Discussion on fuzzy quota harvesting model in fuzzy environment: fuzzy differential equation approach

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Abstract In the recent years much importance has been laid on the role of uncertainty (fuzzy, interval etc.) in mathematical biology. In this paper we tried to study on the quota harvesting model in fuzzy environment. This model is considered in three different ways viz. (1) Initial condition (population density) is a fuzzy number, (2) coefficients of quota harvesting model (intrinsic growth rate and quota harvesting rate) are fuzzy number and (3) both initial condition and coefficients are fuzzy number. We discuss all these fuzzy cases individually. The solution procedure is done by using the concept of fuzzy differential equation approach. We have discussed the equilibrium points and their feasibility in all the three cases. This paper explores the stability analysis of the quota harvesting model at the equilibrium points in fuzzy environment. In order to examine the stability systematically in different fuzzy cases, we have used numerical simulations and discussed them briefly.

Keywords Generalized Hukuhara (gH) derivative · Quota harvesting · Fuzzy differential equation · Equilibrium point · Fuzzy stability analysis

Introduction

Introductory concept on mathematical biology

Mathematical models of population growth have been constructed to provide some significant aspect of true ecological situation. In this paper, we put some models where the parameters of the biological growth model systematically change over time. Verhulst (1838) first proposed three-parameter model for the growth of single-species populations that shows a logistic sigmoid growth curve for time. Pearl and Reed (1920) independently published the same model, which includes mathematically an upper asymptote and a rate constant. Population growth for a single-species population can be modelled in (Lotka 1925; Volterra 1926; Malthus 1959) a variety of ways. Populations will grow exponentially if the per capita reproductive rate is constant and independent of population size. Simple difference equations used to model populations with discrete generations can generate complex behaviors including converging on an equilibrium population size, limit cycles, and chaos. The most common model used to estimate growth in populations with overlapping generations is the Verhulst–Pearl logistic equation where population growth stops at the carrying capacity. The increasing study of realistic and practically useful mathematical models in population biology (see Verhulst 1838; Murray 1990), whether we are dealing with a human population with or without its age distribution, population of an endangered species, bacterial or viral growth and so on, is a reflection of their use in helping to understand the dynamic process involved and in making practical prediction.

Fisheries are classified as renewable resources as the organisms of interest usually produce an annual biological surplus that, with Proper management, can be harvested

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without reducing future productivity. The overall goal of fisheries management is to produce sustainable biological, social, and economic benefits. Fisheries provide a key source of protein, micronutrients, essential fatty acids and minerals. People in coastal countries depend on healthy fisheries for their livelihoods. But with increase demand of fish for world populations, destruction of fish habitats, increasing of fisheries industries, growing of aquaculture fisheries and most importantly the use of modern fishing technology and environmental fluctuations cause the reduction in productivity of world fishery.

In general, in our ecosystem, harvesting is a very frequently used process to exploit biological resources for the necessity of human beings and the society. There are different ways of harvesting have been used in the ecosystem and the most simple and common way to harvest the ecological resources is when the resource population is harvested at a constant rate and mathematically it is represented by $h(t) = h$, where h being a constant. The drawback of the constant rate harvesting is that it is independent of the density of the harvesting stock. Another important harvesting strategy is based on the catch-per-unit-effort (CPUE) hypothesis and mathematically it is written as $h(t) = qEN_1(t)$, where q is the catchability coefficient, E is the constant external effort and $N_1(t)$ is the density of the harvested species at time t . CPUE based harvesting strategy is supposed to be more realistic and productive than its constant rate harvesting counterpart regarding the cause that it is proportional to the density of the harvested stock. In this paper, we have used such a system, where the one species population is harvested following the CPUE based harvesting rate and normalizing the unit of effort by setting.

Basic need for taking imprecise parameter in biological model

Most researchers in theoretical ecology have developed their models based on the assumption that the biological parameters are precisely known. In practical sense the thought can't be often true for various reasons viz.: lack of data, lack of information, mistakes in measurement process and determining the initial conditions. So, it may be easily concluded that to overcome the limitations the models with imprecise parameters are more realistic and helpful. We can work with imprecise parameters in various ways such as fuzzy approach, stochastic approach, fuzzy stochastic approach etc. The imprecise parameters with known membership functions are replaced by fuzzy sets in fuzzy approach. The imprecise parameters with known probability distribution are replaced by random variables in stochastic approach. In fuzzy stochastic approach some parameters are fuzzy in nature and rests are taken as

random variables. However, it is very stiff to construct a suitable probability distribution or a suitable membership function with suitable probability distribution for each of the imprecise biological parameters. So here we consider the fuzzy environment.

Fuzzy differential equation

The topic “fuzzy differential equation” (FDE) has been speedily developing in recent years. The appliance of fuzzy differential equations is an inherent way to model dynamic systems under possibilistic uncertainty (Zadeh (2005)). The concept of the fuzzy derivative was first initiated by Chang and Zadeh (1972). It was followed up by Dubois and Prade (1982). Other methods have been smeared by Puri and Ralescu (1983) and Goetschel and Voxman (1986). The concept of differential equations in a fuzzy environment was first formulated by Kalev (1987). In fuzzy differential equation all derivative is deliberated as either Hukuhara or generalized derivatives. The Hukuhara differentiability has a deficiency (see Bede and Gal 2005; Diamond and Kloeden 1994). The solution turns fuzzier as time goes by. Bede (2006) exhibited that a large class of BVPs has no solution if the Hukuhara derivative was applied. To exceed this difficulty, the concept of a generalized derivative was developed (Chalco-Cano et al. 2008) and fuzzy differential equations were smeared using this concept (Bede et al. 2007; Chalco-Cano et al. 2007, 2008). Khastan and Nieto (2010) set up the solutions for a large enough class of boundary value problems using the generalized derivative. Obviously the disadvantage of strongly generalized differentiability of a function in comparison H-differentiability is that, a fuzzy differential equation has no unique solution (see Bede and Gal (2005)). Recently Stefanini (2008) by the concept of generalization of the Hukuhara difference for compact convex set, introduced generalized Hukuhara differentiability (see Stefanini and Bede 2009) for fuzzy valued function and they displayed that, this concept of differentiability have relationships with weakly generalized differentiability and strongly generalized differentiability.

There are many approaches for solving FDE. Some researchers transform the FDE into equivalent fuzzy integral equation and then solve this (Allahviranloo et al. 2011; Chen et al. 2008; O'Regan et al. 2003). Another one is Zadeh extension principle method. In this method first solve the associated ODE and lastly fuzzify the solution and check whether it is satisfied or not. For details see Buckley and Feuring (2000). In the third approach, the fuzzy problem is converted to a crisp problem. Hüllermeier (1997), uses the concept of differential inclusion. In this way, by taking an α -cut of the initial value and the solution, the given differential equation is converted to a differential

inclusion and the solution is accepted as the α -cut of the fuzzy solution. Laplace transform method is use many where in linear FDE (Allahviranloo and Ahmadi 2010; Tolouti et al. 2010). Recently, Mondal and Roy (2013) solve the first order Linear FDE by Lagrange multiplier method. Using generalized Hukuhara differentiability concept we transform the given FDE into two ODEs. And this ODEs also an differential equation involving the parametric form of a fuzzy number.

Work done using fuzzy differential equation on bio mathematical problem

Barros et al. (2000) take fuzzy population dynamics model with fuzzy initial value and solve the problem. Akin and Oruc (2012) consider a prey predator model with fuzzy initial value. Here they solve the equation by generalized derivative of fuzzy function concept. Zarei et al. (2012) give a formulation on fuzzy HIV modeling. Diniz et al. (2001) consider a fuzzy Cauchy problem and apply in the behavior of decay of the biochemical oxygen demand in water. Nounou et al. (2012) discuss fuzzy intervention in biological phenomena. Fuzzy delay predator–prey system and their existence theorem and oscillation property of solution is nicely delivered by Barzinji (2014). Symptomatic HIV virus infected population with fuzzy concept is done by Jafelice et al. (2004). The interaction of predator prey with uncertain initial population sizes was considered in Omar and Hasan (2011). Predator–prey fuzzy model is solved by Peixoto et al. (2008) by fuzzy rule base method. Ahmad and Hasan (2012) solve a biological population model by numerical method, Euler’s method. Here system of fuzzy differential equation is solved. Ahmad and Baets (2009), solve the predator–prey model with fuzzy initial populations by Runge–Kutta method. Optimal control of HIV infection by using fuzzy dynamical systems is discussed by author Najariyan et al. (2011). Mann et al. (2013) use delay differential equation in predator–prey interaction and analysis on the stability of steady state. Pal and Mahapatra (2014) solve a bio-economic modeling of two-prey and one-predator fishery model with optimal harvesting policy through hybridization approach. They solve the differential equation in interval and fuzzy environment whereas Pal and Mahapatra (2014) take optimal harvesting of prey–predator system with interval biological parameters. Pandit and Sing (2014) solve Prey predator model with fuzzy initial conditions. Quota harvesting model for a single species population under fuzziness is discussed in Pal et al. (2013a). Tapaswini and Chakraverty (2013) solved the fuzzy arbitrary order predator–prey equations by Homotopy perturbation method. A quota harvesting dynamical model with fuzzy intrinsic growth rate and

harvesting quantity is taken by Pal et al. (2013b) but the not use the fuzzy differential equation approach.

Novelties

Although some works have been done in this field but we carried forward the work with some new inputs and concept which makes the model more acceptable and reliable. The new inclusions are described below:-

1. The quota harvesting model is solved in fuzzy environments.
2. The possible all cases are addressed.
3. The model is solved with fuzzy differential equation approach.
4. The fuzzy stability concept is addressed here.
5. The numerical results are taken on each case for study the behavior and stability analysis in fuzzy environment.

Moreover, we can say all these developments can help the researchers who engage with uncertainty modeling, differential equation and mathematical biology. One can model and stability analyze on any biological model with uncertainty and differential equation by same approach.

Preliminaries

Definition 1 Fuzzy Set: A fuzzy set $\tilde{A}$ is defined by a pair $(U, \mu_{\tilde{A}}(x))$, where U be a nonempty universal set and $\mu_{\tilde{A}}(x) : U \rightarrow [0, 1]$

For each $x \in U$, $\mu_{\tilde{A}}(x)$ is called the grade of membership of x in $\tilde{A}$.

Definition 2 α -cut of a fuzzy set: An α -cut of the fuzzy set $\tilde{A}$ of U is a crisp set A_{α} that contains all the elements of U that have membership values in $\tilde{A}$ greater than or equal to α i.e. $\tilde{A} = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in U, 0 < \alpha \leq 1\}$.

Definition 3 Fuzzy Number: The basic definition of fuzzy number is (Hüllermeier 1997): If we denote the set of all real numbers by $\mathcal{R}$ and the set of all fuzzy numbers on $\mathcal{R}$ is indicated by $\mathcal{R}_{\mathcal{F}}$ then a fuzzy number is a mapping such that $u : \mathcal{R} \rightarrow [0, 1]$, which satisfies the following four properties

1. u is upper semi continuous.
2. u is a fuzzy convex i.e., $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathcal{R}, \lambda \in [0, 1]$.
3. u is normal, i.e., $\exists x_0 \in \mathcal{R}$ for which $u(x_0) = 1$.
4. $\text{Supp } u = \{x \in \mathcal{R} | u(x) > 0\}$ is support of u and the closure of $(\text{supp } u)$ is compact.

Definition 4 Parametric form of fuzzy number: (Alahviranloo and Ahmadi 2010) A fuzzy number is represented by an ordered pair of functions $(u_l(\alpha), u_r(\alpha))$, $0 \leq \alpha \leq 1$, that satisfy the following condition:

1. $u_l(\alpha)$ is a bounded left continuous non decreasing function for any $\alpha \in [0, 1]$.
2. $u_r(\alpha)$ is a bounded left continuous non increasing function for any $\alpha \in [0, 1]$.
3. $u_l(\alpha) \leq u_r(\alpha)$ for any $\alpha \in [0, 1]$.

Note: If $u_l(\alpha) = u_r(\alpha) = \alpha$, then α is a crisp number.

Definition 5 Triangular Fuzzy Number: A Triangular fuzzy number (TFN) represented by three points like as $\tilde{A} = (a, b, c)$ and this representation is interpreted as membership function as below

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x = b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x \geq c \end{cases}$$

Definition 6 α -cut of a fuzzy set $\tilde{A}$: The α -cut of $\tilde{A} = (a, b, c)$ is given by

$$A_\alpha = [a + \alpha(b-a), c - \alpha(c-b)], \forall \alpha \in [0, 1]$$

Definition 7 Let $u, v \in E^1$. If there exists $z \in E^1$ such that $u = y + v$, then z is called the Hukuhara-difference of fuzzy numbers u and v , and it denoted by $z = u \ominus v$. Remark that $u \ominus v \neq u + (-1)v$.

Definition 8 Let $f : [a, b] \rightarrow E^1$ and $t_0 \in [a, b]$. We say that f is Hukuhara differential at t_0 , if there exist an element $f'(t_0) \in E^1$ such that for all $h > 0$ sufficiently small, there exists $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ and the limits exists in metric D .

$$\lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0)$$

Definition 9 Let $f : (a, b) \rightarrow E$ and $x_0 \in (a, b)$. We say that f is strongly generalized differential at x_0 (Bede-Gal differential) if there exists an element $f'(x_0) \in E$, such that

- (i) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) -^h f(x_0)$ and $f(x_0) -^h f(x_0 - h)$ and the limits exist in the metric D

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) -^h f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0) -^h f(x_0 - h)}{h} = f'(x_0)$$

Or

- (ii) for all $h > 0$ sufficiently small, there exist $f(x_0) -^h f(x_0 + h)$ and $f(x_0 - h) -^h f(x_0)$ and the limits exist in the metric D

$$\lim_{h \rightarrow 0} \frac{f(x_0) -^h f(x_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) -^h f(x_0)}{-h} = f'(x_0)$$

Or

- (iii) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) -^h f(x_0)$, and $f(x_0 - h) -^h f(x_0)$ and the limits exist in the metric D

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) -^h f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) -^h f(x_0)}{-h} = f'(x_0)$$

Or

- (iv) for all $h > 0$ sufficiently small, there exist $f(x_0) -^h f(x_0 + h)$ and $f(x_0) -^h f(x_0 - h)$ and the limits exists in the metric D

$$\lim_{h \searrow 0} \frac{f(x_0) -^h f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0) -^h f(x_0 - h)}{h} = f'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively).

Definition 10 Generalized Hukuhara difference: (Bede and Gal 2005) The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathfrak{R}_{\mathcal{F}}$ is defined as follows

$$u \ominus_g v = w \Leftrightarrow \begin{cases} (i) u = v \oplus w \\ \text{or} (ii) v = u \oplus (-1)w \end{cases}$$

Consider $[w]_\alpha = [w_1(\alpha), w_2(\alpha)]$, then $w_1(\alpha) = \min\{u_1(\alpha) - v_1(\alpha), u_2(\alpha) - v_2(\alpha)\}$ and $w_2(\alpha) = \max\{u_1(\alpha) - v_1(\alpha), u_2(\alpha) - v_2(\alpha)\}$. Here the parametric representation of a fuzzy valued function $f : [a, b] \rightarrow \mathfrak{R}_{\mathcal{F}}$ is expressed by $[f(t)]_\alpha = [f_1(t, \alpha), f_2(t, \alpha)]$, $t \in [a, b]$, $\alpha \in [0, 1]$.

Definition 11 Generalized Hukuhara derivative on a Fuzzy function: (Bede and Gal 2005) The generalized Hukuhara derivative of a fuzzy valued function $f : (a, b) \rightarrow \mathfrak{R}_{\mathcal{F}}$ at t_0 is defined as

$$f'(t_0) = \lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus_g f(t_0)}{h} \quad (1)$$

If $f'(t_0) \in \mathfrak{R}_{\mathcal{F}}$ satisfying (1) exists, we say that f is generalized Hukuhara differentiable at t_0 . Also we say that $f(t)$ is (i)-gH differentiable at t_0 if

$$[f'(t_0)]_\alpha = [f'_1(t_0, \alpha), f'_2(t_0, \alpha)] \quad (2)$$

and $f(t)$ is (i)-gH differentiable at t_0 if

$$[f'(t_0)]_\alpha = [f'_2(t_0, \alpha), f'_1(t_0, \alpha)] \quad (3)$$

Model formulation

In most cases, the main aspiration of harvesting a natural population is not only population control, but also to yield a substantial harvest from the population. Some common questions which may arise in such cases are: What harvesting strategy one shall implement to provide maximum long term yield? What is the maximum allowable harvesting that will still retain the population? What are the stable equilibrium sizes of the population under harvesting? In this paper, we investigate these questions using the logistic model. Here we extend the work of Pal et al. (2013a, b). We have developed the fuzzy solution of the quota harvesting model in fuzzy environment with the use of generalized Hukuhara differentiability. The prevalent form of crisp quota harvesting model is given by,

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{k} \right) - f(N)n_2 \quad (4)$$

where r, N and k are the intrinsic growth rate, population density and environmental carrying capacity of the resource population (or prey population) respectively. The harvester or consumer of population density is defined by n_2 . Here we consider n_2 as a parameter ignoring entirely the population dynamics of the harvester. $f(N)$ is the per capita harvest rate of N by per individuals of n_2 .

Mathematical models of population growth have been formed to provide an inconceivable significant angle of true ecological situation. The form is similar to the Lotka–Volterra equations for predation in that the equation for each species has one term for self-interaction and one term for the interaction with other species. In the equations for predation, the base population model is exponential. For the competition equations, the logistic equation is the basis. Let us consider that a constant amount or quota of N is harvested by all individuals of n_2 per unit time i.e., $f(N)n_2 = h = \text{constant}$. So the model Eq. (4) becomes crisp quota harvesting model as follows where the meaning of each parameter in the models has been defined biologically,

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{k} \right) - h \quad (5)$$

Population sizes that are less than k , the population will increase in size and at population sizes that are greater than k the population size will decline but at k itself the population neither increases nor decreases. The carrying capacity is therefore a stable equilibrium for the population, and the

model exhibits the regulatory properties classically characteristic of intraspecific competition. For the continuous time model, birth and death are continuous. The net rate of such a population will be denoted by $\frac{dN}{dt}$. This represents the ‘speed’ at which a population increases in size, N , as time, t , progresses. It describes a sigmoidal growth curve approaching a stable carrying capacity.

In the above quota harvesting model Eq. (5), all the parameters are known and have definite values without uncertainty. But in real life situation, it is not so precise. So here we formulate the fuzzy quota harvesting model. Due to imprecise nature of data, here we consider the intrinsic growth rate $\tilde{r}$ is fuzzy in nature and also consider that the fuzziness nature of N is harvested by all individuals of n_2 per unit time i.e., $f(N)n_2 = \tilde{h}$. So the crisp quota harvesting model is converted to a fuzzy quota harvesting model in fuzzy environment as given below,

$$\frac{d\tilde{N}}{dt} = \tilde{r}\tilde{N} \left(1 - \frac{\tilde{N}}{k} \right) - \tilde{h} \quad (6)$$

There arise three possible cases

Case 1 Population density at initially is fuzzy number

Case 2 Intrinsic growth rate and quota harvesting are fuzzy number

Case 3 Population density, intrinsic growth rate and quota harvesting are all fuzzy number

Now we discuss the above cases in different section.

Solution of the model when population density at initially is fuzzy number

In this case $\tilde{N}(0)$ is fuzzy number. Now here two sub cases arise.

When $\tilde{N}(t)$ is (i)-gH differentiable

The model (6) reduces to

$$\begin{aligned} \frac{dN_l(t, \alpha)}{dt} &= rN_l(t, \alpha) \left(1 - \frac{N_r(t, \alpha)}{k} \right) - h \\ \frac{dN_r(t, \alpha)}{dt} &= rN_r(t, \alpha) \left(1 - \frac{N_l(t, \alpha)}{k} \right) - h \end{aligned} \quad (7)$$

With initial condition $N_l(0, \alpha) = N_{l0}(\alpha), N_r(0, \alpha) = N_{r0}(\alpha)$, with the solution $(\tilde{N}(t))_\alpha = [N_l(t, \alpha), N_r(t, \alpha)]$.

(A) Equilibrium points and their feasibility

Here Trivial Equilibrium case does not exist. System (7) possesses only coexistence equilibrium point is

$$E_{1l}^*(N_{1l}^*, N_{1r}^*), \text{ where } N_{1l}^* = \frac{k}{2} \pm \frac{1}{2} \sqrt{\frac{k(kr-4h)}{r}}, \text{ and } N_{1r}^* = \frac{k}{2} \pm \frac{1}{2} \sqrt{\frac{k(kr-4h)}{r}}.$$

Here the coexistence equilibrium point $E_1^*(N_{1l}^*, N_{1r}^*)$ is feasible if,

- (i) $k > \frac{4h}{r}$,
- (ii) $k > \sqrt{\frac{k(kr-4h)}{r}}$.

(B) Stability analysis

The variational matrix of system (7) is

$$V_1 = \begin{pmatrix} r\left(1 - \frac{N_r}{k}\right) & \frac{-rN_l}{k} \\ \frac{-rN_r}{k} & r\left(1 - \frac{N_l}{k}\right) \end{pmatrix}$$

Now the variational matrix at coexistence equilibrium point $E_1^*(N_{1l}^*, N_{1r}^*)$ is,

$$V_1(E_1^*) = \begin{pmatrix} r\left(\frac{1}{2} \mp \frac{1}{2k} \sqrt{\frac{k(kr-4h)}{r}}\right) & -\frac{r}{k}\left(\frac{k}{2} \pm \frac{1}{2} \sqrt{\frac{k(kr-4h)}{r}}\right) \\ \frac{-r}{k}\left(\frac{k}{2} \pm \frac{1}{2} \sqrt{\frac{k(kr-4h)}{r}}\right) & r\left(\frac{1}{2} \mp \frac{1}{2k} \sqrt{\frac{k(kr-4h)}{r}}\right) \end{pmatrix}$$

$$N_{2r}^* = \frac{k}{2} \pm \frac{1}{2} \sqrt{\frac{k(kr-4h)}{r}} \text{ and } N_{2l}^* = \frac{k}{2} \pm \frac{1}{2} \sqrt{\frac{k(kr-4h)}{r}}.$$

Here the coexistence equilibrium point $E_2^*(N_{2r}^*, N_{2l}^*)$ is feasible if $k > \frac{4h}{r}$.

(B) Stability analysis

The variational matrix of system (8) is

$$V_2 = \begin{pmatrix} \frac{-rN_l}{k} & \frac{r(k-N_r)}{k} \\ \frac{r(k-N_l)}{k} & \frac{-rN_r}{k} \end{pmatrix}$$

Now the variational matrix at coexistence equilibrium point $E_2^*(N_{2l}^*, N_{2r}^*)$ is,

$$V_2(E_2^*) = \begin{pmatrix} -r\left(\frac{1}{2} \pm \frac{1}{2k} \sqrt{\frac{k(kr-4h)}{r}}\right) & -r\left(-\frac{1}{2} \pm \frac{1}{2k} \sqrt{\frac{k(kr-4h)}{r}}\right) \\ -r\left(-\frac{1}{2} \pm \frac{1}{2k} \sqrt{\frac{k(kr-4h)}{r}}\right) & -r\left(\frac{1}{2} \pm \frac{1}{2k} \sqrt{\frac{k(kr-4h)}{r}}\right) \end{pmatrix}$$

Therefore, the eigen values of $V_1(E_1^*)$ are, $\lambda_1 = r, \lambda_2 = \pm \frac{r}{k} \sqrt{\frac{k(kr-4h)}{r}}$. Now, the equilibrium point $E_1^*(N_{1l}^*, N_{1r}^*)$ is stable if $\lambda_1 < 0, \lambda_2 < 0$. But here all eigenvalues are not negative. So equilibrium point $E_1^*(N_{1l}^*, N_{1r}^*)$ is unstable.

When $\tilde{N}(t)$ is (ii)-gH differentiable

The model (6) reduces to

$$\begin{aligned} \frac{dN_r(t, \alpha)}{dt} &= rN_l(t, \alpha) \left(1 - \frac{N_r(t, \alpha)}{k}\right) - h \\ \frac{dN_l(t, \alpha)}{dt} &= rN_r(t, \alpha) \left(1 - \frac{N_l(t, \alpha)}{k}\right) - h \end{aligned} \quad (8)$$

(A) Equilibrium points and their feasibility

System (8) gives an coexistence equilibrium point is $E_2^*(N_{2l}^*, N_{2r}^*)$, where,

Therefore, the eigen values of $V_2(E_2^*)$ are, $\lambda_1 = -r, \lambda_2 = \pm \frac{r}{k} \sqrt{\frac{k(kr-4h)}{r}}$.

Now, the equilibrium point $E_2^*(N_{2l}^*, N_{2r}^*)$ is stable if $\lambda_1 < 0, \lambda_2 < 0$. So, $E_2^*(N_{2l}^*, N_{2r}^*)$ becomes locally asymptotically stable at this point $(-r, -\frac{r}{k} \sqrt{\frac{k(kr-4h)}{r}})$ and unstable at this point $(-r, \frac{r}{k} \sqrt{\frac{k(kr-4h)}{r}})$ where the equilibrium point $E_2^*(N_{2l}^*, N_{2r}^*)$ is feasible under $k > \frac{4h}{r}$.

Numerical simulations

Example 1 Now we consider some hypothetical data for quota harvesting model, $r = 0.41, h = 0.01, k = 100$.

Now we consider the Eq. (7) with the initial conditions $N(t_0) = \tilde{N}_0 = (1, 3, 5)$, i.e. $(N(t_0))_\alpha = [1 + 2\alpha, 5 - 3\alpha]$.

Fig. 1 a Crisp solution and fuzzy solution for $\alpha = 0$ and in this figure we see that the crisp solution lies between the fuzzy solutions. So application of fuzziness of coexistence equilibrium of system of

Eq. (7) is justified and hence it is acceptable. Fuzzy solution of system of Eq. (7) for **b** $\alpha = 0$, **c** $\alpha = 0.5$, **d** $\alpha = 1$

Here the coexistence equilibrium $E_1^*(N_{1l}^*, N_{1r}^*)$ is unstable. The dynamics of system (7) according to this condition is graphically presented in Fig. 1, which shows that the species $(N(t))$ is unstable i.e. the graphical result approaches the coexistence equilibrium E_1^* . This result supports our analytical result given in “When $\tilde{N}(t)$ is (i)-gH differentiable”.

Here we observe that $N_l(t, \alpha)$ increases and $N_r(t, \alpha)$ decreases with increasing α , hence the solution is a strong solution. When α increases the difference between N_l and N_r decreases and at $\alpha = 1$ they coincide with one another. So we say that the stability condition of coexistence equilibrium of system of Eq. (7) is depends on the imprecise nature of the biological parameter.

Now we consider the system (8) with the above example. Here the coexistence equilibrium $E_2^*(N_{2l}^*, N_{2r}^*)$ is asymptotically stable at this point $(-r, -\frac{r}{k} \sqrt{\frac{k(kr-4h)}{r}})$ and unstable at this point $(-r, \frac{r}{k} \sqrt{\frac{k(kr-4h)}{r}})$. The dynamics of

system (8) according to this condition is graphically presented in Fig. 2, which shows that only stability condition of E_2^* , i.e. the graphical result approaches the coexistence equilibrium E_2^* . This result supports our analytical result given in “When $\tilde{N}(t)$ is (ii)-gH differentiable”.

Here we observe that $N_l(t, \alpha)$ increases and $N_r(t, \alpha)$ decreases with increasing α , hence the solution is a strong solution. When α increases the difference between N_l and N_r decreases and at $\alpha = 1$, they coincide with one another. So based on this discussion we can conclude that the stability of this equilibrium of system of Eq. (8) depended on the fuzzy nature of the biological parameter.

Solution of the model when intrinsic growth rate and quota harvesting rate are fuzzy numbers

Here $\tilde{r}$ and $\tilde{h}$ are fuzzy numbers (i.e., coefficients are fuzzy numbers)

Fig. 2 a Crisp solution and fuzzy solution for $\alpha = 0$ and in this figure we see that the crisp solution lies between the fuzzy solutions. So application of fuzziness of coexistence equilibrium of system of

Eq. (8) is justified and hence it is acceptable. Fuzzy solution of system of Eq. (8) shows in **b** $\alpha = 0$, **c** $\alpha = 0.5$ and **d** $\alpha = 1$

When $\tilde{N}(t)$ is (i)-gH differentiable

The model (6) reduces to

$$\begin{aligned} \frac{dN_l(t, \alpha)}{dt} &= r_l(\alpha)N_l(t, \alpha) \left(1 - \frac{N_r(t, \alpha)}{k} \right) - h_r(\alpha) \\ \frac{dN_r(t, \alpha)}{dt} &= r_r(\alpha)N_r(t, \alpha) \left(1 - \frac{N_l(t, \alpha)}{k} \right) - h_l(\alpha) \end{aligned} \quad (9)$$

with initial condition $N_r(0, \alpha) = N_{r0}(\alpha)$, $N_l(0, \alpha) = N_{l0}(\alpha)$ and $(\tilde{r})_\alpha = [r_l(\alpha), r_r(\alpha)]$, $(\tilde{h})_\alpha = [h_l(\alpha), h_r(\alpha)]$ with the solution $(N(t))_\alpha = [N_l(t, \alpha), N_r(t, \alpha)]$.

(A) Equilibrium points and their feasibility

Here Trivial Equilibrium case does not exist. System (9) gives only coexistence equilibrium point $E_3^*(N_{3l}^*, N_{3r}^*)$, where,

$$N_{3l}^* = \frac{h_r r_r - h_l r_l + k r_l r_r \pm \sqrt{(h_r r_r - h_l r_l)^2 - k r_l r_r (2 h_l r_l + 2 h_r r_r - k r_l r_r)}}{2 r_l r_r}.$$

and

$$N_{3r}^* = \frac{h_l r_l - h_r r_r + k r_l r_r \pm \sqrt{(h_l r_l - h_r r_r)^2 - k r_l r_r (2 h_l r_l + 2 h_r r_r - k r_l r_r)}}{2 r_l r_r}.$$

Here the coexistence equilibrium point $E^*(N_{3l}^*, N_{3r}^*)$ is feasible if,

- $k^2 r_l^2 r_r^2 - 2k(h_l r_l^2 r_r + h_r r_r^2 r_l) + (h_l r_l - h_r r_r)^2 > 0$,
- $k > \frac{|h_r r_r - h_l r_l|}{r_l r_r}$,
- $h_r r_r - h_l r_l + k r_l r_r > \sqrt{(h_r r_r - h_l r_l)^2 - k r_l r_r (2 h_l r_l + 2 h_r r_r - k r_l r_r)}$,
- $(h_l r_l - h_r r_r) + k r_l r_r > \sqrt{(h_l r_l - h_r r_r)^2 - 2k r_l r_r (h_l r_l + h_r r_r) + k^2 r_l^2 r_r^2}$.

(B) Stability analysis

The variational matrix of system (9) is

$$V_1 = \begin{pmatrix} r_l(1 - \frac{N_r}{k}) & -\frac{r_l N_l}{k} \\ -\frac{r_r N_r}{k} & r_r(1 - \frac{N_l}{k}) \end{pmatrix}$$

Now the variational matrix at coexistence equilibrium point $E_3^*(N_l^*, N_r^*)$ is,

$$V_3(E_3^*) = \begin{pmatrix} r_l(1 - \frac{N_{3r}^*}{k}) & -\frac{r_l N_{3l}^*}{k} \\ -\frac{r_r N_{3r}^*}{k} & r_r(1 - \frac{N_{3l}^*}{k}) \end{pmatrix}$$

Therefore, the characteristic equation of $V_3(E_3^*)$ is given by

$$\lambda^2 + a_1 \lambda + a_2 = 0$$

where,

$$a_1 = \text{trace}(V_3(E_3^*)) = r_l(1 - \frac{N_{3r}^*}{k}) + r_r(1 - \frac{N_{3l}^*}{k}),$$

$$a_2 = \det(V_3(E_3^*)) = r_l r_r \left(1 - \frac{N_{3r}^*}{k}\right) \left(1 - \frac{N_{3l}^*}{k}\right) - \frac{r_l r_r N_{3r}^* N_{3l}^*}{k^2}.$$

Now, if $a_1 < 0, a_2 > 0$, then by R-H condition, the roots of the characteristic equation has a pair of real roots or complex conjugate roots with negative real parts then the equilibrium point the $E_3^*(N_{3l}^*, N_{3r}^*)$ is stable.

Here if $a_1 < 0 \Rightarrow (k - N_{3r}^*) < 0, (N_{3l}^*) < 0$, but it is impossible in reality. Hence the equilibrium point $E_3^*(N_{3l}^*, N_{3r}^*)$ is unstable.

When $\tilde{N}$ is (ii)-gH differentiable

The model (6) reduces to

$$\begin{aligned} \frac{dN_r(t, \alpha)}{dt} &= r_l(\alpha) N_l(t, \alpha) \left(1 - \frac{N_r(t, \alpha)}{k}\right) - h_r(\alpha) \\ \frac{dN_l(t, \alpha)}{dt} &= r_r(\alpha) N_r(t, \alpha) \left(1 - \frac{N_l(t, \alpha)}{k}\right) - h_l(\alpha) \end{aligned} \quad (10)$$

With initial condition $(\tilde{r})_\alpha = [r_l(\alpha), r_r(\alpha)]$, $(\tilde{h})_\alpha = [h_l(\alpha), h_r(\alpha)]$ and $N_r(0, \alpha) = N_{r0}(\alpha), N_l(0, \alpha) = N_{l0}(\alpha)$ with the solution $(N(t))_\alpha = [N_l(t, \alpha), N_r(t, \alpha)]$.

(A) Equilibrium points and their feasibility

System (10) gives an coexistence equilibrium point is $E^*(N_{4l}^*, N_{4r}^*)$, where,

$$N_{4l}^* = \frac{(h_l r_l - h_r r_r) + k r_l r_r \pm \sqrt{(h_l r_l - h_r r_r)^2 - 2k r_l r_r (h_l r_l + h_r r_r) + k^2 r_l^2 r_r^2}}{2 r_l r_r}.$$

And

$$N_{4r}^* = \frac{(h_r r_r - h_l r_l) + k r_l r_r \pm \sqrt{(h_r r_r - h_l r_l)^2 - 2k r_l r_r (h_l r_l + h_r r_r) + k^2 r_l^2 r_r^2}}{2r_l r_r}.$$

Here the coexistence equilibrium point $E^*(N_{4l}^*, N_{4r}^*)$ is feasible if,

- $k^2 r_l^2 r_r^2 - 2k(h_l r_l^2 r_r + h_r r_r^2 r_l) + (h_l r_l - h_r r_r)^2 > 0$,
- $h_r r_r - h_l r_l + k r_l r_r > \sqrt{(h_r r_r - h_l r_l)^2 - k r_l r_r (2h_l r_l + 2h_r r_r - k r_l r_r)}$,
- $(h_l r_l - h_r r_r) + k r_l r_r > \sqrt{(h_l r_l - h_r r_r)^2 - 2k r_l r_r (h_l r_l + h_r r_r) + k^2 r_l^2 r_r^2}$.

(B) Stability analysis

The variational matrix of system (10) is

$$V_2 = \begin{pmatrix} \frac{-r_l N_l}{k} & r_l(\alpha) \left(1 - \frac{N_r}{k}\right) \\ r_r \left(1 - \frac{N_l}{k}\right) & \frac{-r_r N_r}{k} \end{pmatrix}$$

Now the variational matrix at coexistence equilibrium point $E_4^*(N_{4l}^*, N_{4r}^*)$ is,

$$V_4(E_4^*) = \begin{pmatrix} \frac{-r_l N_{4l}^*}{k} & r_l \left(1 - \frac{N_{4r}^*}{k}\right) \\ r_r \left(1 - \frac{N_{4l}^*}{k}\right) & \frac{-r_r N_{4r}^*}{k} \end{pmatrix}$$

Fig. 3 a Crisp solution and fuzzy solution for $\alpha = 0$ and in this figure we see that the crisp solution lies between the fuzzy solutions. So application of fuzziness of coexistence equilibrium in Eq. (5) of

“When $\tilde{N}(t)$ is (i)-gH differentiable” is justified and hence it is acceptable. Fuzzy solution of system of Eq. (5) for **b** $\alpha = 0$, **c** $\alpha = 0.5$, **d** $\alpha = 1$

Therefore, the characteristic equation of $V_4(E_4^*)$ is given by

$$\lambda^2 + b_1\lambda + b_2 = 0$$

where,

$$b_1 = -\left(\frac{r_l N_{4l}^*}{k} + \frac{r_r N_{4r}^*}{k}\right),$$

$$b_2 = \frac{r_l r_r N_{4l}^* N_{4r}^*}{k^2} - r_l r_r \left(1 - \frac{N_{4l}^*}{k}\right) \left(1 - \frac{N_{4r}^*}{k}\right).$$

Here $b_1 < 0$, now if $b_2 > 0$ then by R-H condition the coexistence equilibrium point $E_4^*(N_{4l}^*, N_{4r}^*)$ is stable. Hence the equilibrium point $E_4^*(N_{4l}^*, N_{4r}^*)$ is stable if $N_{4r}^* N_{4l}^* > (k - N_{4r}^*)(k - N_{4l}^*)$. So we have the following theorem,

Theorem 1 The coexistence equilibrium point $E_4^*(N_{4l}^*, N_{4r}^*)$ of the system (7) is locally asymptotically stable if, $N_{4r}^* N_{4l}^* > (k - N_{4r}^*)(k - N_{4l}^*)$.

Numerical simulations

Example 2 Now we consider the system (9) with the initial conditions $N(t_0) = \tilde{N}_0 = 3$. Here we extend the Example 1, where the coefficients are fuzzy numbers and initial condition is crisp number which is given below, $r = [0.3, 0.41, 0.6]$, $h = [0.009, 0.01, 0.02]$, $k = 100$ and $N(t_0) = \tilde{N}_0 = 3$.

Here the coexistence equilibrium $E_3^*(N_{3l}^*, N_{3r}^*)$ is unstable. The dynamics of system (9) according to this condition is graphically presented in Fig. 3, which shows that the species $(N(t))$ exist and unstable i.e. the graphical result approaches the coexistence equilibrium E_3^* . This result supports our analytical result given in “When $\tilde{N}(t)$ is (i)-gH differentiable”.

Here we observe that $N_l(t, \alpha)$ increases and $N_r(t, \alpha)$ decreases with increasing α , hence the solution is a strong solution. When α increases the difference between N_l and N_r is decreases and at $\alpha = 1$ they overlap with one another. So we say that the analytical solution of coexistence equilibrium of “When $\tilde{N}(t)$ is (i)-gH differentiable” is depended on the imprecise nature of the biological parameter.

Fig. 4 a Crisp solution and fuzzy solution for $\alpha = 0$ and in this figure we see that the crisp solution lies between the fuzzy solutions. So application of fuzziness of coexistence equilibrium of system of

Eq. (10) is justified and hence it is acceptable. Fuzzy solution of system of Eq. (10) shows in b $\alpha = 0$, c $\alpha = 0.5$ and d $\alpha = 1$

Now we consider the system of Eq. (10) from **When $\tilde{N}$ is (ii)-gH differentiable** with the above mentioned example.

Here the coexistence equilibrium $E_4^*(N_{2l}^*, N_{2r}^*)$ is stable. The dynamics of system (10) according to this condition is graphically presented in Fig. 4, which shows that the coexistence equilibrium is stable, i.e. the graphical result approaches the coexistence equilibrium E_4^* . This result supports our analytical result given in Theorem 1.

Here we observe that $N_l(t, \alpha)$ increases and $N_r(t, \alpha)$ decreases with increasing α , hence the solution is a strong solution. When α increases the difference between N_l and N_r decreases and at $\alpha = 1$ they coincide with one another. Based on this discourse we can say that the condition of coexistence equilibrium of Eq. (10) is depended on the biological parameter.

Solution of the model when intrinsic growth rate, quota harvesting rate and population density at initially are fuzzy number

Here $\tilde{N}(0)$ and $\tilde{r}, \tilde{h}$ are all fuzzy numbers (i.e., initial condition and coefficients are both are fuzzy numbers)

When $\tilde{N}$ is (i)-gH differentiable

The model (6) reduces to

$$\begin{aligned}\frac{dN_l(t, \alpha)}{dt} &= r_l(\alpha)N_l(t, \alpha)\left(1 - \frac{N_r(t, \alpha)}{k}\right) - h_r(\alpha) \\ \frac{dN_r(t, \alpha)}{dt} &= r_r(\alpha)N_r(t, \alpha)\left(1 - \frac{N_l(t, \alpha)}{k}\right) - h_l(\alpha)\end{aligned}\quad (11)$$

With initial condition $(\tilde{N}(0))_\alpha = [N_l(\alpha), N_r(\alpha)]$ with the solution, $(N(t))_\alpha = [N_l(t, \alpha), N_r(t, \alpha)]$.

Note: The calculation part of Equilibrium points and their feasibility and stability conditions are similar as the condition of system of Eq. (9).

When $\tilde{N}(t)$ is (ii)-gH differentiable

The model (6) reduces to

$$\begin{aligned}\frac{dN_r(t, \alpha)}{dt} &= r_l(\alpha)N_l(t, \alpha)\left(1 - \frac{N_r(t, \alpha)}{k}\right) - h_r(\alpha) \\ \frac{dN_l(t, \alpha)}{dt} &= r_r(\alpha)N_r(t, \alpha)\left(1 - \frac{N_l(t, \alpha)}{k}\right) - h_l(\alpha)\end{aligned}\quad (12)$$

Fig. 5 a Crisp solution and fuzzy solution for $\alpha = 0$ and in this figure we see that the crisp solution lies between the fuzzy solutions. So application of fuzziness of axial equilibrium of system of Eq. (11)

is justified and hence it is acceptable. Fuzzy solution of system of Eq. (11) for **b** $\alpha = 0$, **c** $\alpha = 0.5$, **d** $\alpha = 1$

Fig. 6 a Crisp solution and fuzzy solution for $\alpha = 0$ and in this figure we see that the crisp solution lies between the fuzzy solutions. So application of fuzziness of coexistence equilibrium of system of

With initial condition $(\tilde{N}(0))_\alpha = [N_l(\alpha), N_r(\alpha)]$ with the solution $(\tilde{N}(t))_\alpha = [\tilde{N}_l(t, \alpha), \tilde{N}_r(t, \alpha)]$.

Note: The calculation part of Equilibrium points and their feasibility and stability conditions are similar as the condition of system of Eq. (10).

Numerical simulations

Example 3: Now we consider the system (10) from “When $\tilde{N}$ is (i)-gH differentiable” with the initial conditions $N(t_0) = \tilde{N}_0 = (1, 3, 5)$. Here we extend the example 2, where the coefficients and initial condition all are fuzzy numbers which is given below

$r = [0.3, 0.41, 0.6]$, $h = [0.009, 0.01, 0.02]$, $k = 100$ and $N(t_0) = \tilde{N}_0 = (1, 3, 5)$.

Let the coexistence equilibrium of system of Eq. (11) is $E_5^*(N_{5l}^*, N_{5r}^*)$ and we can say that this equilibrium is unstable because it follows the same condition of “When $\tilde{N}(t)$ is (i)-gH differentiable”. The dynamics of system (11) according to this condition is graphically presented in Fig. 5, which shows that the species $(N(t))$ is unstable i.e. the graphical result approaches the coexistence equilibrium

Eq. (12) is justified and hence it is acceptable. Fuzzy solution of system of Eq. (12) shows in b $\alpha = 0$, c $\alpha = 0.5$ and d $\alpha = 1$

E_5^* . This result supports our analytical result given in “When $\tilde{N}$ is (i)-gH differentiable”.

Here we observe that $N_l(t, \alpha)$ increases and $N_r(t, \alpha)$ decreases with increasing α , hence the solution is a strong solution. When α increases the difference between N_l and N_r decreases and at $\alpha = 1$ they coincide with one another. So we say that the solution of system of Eq. (11) is depended on the imprecise nature of the biologically parameter.

Now we consider the system of Eq. (12) from “When $\tilde{N}$ is (ii)-gH differentiable” with the above mentioned Example 3.

Here we let the coexistence equilibrium is $E_6^*(N_{6l}^*, N_{6r}^*)$ and this is stable. The dynamics of system (12) according to this condition is graphically presented in Fig. 6, which shows that E_6^* is stable, i.e. the graphical result approaches the coexistence equilibrium E_6^* . This result supports our analytical result given in “When $\tilde{N}(t)$ is (ii)-gH differentiable”.

Here we observe that $N_l(t, \alpha)$ is increases and $N_r(t, \alpha)$ is decreases with increasing α , hence the solution is a strong solution. When α increases the difference between N_l and

N_r is decreases and at $\alpha = 1$ they coincide with one another. Based on this discussion we can conclude that the stability of system of Eq. (12) is depends on the imprecise nature of the biological parameter.

Conclusion

Differential equations are very important for modeling various natural behavior of biological system. In any biological problem it is need not to every parameter as crisp or precise. For different behavior of natural phenomena some parameter may be imprecise. Due to presence of the imprecise parameters the differential equations nature are changes. Then the concepts of differentiability on that environment play a significant role. In this paper we take a biological model with fuzzy initial value and discuss using fuzzy differentiability equation approach.

In this paper we describe the quota harvesting model in fuzzy environment. The three cases: (1) Initial population density as a fuzzy number (2) coefficients of quota harvesting model are fuzzy number and (3) both the initial condition and coefficients are fuzzy number. The all cases are solved by fuzzy differential equation approach. We propose the fuzzy stability analysis of this model using fuzzy differential equation approach. The stability of coexistence equilibrium in fuzzy sense of the fuzzy quota harvesting model is calculated by the variational matrix at the corresponding equilibrium points and discusses their feasibility. We also provide some theorems which give us the condition of local stability of our proposed model in fuzzy environment.

The significant mathematical outcomes for the dynamical behavior of the quota harvesting model with fuzzy parameter are numerically established. To demonstrate the ability of calculation on the biological equilibrium points and discuss their feasibility and stability with fuzzy parameter values we verify all the mathematical results using numerical simulation and graphical representation. The corresponding examples are no doubt very realistic and helpful in both mathematical and ecological points of view.

Lastly we can surely say that the approach is very helpful of the researchers who are involved with modeling with impreciseness in any linear and nonlinear differential equation problem in various fields of sciences and engineering.

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