4 ' in an open interval containing $\tilde{q}_4 \in \mathbb{R}^+$ such that following conditions hold:

(i_{*}) $B_1(\tilde{q}_4) > 0$, $\Delta(\tilde{q}_4) = 0$, $B_3(\tilde{q}_4) > 0$; (ii_{*}) $d\Delta(q_4)/dq_4|_{q_4=\tilde{q}_4} \neq 0$

then (i_{*}) and (ii_{*}) are equivalent to conditions (i) and (ii) for the occurrence of a simple Hopf-bifurcation at $q_4 = \tilde{q}_4$. Hence we can propose the following theorem:

Theorem 7 If a critical value $\tilde{q}_4$ of parameter q_4 be found such that $B_1(\tilde{q}_4) > 0$, $B_3(\tilde{q}_4) > 0$ and $\Delta(\tilde{q}_4) = 0$ and further $\Delta' \neq 0$ (where prime denotes differentiation with respect to q_4) then system (12)–(14) undergoes Hopf-bifurcation around E_{33} . (for proof see Appendix 3).

Global stability

Theorem 8 If the following inequalities hold $kq_1\tilde{y} < (1 + \tilde{x})$ and $q_3\tilde{z} < r_1(1 + \tilde{y})$ then the positive equilibrium E_{33} is globally asymptotically stable with respect to all solutions initiating in the interior of positive region Ω . (for proof see Appendix 4).

Numerical simulation

In this section, we demonstrate the dynamical behaviour of the distributed delay on the effect of toxicant in a three species food-chain system for the models with the help of numerical simulations to facilitate the interpretation of our mathematical findings. The figures illustrate the stability behaviour of all the equilibrium points of the models and bifurcation for non-trivial equilibrium points for the given sets of parameters and the graphs have been plotted with the help of MATLAB.

Numerical simulation for Model 3

We choose the following values of parameters for E_{31} :

$$\begin{aligned} s_0 &= 0.9945; \quad k = 25; \quad q_1 = 0.47; \\ q_2 &= 0.334; \quad q_3 = 0.381; \\ q_4 &= 0.12; \quad r_0 = 1.02423; \quad r_1 = 0.0257; \\ d_1 &= 1.0571; \quad d_2 = 0.02483. \end{aligned}$$

It is found that under the above set of parameters, the equilibrium point E_{31} is

$$\ddot{x} = 24.8623, \quad \ddot{y} = 0.0000,$$

is locally asymptotically stable (see Fig. 4).

We choose the following values of parameters for E_{32} :

$$\begin{aligned} s_0 &= 1.07989; \quad r_0 = 0.002423 \quad q_3 = 0.681; \\ q_4 &= 0.672; \quad d_1 = 0.671; \end{aligned}$$

With the above values of parameters and taking the remaining parameters to be the same as considered for E_{31} , it is found that under the above set of parameters, the equilibrium point E_{32} is

$$\ddot{x} = 20.1283, \quad \ddot{y} = 12.2867, \quad \ddot{z} = 0.0000$$

is locally asymptotically stable (see Fig. 5).

Fig. 4 Time graph for the Model 3 (without toxicant and without distributed delay) around the equilibrium point $\tilde{E}_{31}$, showing the stability behavior

Fig. 5 Time graph for the Model 3 (without toxicant and without distributed delay) around the equilibrium point $\tilde{E}_{32}$, showing the stability behavior

We choose the following values of parameters for $\tilde{E}_{33}$:

$$s_0 = 0.7845; \quad r_0 = 0.02423; \quad d_1 = 0.0571.$$

With the above values of parameters and taking the remaining parameters to be the same as considered for $\tilde{E}_{31}$, it is found that under the above set of parameters, the equilibrium point $\tilde{E}_{33}$

$$\tilde{x} = 13.1430, \quad \tilde{y} = 7.7885, \quad \tilde{z} = 1.9833$$

is locally asymptotically stable (see Fig. 6).

Now, we study the Hopf-bifurcation of the Model 3, taking q_4 as the bifurcating parameter. The transversality condition holds with the above set of parameters when $q_4 = \tilde{q}_4 = 0.15$. It is clear that the interior equilibrium point $\tilde{E}_{33}$ of Model 3 is stable when $q_4 > \tilde{q}_4$ and unstable when $q_4 \leq \tilde{q}_4$ for which Hopf-bifurcation occurs (see Fig. 7).

Numerical simulation for Model 2

We choose the following values of parameters for $\hat{E}_{21}$:

$$\begin{aligned} s_0 &= 0.7845; \quad k = 25; \quad q_1 = 0.47; \quad q_2 = 0.334; \\ q_3 &= 0.381; \quad q_4 = 0.12; \\ r_0 &= 1.02423; \quad r_1 = 0.0257; \quad d_1 = 1.0571; \\ d_2 &= 0.02483; \quad s_1 = 0.01; \quad q_0 = .515; \\ a_1 &= 0.71; \quad a_2 = 0.92; \quad v = 0.102; \quad b_1 = 0.52. \end{aligned}$$

It is found that under the above set of parameters, the equilibrium point

$$\begin{aligned} \hat{x} &= 19.4046, \quad \hat{y} = 0.0000, \quad \hat{z} = 0.0000, \\ \hat{C}_E &= 0.1164, \quad \hat{C}_0 = 0.8314 \end{aligned}$$

is locally asymptotically stable (see Fig. 8).

Fig. 6 Time graph for the Model 3 (without toxicant and without distributed delay) around the equilibrium point E_{33}^* , showing the stability behavior

Fig. 7 Time graph for the Model 3 (without toxicant and without distributed delay) around the equilibrium point E_{33}^* , showing the bifurcation behavior at $q_4 = 0.15$

We choose the following values of parameters for E_{22}^* :
 $s_0 = 1.07845$; $r_0 = 0.002423$.

With the above values of parameters and taking the remaining parameters to be the same as considered for E_{21}^* , it is found that under the above set of parameters, the equilibrium point E_{22}^* is

$$\bar{x} = 17.0143, \quad \bar{y} = 11.3319, \quad \bar{z} = 0.0000, \\ \bar{C}_E = 0.1192, \quad \bar{C}_0 = 0.8276$$

is locally asymptotically stable (see Fig. 9).

We choose the following values of parameters for E_{23}^* :
 $r_0 = 0.02423$; $d_1 = 0.0571$.

With the above values of parameters and taking the remaining parameters to be the same as considered for E_{21}^* ,

it is found that under the above set of parameters, the equilibrium point E_{23}^* is

$$x^{**} = 12.8051, \quad y^{**} = 6.7579, \quad z^{**} = 1.9814, \\ C_E^{**} = 0.1267, \quad C_0^{**} = 0.8174$$

is locally asymptotically stable (see Fig. 10).

Now, we study the Hopf-bifurcation of the *Model 2*, taking s_1 as the bifurcating parameter. The transversality condition holds with the above set of parameters when $s_1 = s_1^{**} = 0.06$. It is clear that the interior equilibrium point E_{23}^* of *Model 2* is stable when $s_1 > s_1^{**}$ and unstable when $s_1 \leq s_1^{**}$ for which Hopf-bifurcation occurs (see Fig. 11).

Numerical simulation for Model 1

We choose the following values of parameters for E_{11}^* :

Fig. 8 Time graph for the Model 2 (with toxicant and without distributed delay) around the equilibrium point $\hat{E}_{21}$, showing the stability behavior

Fig. 9 Time graph for the Model 2 (with toxicant and without distributed delay) around the equilibrium point $\bar{E}_{22}$, showing the stability behavior

$$\begin{aligned} s_0 &= 0.7845; \quad k = 25; \quad q_1 = 0.47; \quad q_2 = 0.334; \\ q_3 &= 0.381; \quad q_4 = 0.12; \\ r_0 &= 1.02423; \quad r_1 = 0.0257; \quad d_1 = 1.0571; \\ d_2 &= 0.02483; \quad s_1 = 0.05; \quad q_0 = 0.515; \\ a_1 &= 0.71; \quad a_2 = 0.0042; \quad v = 0.102; \quad b_1 = 0.52; \\ \gamma_1 &= 0.59. \end{aligned}$$

It is found that under the above set of parameters, the equilibrium point $\hat{E}_{11}$

$$\begin{aligned} \hat{x} &= 18.5747, \quad \hat{y} = 0.0000, \quad \hat{z} = 0.0000, \\ \hat{C}_0 &= 0.8302, \quad \hat{R}_1 = 0.1173 \end{aligned}$$

is locally asymptotically stable (see Fig. 12).

We choose the following values of parameters for $\bar{E}_{12}$:

$$\begin{aligned} s_0 &= 0.9845; \quad r_0 = 0.002423; \quad a_1 = 0.61; \\ \gamma_1 &= 0.59; \quad a_2 = 0.92. \end{aligned}$$

With the above values of parameters and taking the remaining parameters to be the same as considered for $\hat{E}_{11}$, it is found that under the above set of parameters, the equilibrium point $\bar{E}_{12}$

$$\begin{aligned} \bar{x} &= 14.3096, \quad \bar{y} = 12.0529, \quad \bar{z} = 0.0000, \\ \bar{C}_0 &= 0.8419, \quad \bar{R}_1 = 0.1268 \end{aligned}$$

is locally asymptotically stable (see Fig. 13).

We choose the following values of parameters for E_{13}^* :

$$\begin{aligned} r_0 &= 0.02423; \quad d_1 = 0.0571; \quad s_1 = 0.195; \\ q_0 &= 0.0989; \quad a_1 = 0.89; \\ a_2 &= 0.23; \quad v = 0.103; \quad b_1 = 0.07; \quad \gamma_1 = 1.32; \end{aligned}$$

Fig. 10 Time graph for the Model 2 (with toxicant and without distributed delay) around the equilibrium point E_{23}^* , showing the stability behavior

Fig. 11 Time graph for the Model 2 (with toxicant and without distributed delay) around the equilibrium point E_{23}^* , showing the bifurcation behavior at $s_1 = 0.06$

With the above values of parameters and taking the remaining parameters to be the same as considered for $\hat{E}_{11}$, it is found that under the above set of parameters, the equilibrium point E_{13}^*

$$x^* = 11.7650, y^* = 6.0464, z^* = 1.4017, C_E^* = 0.4176, \\ C_0^* = 0.8819, R_1^* = 0.4179$$

is locally asymptotically stable (see Fig. 14).

These calculations prove that the equilibrium E_{13}^* is stable when $\gamma_1 > \gamma_1^*$ as is illustrated by the computer simulations (Fig. 14 and $\gamma_1 = 1.92$). When γ_1 passes through the critical value $\gamma_1^* = 1.22$, i.e., when $\gamma_1 < \gamma_1^*$, E_{13}^* loses its stability and a Hopf bifurcation occurs (see Fig. 15); i.e., a family of periodic solutions bifurcate from the positive equilibrium E_{13}^* .

Conclusion

In this paper we have proposed and analyzed a nonlinear mathematical model to study the role of distributed delay in toxicant uptake process by exposed prey population in a three-species food-chain system under toxicant stress. It is assumed in the model that the growth rate of prey population is adversely affected by the toxicant concentration present in the prey population, which is indirectly affecting the predator populations.

It is concluded from the stability of E_{31}^* of Model 3 that only the prey population would survive and both the predator populations would tend to extinction (see Fig. 4). From the stability of E_{32}^* of Model 3 it may be concluded that the prey and intermediate predator populations would

Fig. 12 Time graph for the Model 1 (toxicant and distributed delay) around the equilibrium point E_{11} , showing the stability behavior

Fig. 13 Time graph for the Model 1 (toxicant and distributed delay) around the equilibrium point E_{12} , showing the stability behavior

Fig. 14 Time graph for the Model 1 (toxicant and distributed delay) around the equilibrium point E_{13}^* , showing the stability behavior

Fig. 15 Time graph for the Model 1 (toxicant and distributed delay) around the equilibrium point E_{13}^* , showing the bifurcation behavior at $\gamma_1 = 1.22$

Fig. 16 Time series graph of Prey population for Model 3, Model 2 and Model 1 around the equilibrium points $\tilde{E}_{33} = (\tilde{x}, \tilde{y}, \tilde{z})$, $E_{23}^{**} = (x^{**}, y^{**}, z^{**}, C_E^{**}, C_0^{**})$ and $E_{13}^* = (x^*, y^*, z^*, C_E^*, C_0^*, R_1^*)$

survive and the top predator population would tend to extinction (see Fig. 5). The stability of $\tilde{E}_{33}$ of Model 3 ensures the coexistence of all the three populations (see Fig. 6). From the stability of $\hat{E}_{21}$ of Model 2, it is concluded that only prey population will survive in the presence of toxicant, but at reduced equilibrium. It may be noted that due to decrease in prey population, both the predator populations may get extinct (see Remark 3 and Fig. 8). From the stability of $\bar{E}_{22}$ of Model 2 it may be concluded that the prey and intermediate predator population both will survive in the presence of toxicant, but their equilibrium levels decrease due to toxicant (see Fig. 9). Due to the decrease in intermediate predator population, the top predator population also tend to extinction. The stability of E_{23}^{**} of Model 2 ensures the coexistence of all the three populations but their

equilibrium levels decrease due to the presence of toxicant (see Fig. 10). All the equilibrium points of Model 1 show the similar stability behavior as it is being shown in Model 2 but the equilibrium levels of each population in respective equilibrium points of Model 1 decreases further on account of distributed delay considered in toxicant uptake process by prey population (see Figs. 12, 13, 14, 16, 17, 18, 19). After introducing the distributed delay in Model 2 the Hopf-bifurcation phenomenon is observed in Model 1 with respect to the delay parameter γ_1 and it is shown that when γ_1 is greater than its critical value, the stability behavior is observed and when γ_1 is less than its critical value then Hopf-bifurcation occurs and periodic solution is obtained (see Fig. 15). This phenomenon has been observed by many authors (Ruan and Wolkowicz (1996); Tabares et al. 2011).

Fig. 17 Time series graph of intermediate predator population for Model 3, Model 2 and Model 1 around the equilibrium points

$$E_{33} = (\tilde{x}, \tilde{y}, \tilde{z}), E_{23}^* = (x^{**}, y^{**}, z^{**}, C_E^{**}, C_0^{**}) \text{ and } E_{13}^* = (x^*, y^*, z^*, C_E^*, C_0^*, R_1^*)$$

Fig. 18 Time series graph of top predator population for Model 3, Model 2 and Model 1 around the equilibrium points

$$E_{33} = (\tilde{x}, \tilde{y}, \tilde{z}), E_{23}^* = (x^{**}, y^{**}, z^{**}, C_E^{**}, C_0^{**}) \text{ and } E_{13}^* = (x^*, y^*, z^*, C_E^*, C_0^*, R_1^*)$$

Fig. 19 Time graph for Model 3, Model 2 and Model 1 around the equilibrium points

$$E_{33} = (\tilde{x}, \tilde{y}, \tilde{z}), E_{23}^* = (x^{**}, y^{**}, z^{**}, C_E^{**}, C_0^{**}) \text{ and } E_{13}^* = (x^*, y^*, z^*, C_E^*, C_0^*, R_1^*)$$

Appendix A: Proof of Theorem 5

Now, we will study the Hopf-bifurcation (Khare et al. 2011; Misra et al. 2013) of the system of Eqs. (7)–(11), taking s_1 as the bifurcation parameter. Now, the necessary and sufficient condition for the existence of the Hopf-bifurcation, if it exists is $s_1 = s_1^*$ such that

- (i) $Q_i(s_1^*) > 0, i = 1, 2, 3, 4, 5$,
- (ii) $Q_1(s_1^*)Q_2(s_1^*) > Q_3(s_1^*)$,
- (iii) $Q_1(s_1^*)Q_2(s_1^*)Q_3(s_1^*) > (Q_3(s_1^*))^2 + Q_1(s_1^*)^2 Q_4(s_1^*)$,
- (iv) $(Q_3(s_1^*)Q_4(s_1^*) - Q_2(s_1^*)Q_5(s_1^*))(Q_1(s_1^*)Q_2(s_1^*) - Q_3(s_1^*)) - (Q_1(s_1^*)Q_4(s_1^*) - Q_5(s_1^*))^2 = 0$ and
- (v) if we consider the eigen values of the characteristic Eq. (90) is of the form $\lambda_i = u_i + iv_i$, then $\frac{du_i}{ds_1} \neq 0, i = 1, 2, 3, 4, 5$. After substituting the values, the condition $(Q_1Q_2 - Q_3)(Q_3Q_4 - Q_2Q_5) - (Q_1Q_4 - Q_5)^2$ becomes

$$D_{11}s_1^3 + D_{12}s_1^2 + D_{13}s_1 + D_{14} = 0 \quad (118)$$

where,

$$\begin{aligned} D_{11} &= (R_{14}R_{16} - R_{12}R_{18})(Q_1R_{12} + R_{14}), \\ D_{12} &= Q_1R_{12}(R_{13}R_{16} - R_{14}R_{15}) + R_{14}(R_{11}R_{18} - R_{14}R_{15}) \\ &\quad + 2R_{16}(R_{13}R_{14} - Q_1R_{18}) \\ &\quad + Q_1R_{11}(2R_{12}R_{18} - R_{14}R_{16}) - Q_1(Q_1R_{16}^2 + R_{12}^2R_{17}) \\ &\quad - R_{12}(R_{13}R_{18} + R_{14}R_{17}) - R_{18}^2, \\ D_{13} &= 2Q_1R_{17}(R_{11}R_{12} - R_{16}) + (Q_1R_{15} + R_{17})(R_{11}R_{14} - R_{12}R_{13}) \\ &\quad + 2R_{18}(Q_1R_{15} - R_{17}) + 2R_{15}(Q_1^2R_{16} - R_{13}R_{14}) \\ &\quad - R_{11}R_{13}(Q_1R_{16} - R_{18}) + R_{13}^2R_{16} - Q_1R_{11}^2R_{18}, \\ D_{14} &= (Q_1R_{11} - R_{13})(R_{13}R_{15} - R_{11}R_{17}) \\ &\quad + Q_1R_{15}(2R_{17} - Q_1R_{15}) - R_{17}^2, \end{aligned}$$

$$\begin{aligned} R_{11} &= q_{12}q_{13} + x^{**}q_{11}y^{**}q_{14} + q_{15}q_{16} - (x^{**}q_{11} + y^{**}q_{14}) \\ &\quad d_2z^{**} - (x^{**}q_{11} + y^{**}q_{14} - d_2z^{**})q_{17} - va_2(x^{**})^2 + (d_2z^{**} \\ &\quad + q_{17} - x^{**}q_{11} - y^{**}q_{14})q_{18}, \end{aligned}$$

$$\begin{aligned} R_{12} &= vx^{**}C_0^{**}, \quad R_{13} = (q_{12}q_{13} + x^{**}q_{11}y^{**}q_{14})d_2z^{**} \\ &\quad - x^{**}q_{11}q_{15}q_{16} + (q_{12}q_{13} + x^{**}q_{11}y^{**}q_{14} + q_{15}q_{16} - x^{**}q_{11} \\ &\quad d_2z^{**} - y^{**}q_{14}d_2z^{**})q_{17} + (q_{12}q_{13} + x^{**}q_{11}y^{**}q_{14} + q_{15}q_{16} - \\ &\quad x^{**}q_{11}d_2z^{**} - y^{**}q_{14}d_2z^{**} - x^{**}q_{11}q_{17} - y^{**}q_{14}q_{17} + d_2z^{**} \\ &\quad q_{17})q_{18} + (q_{11}v(x^{**})^2 + y^{**}q_{14}vx^{**} - d_2z^{**}vx^{**})a_2x^{**}, \end{aligned}$$

$$R_{14} = a_2x^{**2}(vC_0^{**} - a_2C_E^{**}) + (y^{**}q_{14} - d_2z^{**} - q_{17})x^{**}vC_0^{**},$$

$$\begin{aligned} R_{15} &= q_{12}q_{13}d_2z^{**}q_{17} + x^{**}q_{11}y^{**}q_{14}d_2z^{**}q_{17} - x^{**}q_{11} \\ &\quad q_{15}q_{16}q_{17} + (x^{**}q_{11}d_2z^{**} + y^{**}q_{14}d_2z^{**} - q_{12}q_{13} - x^{**}q_{11} \\ &\quad y^{**}q_{14} - q_{15}q_{16})va_2x^{**2} - x^{**}q_{11}q_{15}q_{16}q_{18} + (q_{12}q_{13} + x^{**} \end{aligned}$$

$$q_{11}y^{**}q_{14})d_2z^{**}q_{18} + (q_{12}q_{13} + x^{**}q_{11}y^{**}q_{14} + q_{15}q_{16} - x^{**}q_{11}d_2z^{**} - y^{**}q_{14}d_2z^{**})q_{17}q_{18},$$

$$R_{16} = (d_2z^{**} - y^{**}q_{14})x^{**}q_{17}vC_0^{**} - (y^{**}q_{14}d_2z^{**} - q_{15}q_{16})x^{**}vC_0^{**} - (d_2z^{**} - y^{**}q_{14})(vC_0^{**} - a_2C_E^{**})a_2(x^{**})^2,$$

$$R_{17} = (x^{**}q_{11}q_{15}q_{16} - q_{12}q_{13}d_2z^{**} - x^{**}q_{11}y^{**}q_{14}d_2z^{**})vx^{**}a_2x^{**} - x^{**}q_{11}q_{15}q_{16}q_{17}q_{18} + (q_{12}q_{13} + x^{**}q_{11}y^{**}q_{14})d_2z^{**}q_{17}q_{18},$$

$$R_{18} = x^{**}(y^{**}q_{14}d_2z^{**} - q_{15}q_{16})(a_1vC_0^{**} + a_2^2C_E^{**}x^{**}).$$

The Eq. (118) has at least one positive root say $s_1 = s_1^*$. Therefore, one pair of eigenvalues of the characteristic Eq. (90) at $s_1 = s_1^*$ are of the form $\lambda_{1,2} = \pm iv$, where v is positive real number. Now, we will verify the Hopf-bifurcation condition (v), putting $\lambda = u + iv$ in (90) and separating real and imaginary parts, we have

$$\begin{aligned} u^5 + Q_1u^4 + (Q_2 - 10v^2)u^3 + (Q_3 - 6Q_1v^2)u^2 \\ + (5v^4 - 3Q_2v^2 + Q_4)u + (Q_1v^4 - Q_3v^2 + Q_5) = 0 \end{aligned} \quad (119)$$

$$\begin{aligned} (v^2)^2 - (10u^2 + 4Q_1u + Q_2)v^2 + (5u^4 + 4Q_1u^3 \\ + 3Q_2u^2 + 2Q_3u + Q_4) = 0 \end{aligned} \quad (120)$$

Substituting the value of v^2 from (120) in (119), we get

$$\begin{aligned} u^5 + Q_1u^4 + (Q_2 - 10p(u))u^3 + (Q_3 - 6Q_1p(u))u^2 \\ + (5(p(u))^2 - 3Q_2p(u) + Q_4)u + (Q_1(p(u))^2 \\ - Q_3p(u) + Q_5) = 0 \end{aligned} \quad (121)$$

where $p(u) = \frac{1}{2}((10u^2 + 4Q_1u + Q_2) - G_{20})$ and $G_{20} = \sqrt{(10u^2 + 4Q_1u + Q_2)^2 - 4(5u^4 + 4Q_1u^3 + 3Q_2u^2 + 2Q_3u + Q_4)}$, differentiating with respect to s_1 and putting $s_1 = s_1^*$, we have

$$\begin{aligned} \left[\frac{du}{ds_1} \right]_{s_1=s_1^*} \\ = \frac{p(0)\frac{dQ_3}{ds_1} - (p(0))^2\frac{dQ_1}{ds_1} - \frac{dQ_5}{ds_1}}{5(p(0))^2 + 2Q_1p(0)p'(0) - 3Q_2p(0) - Q_3p'(0) + Q_4} \neq 0 \end{aligned} \quad (122)$$

since,

$$\begin{aligned} p(s_1)\frac{dQ_3}{ds_1} - (p(s_1))^2\frac{dQ_1}{ds_1} - \frac{dQ_5}{ds_1} = p(u)[vx^{**}C_0^{**}(a_2x^{**} \\ + y^{**}q_{14} - z^{**}d_1 - q_{17}) - a_2^2x^{**2}C_E^{**}] \\ - [x^{**}(y^{**}z^{**}d_2q_{14} - q_{15}q_{16})(va_1C_0^{**} + x^{**}C_E^{**}a_2^2)] \neq 0 \end{aligned}$$

For example, taking $s_0 = 0.7845, k = 25, q_1 = 0.47, q_2 = 0.334, q_3 = 0.381, q_4 = 0.12, r_0 = 0.02423, r_1 = 0.0257, s_1 = 0.06, d_1 = 0.0571, d_2 = 0.02483, q_0 = 0.515, a_1 = 0.71, a_2 = 0.92, v = 0.102$, and $b_1 = 0.52$, in the Eq. (122), we get a positive value i.e., 0.313506.

This ensures that the above system has a Hopf-bifurcation around the positive interior equilibrium E_{23}^{**} .

Appendix B: Proof of Theorem 6

We consider the following positive definite function about E_{23}^{**} :

$$\begin{aligned} V_{24} &= \left(x - x^{**} - x^{**} \ln\left(\frac{x}{x^{**}}\right)\right) + H_1 \left(y - y^{**} - y^{**} \ln\left(\frac{y}{y^{**}}\right)\right) \\ &\quad + \left(z - z^{**} - z^{**} \ln\left(\frac{z}{z^{**}}\right)\right) + \frac{H_2}{2} (C_E - C_E^{**})^2 \\ &\quad + \frac{H_3}{2} (C_0 - C_0^{**})^2 \\ \frac{dV_{24}}{dt} &= \left(\frac{x - x^{**}}{x}\right) \frac{dx}{dt} + H_1 \left(\frac{y - y^{**}}{y}\right) \frac{dy}{dt} + \left(\frac{z - z^{**}}{z}\right) \frac{dz}{dt} \\ &\quad + H_2 (C_E - C_E^{**}) \frac{dC_E}{dt} + H_3 (C_0 - C_0^{**}) \frac{dC_0}{dt} \\ \frac{dV_{24}}{dt} &= -(x - x^{**})^2 \left(\frac{1}{k} - \frac{q_1 y^*}{\pi_{21}}\right) - (y - y^{**})^2 \\ &\quad H_1 \left(r_1 + \frac{q_3 z^{**}}{\pi_{22}}\right) - (z - z^{**})^2 d_2 \\ &\quad - (C_E - C_E^{**})^2 H_2 [a_1 + (a_2 - v)x^{**}] \\ &\quad - (C_0 - C_0^{**})^2 H_3 (b_1 + vx^{**}) \\ &\quad - (x - x^{**})(y - y^{**})[(1 + x^{**})q_1 - H_1 q_2]/\pi_{21} \\ &\quad - (x - x^{**})(C_E - C_E^{**})H_2 [a_2 C_E - vC_0] \\ &\quad + (C_E - C_E^{**})(C_0 - C_0^{**})H_3 a_2 x^{**} \\ &\quad + (x - x^{**})(C_0 - C_0^{**})[H_3 (a_2 C_E - vC_0) - s_1] \\ &\quad + (y - y^{**})(z - z^{**})[(1 + y^{**})H_1 q_3 + H_2 q_4]/\pi_{22} \end{aligned}$$

Now using the sylvester's criterion in the quadratic forms

$$\begin{aligned} \frac{dV_{24}}{dt} &\leq -[(h_{11}/3)(x - x^{**})^2 + h_{12}(x - x^{**})(y - y^{**}) \\ &\quad + (h_{22}/2)(y - y^{**})^2] \\ &\quad + ((h_{11}/3)(x - x^{**})^2 + h_{14}(x - x^{**})(C_E - C_E^{**}) \\ &\quad + (h_{44}/2)(C_E - C_E^{**})^2) \\ &\quad + ((h_{11}/3)(x - x^{**})^2 - h_{15}(x - x^{**})(C_0 - C_0^{**}) \\ &\quad + (h_{55}/2)(C_0 - C_0^{**})^2) \\ &\quad + ((h_{22}/2)(y - y^{**})^2 - h_{23}(y - y^{**})(z - z^{**}) \\ &\quad + h_{33}(z - z^{**})^2) \\ &\quad + ((h_{44}/2)(C_E - C_E^{**})^2 - h_{45}(C_E - C_E^{**}) \\ &\quad (C_0 - C_0^{**}) + (h_{55}/2)(C_0 - C_0^{**})^2) \end{aligned}$$

where,

$$\begin{aligned} h_{11} &= \left(\frac{1}{k} - \frac{q_1 y^*}{\pi_{21}}\right)/3, h_{12} = [(1 + x^{**})q_1 - H_1 q_2]/\pi_{21}, \\ h_{14} &= H_2 [a_2 C_E - vC_0], \\ h_{15} &= H_3 (a_2 C_E - vC_0) - s_1, h_{22} = H_1 (r_1 + \frac{q_3 z^{**}}{\pi_{22}})/2, \\ \pi_{22} &= (1 + y)(1 + y^{**}) \\ h_{23} &= [(1 + y^{**})H_1 q_3 + H_2 q_4]/\pi_{22}, h_{33} = d_2, \\ h_{44} &= H_2 [a_1 + (a_2 - v)x^{**}]/2, \\ h_{45} &= H_3 a_2 x^{**}, h_{55} = H_3 (b_1 + vx^{**})/2, \\ \pi_{21} &= (1 + x)(1 + x^{**}). \end{aligned}$$

Sufficient conditions for dV_{34}/dt to be negative definite are that the following inequalities hold:

$$h_{11} > 0 \quad (123)$$

$$h_{44} > 0 \quad (124)$$

$$h_{11}h_{22} > h_{12}^2 \quad (125)$$

$$h_{11}h_{44} > h_{14}^2 \quad (126)$$

$$h_{11}h_{55} > h_{15}^2 \quad (127)$$

$$h_{22}h_{33} > h_{23}^2 \quad (128)$$

$$h_{44}h_{55} > h_{45}^2 \quad (129)$$

We note that the fifth, sixth and eighth inequalities, i.e., $h_{11}h_{22} > h_{12}^2$, $h_{11}h_{44} > h_{14}^2$ and $h_{11}h_{55} > h_{15}^2$ are satisfied due to the proper choice of H_1 , H_2 and H_3 and other inequalities, (91) $\Rightarrow$ (123), (92) $\Rightarrow$ (124), (93) $\Rightarrow$ (128) and (94) $\Rightarrow$ (129). Hence V_{24} is a Liapunov function with respect to E_{23}^{**} , whose domain contains the region of attraction Ω , proving the theorem.

Appendix C: Proof of Theorem 7

Now, we will study the Hopf-bifurcation (Khare et al. 2011; Misra et al. 2013) of the system of Eqs. (12)–(14) taking q_4 as the bifurcation parameter. The necessary and sufficient conditions for the existence of the Hopf-bifurcation for $q_4 = \tilde{q}_4$, if it exist are

- (i) $B_i(\tilde{q}_4) > 0, i = 1, 2, 3$
- (ii) $B_1(\tilde{q}_4)B_2(\tilde{q}_4) - B_3(\tilde{q}_4) = 0$ and
- (iii) the eigenvalues of the characteristic equation should be of the form $\lambda_i = u_i + iv_i$, and $\frac{du_i}{dq_4} \neq 0$,

$i = 1, 2, 3$. After substituting the values, the condition $G_{10} = B_1 B_2 - B_3$ becomes

$$-D_1 q_4^3 + D_2 q_4^2 + D_3 q_4 + D_4 = 0 \quad (130)$$

where, $D_1 = V_1 V_3, D_2 = V_6 + V_1 V_4 - V_2 V_3, D_3 = V_2 V_4 + V_1 V_5 - V_7, D_4 = V_2 V_5 + V_8, V_1 = \frac{\tilde{y}}{1+\tilde{y}}$
 $(1 - \frac{q_3 \tilde{y}}{d_2(1+\tilde{y})^2}), V_2 = \tilde{x}(\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2}) + r_1 \tilde{y} - d_1 + \frac{d_1 q_3 \tilde{y}}{d_2(1+\tilde{y})^2}, V_3 = (\tilde{y} - 1) \frac{q_3 \tilde{y}^2}{d_2(1+\tilde{y})^4}, V_4 = \frac{\tilde{y}^2}{1+\tilde{y}} (r_1 + \frac{d_1 q_3}{d_2(1+\tilde{y})^2}) - \frac{d_1 q_3 \tilde{y}}{d_2(1+\tilde{y})^3} + \frac{\tilde{x} \tilde{y}}{1+\tilde{y}} (1 - \frac{q_3 \tilde{y}}{d_2(1+\tilde{y})^2})(\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2}),$
 $V_5 = \tilde{x}(r_1 \tilde{y} - d_1)(\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2}) + \frac{q_1 q_2 \tilde{x} \tilde{y}^2}{(1+\tilde{x})^3} + \frac{d_1 q_3 \tilde{y}}{d_2(1+\tilde{y})^2}, V_6 = \frac{(\tilde{y}-1)q_3 \tilde{x} \tilde{y}^2}{d_2(1+\tilde{y})^4} (\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2}), V_7 = \frac{q_1 q_2 \tilde{x} \tilde{y}^2}{(1+\tilde{x})^3(1+\tilde{y})} + \frac{q_1 \tilde{x} \tilde{y}^2}{d_2(1+\tilde{y})^2}$
 $(r_1 d_2 + \frac{d_1 q_3}{(1+\tilde{y})^2}) (\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2}) + (\tilde{y} - 1) \frac{\tilde{x} \tilde{y} d_1 q_3}{(1+\tilde{y})^3} (\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2}), V_8 = \frac{d_1 q_1 q_2 \tilde{x} \tilde{y}}{(1+\tilde{x})^3} + \frac{d_1 \tilde{x} \tilde{y}}{d_2} (r_1 d_2 + \frac{d_1 q_3}{(1+\tilde{y})^2}) (\frac{1}{k} - \frac{q_1 \tilde{y}}{(1+\tilde{x})^2}).$ The Eq. (130) has at least one positive root say $q_4 = \tilde{q}_4$. Therefore, one pair of eigenvalues of the characteristic Eq. (115) at $q_4 = \tilde{q}_4$ are of the form $\lambda_{1,2} = \pm iv$, where v is positive real number.

Now, we will verify the Hopf-bifurcation condition (iii), putting $\lambda = u + iv$ in (115), we get

$$(u + iv)^3 + B_1(u + iv)^2 + B_2(u + iv) + B_3 = 0 \quad (131)$$

On separating the real and imaginary parts, we have

$$u^3 - 3uv^2 + B_1 u^2 - B_1 v^2 + B_2 u + B_3 = 0 \quad (132)$$

$$-v^2 + 3u^2 + 2B_1 u + B_2 = 0 \quad (133)$$

Substituting the value of v^2 from (133) in (132), we get

$$8u^3 + 8B_1 u^2 + 2(B_1^2 + B_2)u + (B_1 B_2 - B_3) = 0 \quad (134)$$

It is clear from the above that $u(\tilde{q}_4) = 0$ iff $B_1(\tilde{q}_4)B_2(\tilde{q}_4) - B_3(\tilde{q}_4) = 0$. Further, at $q_4 = \tilde{q}_4$, $u(\tilde{q}_4)$ is the only root, since the discriminate of $8u^2 + 8B_1 u + 2(B_1^2 + B_2) = 0$ is $64B_1^2 - 64(B_1^2 + B_2) < 0$. Again, differentiating (134) with respect to q_4 , we have

$$24u^2 \frac{du}{dq_4} + 16B_1 u \frac{du}{dq_4} + 8u^2 \frac{dB_1}{dq_4} + 2(B_1^2 + B_2) \frac{du}{dq_4} + 2u \left[2B_1 \frac{dB_1}{dq_4} + \frac{dB_2}{dq_4} \right] + \frac{dG_{10}}{dq_4} = 0$$

Now, since at $q_4 = \tilde{q}_4$, $u(\tilde{q}_4) = 0$ we get

$$\left[\frac{du}{dq_4} \right]_{q_4=\tilde{q}_4} = \frac{-\frac{dG_{10}}{dq_4}}{2(B_1^2 + B_2)} \neq 0 \quad (135)$$

This ensures that the above system has a Hopf-bifurcation around the positive interior equilibrium E_{33} .

Appendix D: Proof of Theorem 8

We consider the following positive definite function about E_{33} :

$$V_{14} = \left(x - \tilde{x} - \tilde{x} \ln \left(\frac{x}{\tilde{x}} \right) \right) + I_1 \left(y - \tilde{y} - \tilde{y} \ln \left(\frac{y}{\tilde{y}} \right) \right) + I_2 \left(z - \tilde{z} - \tilde{z} \ln \left(\frac{z}{\tilde{z}} \right) \right)$$

Differentiating V_{14} with respect to time t , we get

$$\begin{aligned} \frac{dV_{14}}{dt} &= \left(\frac{x - \tilde{x}}{x} \right) \frac{dx}{dt} + I_1 \left(\frac{y - \tilde{y}}{y} \right) \frac{dy}{dt} + I_2 \left(\frac{z - \tilde{z}}{z} \right) \frac{dz}{dt} \\ \frac{dV_{14}}{dt} &= -(x - \tilde{x})^2 \left(\frac{1}{k} - \frac{q_1 \tilde{y}}{\sigma_1} \right) - (y - \tilde{y})^2 I_1 \left(r_1 - \frac{\tilde{z} q_3}{\sigma_2} \right) - I_2 (z - \tilde{z})^2 d_2 \\ &\quad + (x - \tilde{x})(y - \tilde{y}) \frac{1}{\sigma_1} (q_2 I_1 - q_1(1 + \tilde{x})) \\ &\quad + (y - \tilde{y})(z - \tilde{z}) \frac{1}{\sigma_2} (q_4 I_2 - I_1 q_3(1 + \tilde{y})). \end{aligned}$$

Assuming $I_1 = \frac{q_1}{q_2}(1 + \tilde{x})$ and $I_2 = \frac{q_1 q_3}{q_2 q_4}(1 + \tilde{x})(1 + \tilde{y})$, where $\sigma_1 = (1 + x)(1 + \tilde{x}), \sigma_2 = (1 + y)(1 + \tilde{y})$.

$$\begin{aligned} \frac{dV_{14}}{dt} &= -(x - \tilde{x})^2 \left(\frac{1}{k} - \frac{q_1 \tilde{y}}{\sigma_1} \right) - (y - \tilde{y})^2 I_1 \left(r_1 - \frac{\tilde{z} q_3}{\sigma_2} \right) \\ &\quad - I_2 (z - \tilde{z})^2 d_2 \end{aligned}$$

which is negative definite if in Ω requiring that: $\frac{1}{k} - \frac{q_1 \tilde{y}}{1+\tilde{x}} > 0$ and $r_1 - \frac{\tilde{z} q_3}{1+\tilde{y}} > 0$. Hence V_{14} is a Liapunov function with respect to E_{33} , whose domain contains the region of attraction Ω , proving the theorem.

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