Photosynthesis rate, salinity induced grazing rate and gestation delay as determinants of the dynamics of a realistic food chain model: a stochastic approach

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Abstract Impact of photosynthesis rate of phytoplankton, salinity induced grazing rate of zooplankton and gestation delay of zooplankton and fish population are considered in the present study. Models with delay are much more realistic, as in reality time delays occur in almost every biological situation and assume to be one of the reasons of regular fluctuations in population density. Reproduction of predator after consuming prey is not instantaneous, but mediated by some time lag required for gestation. It is observed that there is stability switches and Hopf bifurcation occurs when the delay crosses some critical value. It is observed that the quantitative level of abundance of system population depends crucially on the delay parameter if the gestation period exceeds some critical value. Also, environmental stochasticity in the form of Gaussian white-noise, plays a significant role to describe the system and its values. Numerical examples are also support the model assumption and analytical results.

Keywords Photosynthesis rate · Salinity induced grazing rate · Gestation delay · Hopf-bifurcation · Gaussian white-noise

Introduction

Ecosystem is composed of dynamically interacting parts including organisms, the communities they make up and the non-living components of their environment. Understanding the dynamical relationship between prey and predator through mathematical modelling in theoretical ecology has gained a lot of importance during the last few decades (Berryman 1992). On the purpose of observation and study the effect of environmental noise on population dynamics, researchers have used Gaussian white-noise (Lande 1993; May 1973; Jana et al. 2015) and color noise (Samanta and Maiti 2003; Jana and Bairagi 2013) as a model of environmental variations. May (1973) studies a population model considering white noise and shows that when the population deviates more from equilibrium point, the system shows irregular behavior (i.e., instability). Where as Jana et al. (2015) show a three species food chain model where the interacting processes among the species are influenced by Gaussian white-noise and with increasing noise intensity system gradually goes into chaos from stable coexistence through instability stochastically. Ripa et al. (1998) examine the effect of environmental noise on population and present a general theory of environmental noise in ecological food webs. On a fairly realistic ecological model with generalist top predator, Upadhyay et al. (2007) investigate the influence of environmental noise and show the importance of the noise amplitude on the trophic level and the susceptibility of populations to environmental noise. Complex dynamical behavior of a biological system arises as a consequence of time delay (with significant time delay) which may exhibit limit cycle oscillations (Ruan 2009) and chaos (Mackey and Glass 1977; Jana 2014; Jana et al. 2014). In reality, time delays occur in almost every ecological situation and assume to be one of the most

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prominent reason of regular fluctuations on population biomass (Jana 2014; Jana et al. 2014, 2015). Ruan (2009) presents a brief review on prey–predator models with discrete delay for both Kolmogorov and non-Kolmogorov type model system. It often represents gestation time (Jana 2014; Jana et al. 2014), incubation period, transport delay or can simply lump complicated biological process together, accounting only for the time requires for these process to occur. Delay parameter emphasizes on Hopf-bifurcation (Ruan 2009; Jana 2014; Jana et al. 2014, 2015) and also it causes some complex dynamical behaviors, such as double Hopf bifurcation, quasi-periodic solutions and chaos (Lü 2002; Ma et al. 2008; Matouk 2008; Ding and Jiang 2011; Wang et al. 2014; Ding et al. 2013; Song and Xu 2013). Along with different studies of environmental stochasticity and biological delays in prey–predator system, influence of noise on a delayed system is very significant for advance study of biological system. Schwartz et al. (2015) and Jana et al. (2015) study the effect of delay on the rates of noise-induced switching between co-existing stable states and noise-induced extinction in a population dynamic model.

In the present work, we have investigated the complex dynamical behavior of a phytoplankton–zooplankton–fish population delayed food chain model with Holling type II functional response as to describe feeding process of both zooplankton and fish population. Here we consider gestation delays in the numerical response terms of both zooplankton and fish population. The paper is structured as follows: in “Model and analysis”, model system and their stability and Hopf-bifurcation has been discussed, followed by “Stochastic scenario of non-delayed system”, “Stochastic scenario of delayed system” and “Numerical results”. At last, in “Discussion”, conclusions and discussions have been made.

Model and analysis

Let P , Z and F denote the population of phytoplankton, zooplankton and fish population respectively. The model is

$$\begin{aligned}\frac{dP}{dt} &= m_1 P \left(1 - \frac{P}{K_P}\right) - g_s \frac{PZ}{P + K_Z}, \\ \frac{dZ}{dt} &= ag_s \frac{PZ}{P + K_Z} - m_2 Z - g_w \frac{ZF}{Z + K_F}, \\ \frac{dF}{dt} &= g_w \frac{ZF}{Z + K_F} - m_3 F.\end{aligned}\quad (1)$$

In this present PZF model, light and temperature dependant photosynthesis rate (m_1) of phytoplankton and salinity induced grazing rate (g_s) of zooplankton are incorporated, these equations are taken from the model of Mandal et al. (2012).

$$m_1 = (r_{ph} + r_s + M_{ph}), \quad (2)$$

where, r_{ph} , r_s , M_{ph} are respectively respiratory rate of phytoplankton, rate of settling and mortality rate of phytoplankton.

$$g_s = \delta g_z \quad (3)$$

where

$$\delta = \frac{S_u}{S_u - S_d}, \quad (4)$$

where, g_s , S_d , S_u , g_z , g_w , K_Z , K_F and K_P are salinity induced grazing of zooplankton, upstream salinity, downstream salinity, grazing rate of zooplankton, grazing rate of fish, half saturation constant of zooplankton on phytoplankton, half saturation constant of fish on zooplankton and environmental carrying capacity of phytoplankton. Since estuary is a transition zone of river and sea, there is always fluctuation of salinity throughout the year, which is due to dilution by upstream river water and/or mixing by downstream sea tidal water. δ is the dilution factor, is calculated following the equation of Ketchum (1951). It is the function of S_d and S_u . Besides grazing the abundance of zooplankton is also dependent on loss due to $E_{zoo} = ZE_{zo}$, respiration $R_{zoo} = Zr_{zo}$, fish predation $F_p = Zr_{rf}$, and mortality $M_{zoo} = ZM_z$, R_{zoo} is governed by r_{zo} .

$$m_2 = (E_{zo} + r_{zo} + r_{fp} + M_z). \quad (5)$$

The abundance of F is governed by many processes in the estuary. Fish predation F_p on Z follows Michaelis–Menten kinetics (type II) which enriches the fish pool of the estuarine system. F_p depends on K_Z and g_f . F population is reduced by mortality rate of fish M_f , respiration Fr_f , harvest by fishing FH_{cf} and excretion of fish FE_f and FH_f are controlled by R_f and H_f respectively. Respiration rate of fish and H_f harvest rate of carnivorous fish respectively.

$$m_3 = (E_f + M_f + r_f + H_f). \quad (6)$$

It is already mentioned that time-delay is an important factor in biological systems. As a starting point of this section, we consider the following generalization of the model

$$\begin{aligned}\frac{dP}{dt} &= m_1 P \left(1 - \frac{P}{K_P}\right) - g_s \frac{PZ}{P + K_Z}, \\ \frac{dZ}{dt} &= ag_s \frac{P(t-\tau)Z}{P(t-\tau) + K_Z} - m_2 Z - g_w \frac{ZF}{Z + K_F}, \\ \frac{dF}{dt} &= g_w \frac{Z(t-\tau)F}{Z(t-\tau) + K_F} - m_3 F,\end{aligned}\quad (7)$$

where τ represents the gestation period of zooplankton and fish population respectively.

The system (7) has only one interior equilibrium point given by $E^* = (P^*, Z^*, F^*)$; where

$$P^* = \frac{-m_1(g_f - m_3)(k - k_z) + \sqrt{(m_1(g_f - m_3)(k - k_z))^2 - 4m_1(m_3 - g_f)(kk_z m_1(g_f - m_3) - m_3 k_f g_s k)}}{2m_1(m_3 - g_f)},$$

$Z^* = \frac{m_3 k_f}{(g_f - m_3)}$ and $F^* = \frac{m_1 k_f a(P^* - k)}{m_3 k} + \frac{k_f(a g_s - m_2)}{(g_f - m_3)}$. The equilibrium point E^* is biologically feasible if $(g_f - m_3) > 0$. Linearizing the system (7) at E^* , we get

$$\begin{aligned} \frac{dP}{dt} &= a_{11}P(t) + a_{12}Z(t), \\ \frac{dZ}{dt} &= a_{22}Z(t) + a_{23}F(t) + b_{21}P(t - \tau), \\ \frac{dF}{dt} &= b_{32}Z(t - \tau) \end{aligned} \quad (8)$$

where, $a_{11} = -m_1 \frac{P^*}{K_P} + g_s \frac{P^* Z^*}{(P^* + K_Z)^2}$, $a_{12} = g_s \frac{P^*}{P^* + K_Z}$, $a_{22} = g_w \frac{Z^* F^*}{(Z^* + K_F)^2}$, $a_{23} = g_w \frac{Z^*}{Z^* + K_F}$, $b_{21} = a g_s \frac{K_Z Z^*}{(Z^* + K_Z)^2}$, $b_{32} = g_w \frac{F^* K_F}{(Z^* + K_F)^2}$. The corresponding characteristic equation is given by

$$\lambda^3 + A\lambda^2 + B\lambda + e^{-\lambda\tau}(C\lambda + D) = 0, \quad (9)$$

where $A = -(a_{11} + a_{22})$, $B = a_{11}a_{22}$, $C = -(a_{23}b_{32} + a_{12}b_{21})$ and $D = a_{11}a_{23}b_{32}$. We now discuss the following cases.

Case I: $\tau = 0$

For non-delayed system i.e., when $\tau = 0$, characteristic Eq. (9) becomes

$$\lambda^3 + A\lambda^2 + (B + C)\lambda + D = 0.$$

Hence, by Routh–Hurwitz criterion, we get the stability conditions of E^* .

Theorem 1 *The coexistence equilibrium point E^* of system (1) is asymptotically stable if $A, B + C, D > 0$ and $A(B + C) > D$.*

Case II: $\tau > 0$

We know that $i\omega$ ($\omega > 0$) is a root of (9) if and only if ω satisfies

$$-i\omega^3 - A\omega^2 + Bi\omega + (iC\omega + D)(\cos \omega\tau - i\sin \omega\tau) = 0. \quad (10)$$

Separating real and imaginary parts, we get

$$\begin{aligned} A\omega^2 &= C\omega \sin \omega\tau + D \cos \omega\tau, \\ S\omega^3 + B\omega &= D \sin \omega\tau - C\omega \cos \omega\tau. \end{aligned} \quad (11)$$

These two equations give the positive values of τ and ω for which Eq. (9) can have purely imaginary roots. Squaring and adding these equations, we have

$$\omega^6 + p\omega^4 + q\omega^2 + r = 0, \quad (12)$$

where

$$\begin{aligned} p &= A^2 + 2B, \\ q &= B^2 - C^2, \\ r &= -D^2. \end{aligned} \quad (13)$$

Now we shall measure the value of τ for which system (7) will remain stable. From Eq. (11) we have,

$$\tau_k = \frac{1}{\omega_k} \left[\cos^{-1} \left\{ \frac{(AD - BC)\omega_k^2 - C\omega_k^4}{D^2 + C^2\omega_k^2} \right\} + 2k\pi \right], \quad (14)$$

$k = 0, 1, 2, 3, \dots$, where ω_k is the real positive root of Eq. (12). Also, differentiating Eq. (9) with respect to τ we get

$$\left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\tau=\tau_0} = \frac{U_1 V_1 + U_2 V_2}{V_1^2 + V_2^2}, \quad (15)$$

where,

$$\begin{aligned} U_1 &= C + \sin \omega\tau(-2A\omega + \tau\omega^3 - B\tau\omega) \\ &\quad + \cos \omega\tau(B - 3\omega^2 - A\tau\omega^2), \\ U_2 &= \sin \omega\tau(B - 3\omega^2 - A\tau\omega^2) \\ &\quad + \cos \omega\tau(2A\omega - \tau\omega^3 + B\tau\omega), \\ V_1 &= -A\omega^3 \sin \omega\tau + \cos \omega\tau(B\omega^2 - \omega^4), \\ V_2 &= \sin \omega\tau(B\omega^2 - \omega^4) + A\omega^3 \cos \omega\tau. \end{aligned} \quad (16)$$

Hence the transversality condition is satisfied if $(U_1 V_1 + U_2 V_2) > 0$.

Theorem 2 *Assume $\tau > 0$ and conditions of the Theorem 1 are satisfied, then the equilibrium $E^*(P^*, Z^*, F^*)$ is locally asymptotically stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$. Furthermore, the system (7) undergoes a Hopf bifurcation at $E^*(P^*, Z^*, F^*)$ when $\tau = \tau_0$, where*

$$\tau_0 = \frac{1}{\omega_0} \left[\cos^{-1} \left\{ \frac{(AD - BC)\omega_0^2 - C\omega_0^4}{D^2 + C^2\omega_0^2} \right\} \right].$$

Stochastic scenario of non-delayed system

Deterministic models in ecology do not usually incorporate environmental fluctuation; they are often justified by the implicit assumption that in large population, stochastic deviations are small enough to be ignored. Deterministic model will prove ecologically useful only if the dynamical patterns they reveal are still in evidence when stochastic effects are introduced. For terrestrial system, the environmental variability is large at both short and long time periods and could be expected to develop internal mechanisms to the system which would cope with short term variability and minimize the effects of long term variations, hence analysis of the system with white noise gives better results. Uncertain growth of populations is usually considered as an effect of environmental stochasticity. Reproduction of species depends on various factors, such as temperature, humidity, parasites and pathogens, environmental pollution etc. (Iwasa et al. 2000). Since physical and biological environments of populations are not totally predictable, the growth of populations should be considered as a stochastic process rather than a deterministic one (Turelli 1986). In spite of some shortcomings, Gaussian white noise has been proved extremely useful to model rapidly fluctuating phenomena (Valsakumar et al. 1983; Samanta and Maiti 2003; Jana et al. 2015). Therefore, in the non-delayed counterpart of model system (1) by introducing the environmental stochasticity in the form of Gaussian white noise is represented by:

$$\begin{aligned} \frac{dP}{dt} &= m_1 P \left(1 - \frac{P}{K_P} \right) - g_s \frac{PZ}{P + K_Z} + \alpha_1 \eta_1(t), \\ \frac{dZ}{dt} &= ag_s \frac{PZ}{P + K_Z} - m_2 Z - g_w \frac{ZF}{Z + K_F} + \alpha_2 \eta_2(t), \\ \frac{dF}{dt} &= g_w \frac{ZF}{Z + K_F} - m_3 F + \alpha_3 \eta_3(t), \end{aligned} \quad (17)$$

where the perturbed terms $\eta_1(t), \eta_2(t)$ and $\eta_3(t)$ are assumed to be the independent Gaussian white noise satisfying the conditions:

$$\langle \eta_j(t) \rangle = 0 \quad \text{and} \quad \langle \eta_j(t_1) \eta_j(t_2) \rangle = \epsilon_{ij} \delta(t_1 - t_2)$$

for $j = 1, 2, 3$.

Here $\epsilon_{ij} > 0 (j = 1, 2, 3)$ are the intensities or strengths of the random perturbations (Kronecker symbol), δ is the Dirac delta function defined by

$$\begin{cases} \delta(x) = 0, & \text{for } x \neq 0, \\ \int_{-\infty}^{\infty} \delta(x) dx = 1, \end{cases}$$

$\langle \cdot \rangle$ represents the ensemble average of the stochastic process and $\alpha_i (i = 1, 2, 3)$ are real constants. It is noted that $\eta_i(t)$ are not defined in the ordinary sense. It can be proved that $\eta_i(t)$ are the derivatives of the Wiener process $W_i(t)$ in the generalized functions sense. To study the behavior of the system (17) about the steady state E^* , let us substitute $P' = \ln P$, $Z' = \ln Z$, $F' = \ln F$; $P = u + P^*$, $Z = v + Z^*$ and $F = w + F^*$. Then the system (17) reduces to the following linear stochastic differential equations in terms of deviation variables (u, v, w):

$$\begin{aligned} \frac{du}{dt} &= \alpha_{11} P^* u + \alpha_{12} P^* v + \alpha_{13} P^* w + \alpha_1 \eta_1(t), \\ \frac{dv}{dt} &= \alpha_{21} Z^* u + \alpha_{22} Z^* v + \alpha_{23} Z^* w + \alpha_2 \eta_2(t), \\ \frac{dw}{dt} &= \alpha_{31} F^* u + \alpha_{32} F^* v + \alpha_{33} F^* w + \alpha_3 \eta_3(t), \end{aligned} \quad (18)$$

where

$$\begin{aligned} \alpha_{11} &= -m_1 \frac{P^*}{K_P} + g_s \frac{P^* Z^*}{(P^* + K_Z)^2}, \quad \alpha_{12} = g_s \frac{1}{P^* + K_Z}, \quad \alpha_{13} = 0, \\ \alpha_{21} &= ag_s \frac{K_Z}{(Z^* + K_Z)^2}, \quad \alpha_{22} = g_w \frac{F^*}{(Z^* + K_F)^2}, \quad \alpha_{23} = g_w \frac{1}{Z^* + K_F}, \\ \alpha_{31} &= 0, \quad \alpha_{32} = g_w \frac{K_F}{(Z^* + K_F)^2}, \quad \alpha_{33} = 0. \end{aligned} \quad (19)$$

The solutions ($u(t), v(t), w(t)$) of system (18) subject to known initial values ($u(t_0), v(t_0), w(t_0)$) determine the statistical behavior of the model system (17) near the steady state E^* at time $t > t_0$.

Taking the Fourier transform on both sides of (18), we get,

$$\begin{aligned} \alpha_1 \tilde{\eta}_1(\omega) &= (i\omega + \alpha_{11} P^*) \tilde{u}(\omega) + \alpha_{12} P^* \tilde{v}(\omega) + \alpha_{13} P^* \tilde{w}(\omega), \\ \alpha_2 \tilde{\eta}_2(\omega) &= \alpha_{21} Z^* \tilde{u}(\omega) + (i\omega + \alpha_{22} Z^*) \tilde{v}(\omega) + \alpha_{23} Z^* \tilde{w}(\omega), \\ \alpha_3 \tilde{\eta}_3(\omega) &= \alpha_{31} F^* \tilde{u}(\omega) + \alpha_{32} F^* \tilde{v}(\omega) + (i\omega + \alpha_{33} F^*) \tilde{w}(\omega). \end{aligned} \quad (20)$$

The matrix of (20) is

$$M(\omega) \tilde{U}(\omega) = \tilde{\eta}(\omega), \quad (21)$$

where

$$\begin{aligned} M(\omega) &= \begin{pmatrix} i\omega + \alpha_{11} P^* & \alpha_{12} P^* & \alpha_{13} P^* \\ \alpha_{21} Z^* & i\omega + \alpha_{22} Z^* & \alpha_{23} Z^* \\ \alpha_{31} F^* & \alpha_{32} F^* & i\omega + \alpha_{33} F^* \end{pmatrix}, \\ \tilde{U}(\omega) &= \begin{pmatrix} \tilde{u}(\omega) \\ \tilde{v}(\omega) \\ \tilde{w}(\omega) \end{pmatrix}, \quad \tilde{\eta}(\omega) = \begin{pmatrix} \tilde{\eta}_1(\omega) \\ \tilde{\eta}_2(\omega) \\ \tilde{\eta}_3(\omega) \end{pmatrix}. \end{aligned}$$

Equation (21) can also be written as $\tilde{U}(\omega) = [M(\omega)]^{-1} \tilde{\eta}(\omega)$. Let $[M(\omega)]^{-1} = K(\omega)$, therefore

$$\tilde{U}(\omega) = K(\omega)\tilde{\eta}(\omega), \quad (22)$$

where

$$K(\omega) = \frac{\text{Adj}(M(\omega))}{|M(\omega)|}. \quad (23)$$

If the function $Y(t)$ has a zero mean value, then the fluctuation intensity (variance) of it in the frequency interval $[\omega, \omega + d\omega]$ is $S_Y(\omega)d\omega$. Where $S_Y(\omega)$ is spectral density of Y and is defined as

$$S_Y(\omega) = \lim_{\tilde{T} \rightarrow \infty} \frac{|\tilde{Y}(\omega)|^2}{\tilde{T}}. \quad (24)$$

The power spectrum of a time series describes how the variance of the data is distributed over the frequency components into which the data may be decomposed. Intuitively, the spectrum decomposes the content of a signal or of a stochastic process into the different frequencies present in that process and helps to identify periodicities. If Y has a zero mean value, the inverse transform of $S_Y(\omega)$ is the auto covariance function

$$C_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) e^{i\omega\tau} d\omega. \quad (25)$$

The corresponding variance of fluctuations in $Y(t)$ is given by

$$\sigma_Y^2 = C_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) d\omega \quad (26)$$

and the auto correlation function is the normalized auto covariance

$$P_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)}. \quad (27)$$

For a Gaussian white noise process, it is

$$\begin{aligned} S_{\eta_i\eta_j}(\omega) &= \lim_{\tilde{T} \rightarrow +\infty} \frac{\langle \tilde{\eta}_i(\omega)\tilde{\eta}_j(\omega) \rangle}{\tilde{T}} \\ &= \lim_{\tilde{T} \rightarrow +\infty} \frac{1}{\tilde{T}} \int_{-\frac{\tilde{T}}{2}}^{\frac{\tilde{T}}{2}} \int_{-\frac{\tilde{T}}{2}}^{\frac{\tilde{T}}{2}} \langle \tilde{\eta}_i(t)\tilde{\eta}_j(t') \rangle e^{-i\omega(t-t')} dt dt' = \delta_{ij}. \end{aligned} \quad (28)$$

From (22), we have

$$\tilde{U}(\omega) = \sum_{j=1}^2 K_{ij}(\omega)\tilde{\eta}_j(\omega), \quad i = 1, 2, 3. \quad (29)$$

And from (24), we have

$$S_U(\omega) = \sum_{j=1}^2 \xi_j |K_{ij}(\omega)|^2, \quad i = 1, 2, 3. \quad (30)$$

Intensities of fluctuations or variance measures how far a set of numbers is spread out. A variance of zero indicates

that all the values are identical. A small variance indicates that the data points tend to be very close to the mean (expected value, here the equilibrium density of prey and predator population) and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other. Hence by (26) and (30), the intensities of fluctuations in the variable U are given by

$$\sigma_U^2 = \frac{1}{2\pi} \sum_{j=1}^2 \int_{-\infty}^{\infty} \alpha_j |K_{ij}(\omega)|^2 d\omega, \quad i = 1, 2, 3. \quad (31)$$

and by (23), we obtain

$$\begin{aligned} \sigma_u^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \alpha_1 \left| \frac{A_1}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \left| \frac{B_1}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_3 \left| \frac{C_1}{|M(\omega)|} \right|^2 d\omega \right\}, \\ \sigma_v^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \alpha_1 \left| \frac{A_2}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \left| \frac{B_2}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_3 \left| \frac{C_2}{|M(\omega)|} \right|^2 d\omega \right\}, \\ \sigma_w^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \alpha_1 \left| \frac{A_3}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \left| \frac{B_3}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_3 \left| \frac{C_3}{|M(\omega)|} \right|^2 d\omega \right\} \end{aligned} \quad (32)$$

where $|M(\omega)| = |R(\omega) + i|l(\omega)|$ and real part of $|M(\omega)|$ is

$$\begin{aligned} R^2(\omega) &= -\alpha_{33}\omega^2 F^* - \alpha_{22}\omega^2 Z^* - \alpha_{11}P^* + (\alpha_{11}\alpha_{22}\alpha_{33} \\ &\quad + \alpha_{11}\alpha_{32}\alpha_{23} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{31}\alpha_{23} \\ &\quad + \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{13}\alpha_{31}\alpha_{22})P^*Z^*F^*. \end{aligned} \quad (33)$$

Imaginary part of $|M(\omega)|$ is

$$\begin{aligned} l^2(\omega) &= -\omega^3 + \omega(\alpha_{22}\alpha_{33} + \alpha_{32}\alpha_{23})Z^*F^* \\ &\quad + \omega(\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31})P^*F^* + \omega(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})P^*Z^*, \end{aligned} \quad (34)$$

and

$$\begin{aligned} |A_1|^2 &= X_1^2 + Y_1^2, \quad |B_1|^2 = X_2^2 + Y_2^2, \quad |C_1|^2 = X_3^2 + Y_3^2, \\ |A_2|^2 &= X_4^2 + Y_4^2, \quad |B_2|^2 = X_5^2 + Y_5^2, \quad |C_2|^2 = X_6^2 + Y_6^2, \\ |A_3|^2 &= X_7^2 + Y_7^2, \quad |B_3|^2 = X_8^2 + Y_8^2, \quad |C_3|^2 = X_9^2 + Y_9^2, \end{aligned}$$

where,

$$\begin{aligned} X_1^2 &= -\omega^2 + \alpha_{32}\alpha_{23}Z^*F^*, \quad Y_1^2 = \omega\alpha_{33}F^* + \omega\alpha_{22}Z^*, \\ X_2^2 &= -(\alpha_{12}\alpha_{33} + \alpha_{13}\alpha_{32})P^*F^*, \quad Y_2^2 = -\omega\alpha_{12}P^*, \\ X_3^2 &= (\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22})P^*Z^*, \quad Y_3^2 = -\omega\alpha_{13}P^*, \\ X_4^2 &= -(\alpha_{21}\alpha_{33} + \alpha_{23}\alpha_{31})Z^*F^*, \quad Y_4^2 = -\omega\alpha_{21}Z^*, \\ X_5^2 &= -\omega^2 + (\alpha_{11}\alpha_{33} + \alpha_{31}\alpha_{13})P^*F^*, \quad Y_5^2 = \omega(\alpha_{33}F^* + \alpha_{11}P^*), \\ X_6^2 &= (\alpha_{21}\alpha_{13} - \alpha_{11}\alpha_{23})P^*Z^*, \quad Y_6^2 = -\omega\alpha_{23}Z^*, \\ X_7^2 &= (\alpha_{31}\alpha_{22} - \alpha_{21}\alpha_{32})Z^*F^*, \quad Y_7^2 = \omega\alpha_{31}F^*, \\ X_8^2 &= (\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31})P^*F^*, \quad Y_8^2 = \omega\alpha_{32}F^*, \\ X_9^2 &= -\omega^2 + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})P^*Z^*, \quad Y_9^2 = \omega(\alpha_{22}Z^* + \alpha_{11}P^*). \end{aligned}$$

Thus (32) becomes

$$\begin{aligned}\sigma_u^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} [\alpha_1(X_1^2 + Y_1^2) \right. \\ &\quad \left. + \alpha_2(X_2^2 + Y_2^2) + \alpha_3(X_3^2 + Y_3^2)] d\omega \right\}, \\ \sigma_v^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} [\alpha_1(X_4^2 + Y_4^2) \right. \\ &\quad \left. + \alpha_2(X_5^2 + Y_5^2) + \alpha_3(X_6^2 + Y_6^2)] d\omega \right\}, \\ \sigma_w^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} [\alpha_1(X_7^2 + Y_7^2) \right. \\ &\quad \left. + \alpha_2(X_8^2 + Y_8^2) + \alpha_3(X_9^2 + Y_9^2)] d\omega \right\}.\end{aligned}\quad (35)$$

If we are interested in the dynamics of system (17) with either $\alpha_1 = 0$ or $\alpha_2 = 0$, then the population variances are:

If $\alpha_1 = 0$, then

$$\begin{aligned}\sigma_u^2 &= \frac{\alpha_2(X_2^2 + Y_2^2)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega, \\ \sigma_v^2 &= \frac{\alpha_2(X_4^2 + Y_4^2)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega.\end{aligned}\quad (36)$$

If $\alpha_2 = 0$, then

$$\begin{aligned}\sigma_u^2 &= \frac{\alpha_1(X_1^2 + Y_1^2)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega, \\ \sigma_v^2 &= \frac{\alpha_1(X_3^2 + Y_3^2)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega.\end{aligned}\quad (37)$$

Finally we conclude that Fourier transform method which has been used to study the effects of stochasticity on the positive equilibrium of our model leading to chaos in a realistic ecological situation.

Stochastic scenario of delayed system

The prey–predator system (1) with delay in a random environment will be considered as:

$$\begin{aligned}\frac{dP}{dt} &= m_1 P \left(1 - \frac{P}{K_P}\right) - g_s \frac{PZ}{P + K_Z} + \alpha_1 \eta_1(t), \\ \frac{dZ}{dt} &= ag_s \frac{P(t-\tau)Z}{P(t-\tau) + K_Z} - m_2 Z - g_w \frac{ZF}{Z + K_F} + \alpha_2 \eta_2(t), \\ \frac{dF}{dt} &= g_w \frac{Z(t-\tau)F}{Z(t-\tau) + K_F} - m_3 F + \alpha_3 \eta_3(t).\end{aligned}\quad (38)$$

Again using the transformations: $P' = \ln P$, $Z' = \ln Z$, $F' = \ln F$; $P = u + P^*$, $Z = v + Z^*$, $F = w + F^*$ and

assuming the delay to be very small, the system (38) (to a first approximation) can be written as

$$\begin{aligned}\frac{du}{dt} &= a_{11}u(t) + a_{12}v(t) + \alpha_1 \eta_1(t), \\ \frac{dv}{dt} &= a_{22}v(t) + a_{23}w(t) + b_{21}u(t-\tau) + \alpha_2 \eta_2(t), \\ \frac{dw}{dt} &= b_{32}v(t-\tau) + \alpha_3 \eta_3(t)\end{aligned}\quad (39)$$

$$\begin{aligned}\text{where, } a_{11} &= -m_1 \frac{P^*}{K_P} + g_s \frac{P^* Z^*}{(P^* + K_Z)^2}, \quad a_{12} = g_s \frac{P^*}{P^* + K_Z}, \\ a_{22} &= g_w \frac{Z^* F^*}{(Z^* + K_F)^2}, \quad a_{23} = g_w \frac{Z^*}{Z^* + K_F}, \quad b_{21} = ag_s \frac{K_Z Z^*}{(Z^* + K_Z)^2}, \quad b_{32} = \\ &g_w \frac{F^* K_F}{(Z^* + K_F)^2}.\end{aligned}$$

Taking Fourier transform of both sides of each of the equation in (39), we obtain

$$\begin{aligned}\alpha_1 \tilde{\eta}_1(\omega) &= (i\omega + a_{11})\tilde{u}(\omega) + a_{12}\tilde{v}(\omega), \\ \alpha_2 \tilde{\eta}_2(\omega) &= b_{21}\tilde{u}(\omega)e^{-i\omega\tau} + (i\omega + a_{22})\tilde{v}(\omega) + a_{23}\tilde{w}(\omega), \\ \alpha_3 \tilde{\eta}_3(\omega) &= b_{32}\tilde{v}(\omega)e^{-i\omega\tau} + i\omega\tilde{w},\end{aligned}\quad (40)$$

where $\tilde{n}(\omega) = \int_{-\infty}^{+\infty} n(t)e^{-i\omega t} dt$.

The system of Eq. (40) can be written in matrix form as

$$AX = B, \quad (41)$$

where

$$\begin{aligned}A &= \begin{pmatrix} i\omega + a_{11} & a_{12} & 0 \\ b_{21}e^{-i\omega\tau} & i\omega + a_{22} & a_{23} \\ 0 & b_{32}e^{-i\omega\tau} & i\omega \end{pmatrix}, \\ X &= \begin{pmatrix} \tilde{u}(\omega) \\ \tilde{v}(\omega) \\ \tilde{w}(\omega) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha_1 \tilde{\eta}_1 \\ \alpha_2 \tilde{\eta}_2 \\ \alpha_3 \tilde{\eta}_3 \end{pmatrix}.\end{aligned}$$

Now assuming A^{-1} exists, we have $A^{-1} = (A_{ij})_{3 \times 3}$, then the solution of (41) can be written as

$$\begin{aligned}\tilde{u}(s) &= \sum_{j=1}^3 A_{1j} \alpha_j \tilde{\eta}_j, \quad \tilde{v}(s) = \sum_{j=1}^3 A_{2j} \alpha_j \tilde{\eta}_j \\ \tilde{w}(s) &= \sum_{j=1}^3 A_{3j} \alpha_j \tilde{\eta}_j.\end{aligned}\quad (42)$$

Now following Samanta (1996) and Nisbet and Gurney (1982) and using Eq. (26), the spectral density of u is given by

$$\begin{aligned}S_u(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \langle u(t)u(t') \rangle \exp\{i\omega(t' - t)\} dt dt' \\ &= \sum_{j=1}^3 |A_{1j}|^2 S_{\eta_j}(\omega).\end{aligned}$$

Similarly the spectral density of v and w given by

Fig. 1 Time series (a) and phase space (b) diagram of system (1), figures show that system is asymptotically stable. Other parameters are as in the text

Fig. 2 Time series solutions of system (7), a, b show that system (7) is stable and unstable for $\tau = 0.3 < \tau_0 = 0.845$ and $\tau = 1.2 > \tau_0 = 0.845$ respectively. Other parameters are as in the text

$$S_v = \sum_{j=1}^3 |A_{2j}|^2 S_{\eta_j}(\omega) \quad S_w = \sum_{j=1}^3 |A_{3j}|^2 S_{\eta_j}(\omega).$$

Therefore the fluctuation intensity (variance) of u is given by

$$\begin{aligned} \sigma_u^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_u(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^3 |A_{1j}|^2 S_{\eta_j}(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^3 |A_{1j}|^2 d\omega, \end{aligned}$$

since $S_{\eta_j}(\omega) = 1$.

Similarly the fluctuation intensity of v and w are given by

$$\sigma_v^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^3 |A_{2j}|^2 d\omega \quad \sigma_w^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^3 |A_{3j}|^2 d\omega.$$

Numerical results

In this study, we choose a parameter set as $m_1 = 0.6, m_2 = 0.0698, m_3 = 0.324, g_s = 0.75, g_w = 0.6894, s_e = 12.30, s_r = 8.25, k_p = 12, k_z = 38, k_f = 10, a = 0.8$, initial data as $(3, 6, 5)$. By this set of parameters we get equilibrium abundance as $P^* = 1.883, Z^* = 8.892, F^* = 3.798$. By this set of parameters coexistence equilibrium point $E^*(P^*, Z^*, F^*)$ of non-delayed system (1) is asymptotically stable. We can show this dynamical aspect by time series in Fig. 1a and phase space diagram in Fig. 1b. All the solutions with initial data converges to coexistence equilibrium point $E^*(P^*, Z^*, F^*)$ which is denoted by green bullet in Fig. 1b. Now we move on to the delayed system (2) with the same parameter set. Here we can calculate $\omega_0 = 0.327$ and critical value of delay parameter $\tau_0 = 0.845$. Now we choose $\tau = 0.3$ and $\tau = 1.2$, here we observe by time series solution that system (2) is stable for $\tau = 0.3$ (Fig. 2a) and

Fig. 3 Bifurcation diagram of system (7) with respect to τ , system (7) is stable and unstable when $\tau < \tau_0 = 0.845$ and $\tau > \tau_0 = 0.845$ respectively and system experiences Hopf bifurcation when $\tau = \tau_0 = 0.845$. Other parameters are as in the text

unstable for $\tau = 1.2$ (Fig. 2b). Broadly we show the system (2) with respect to long range of τ by bifurcation diagram in Fig. 3, this figure indicates that system (2) is stable for $\tau < \tau_0 = 0.845$ and unstable for $\tau > \tau_0 = 0.845$ and system experiences Hopf bifurcation at $\tau = \tau_0 = 0.845$.

In the numerical experiments of the stochastic system, we consider the set of parameter values previously with $\alpha_1 = 0.3$, $\alpha_2 = 0.2$, $\alpha_3 = 0.27$ and approximate the solutions of the system (38) by Euler–Maruyama method.

Fig. 4 Plots a and b depict, respectively, the time evolution and corresponding phase trajectory of the delay-stochastic model system (38). For $\alpha_1 = 0.3$, $\alpha_2 = 0.2$, $\alpha_3 = 0.27$, system is stochastically stable for $\tau = 0.3$. Other parameters are as in the text

Delay-stochastic system (38) is stochastically stable for $\tau_1 = 0.3$ (see time series in Fig. 4a and phase portrait in Fig. 4b) and stochastically unstable for $\tau_2 = 1.2$ (see time series in Fig. 5a and phase portrait in Fig. 5b).

Discussion

In this present work, we consider a *PZF*-model with Holling type II functional response where photosynthesis rate of phytoplankton, salinity induced grazing rate of zooplankton and gestation delay of zooplankton and fish population are considered. In the later part of the study, we study the impact of stochastic perturbation of the environment on the system by using Gaussian white noise. Delay induced deterministic system shows that there is a critical value of delay parameter below which system is stable and above which system is unstable and when the delay parameter crosses the critical value then system experiences Hopf-bifurcation.

The main purpose of this work is to observe the effect of delay along with photosynthesis rate of phytoplankton, salinity induced grazing rate of zooplankton under fluctuating environment. Results show that these biological parameters play a significant role to change the stochastic stability of the system. To study the effect of environmental fluctuation on the non-delayed *PZF* system, we have superimposed Gaussian white noises and then studied non-equilibrium fluctuation and stability of the resulting

Fig. 5 Plots **a** and **b** depict, respectively, the time evolution and corresponding phase trajectory of the delay-stochastic model system (38). For $\alpha_1 = 0.3$, $\alpha_2 = 0.2$, $\alpha_3 = 0.27$, system is stochastically unstable for $\tau = 1.2$. Other parameters are as in the text

stochastic model by using Fourier transform technique. Also the deterministic system and the noise-induced stochastic system may be have a like with respect to stability.

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