Modeling the effect of toxicant on the deformity in a subclass of a biological species

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Abstract In this paper, a nonlinear mathematical model to study the effect of a toxicant on a biological population is proposed and analyzed. We have taken the case in which some members of a biological population get severely affected by the toxicant and show abnormal symptoms, like deformity, fecundity, necrosis, etc. It has been assumed that the toxicant is produced by the population itself. This model can be applied to the human population which creates pollution and affects itself. The analysis of the model suggests the need of a regulatory agency to control the emission of toxicant from manmade projects. The stability analysis of the equilibria of the proposed model and existence of Hopf-bifurcation are determined. We have also determined the direction and stability of bifurcating periodic solutions to clearly understand the effect of emission rate of the toxicant on the biological species. Finally, numerical simulation has been given to illustrate the mathematical results.

Keywords Mathematical model · Toxicants · Deformity · Biological species · Stability · Hopf-bifurcation

Introduction

Environment is where we live, breath, eat, drink and survive. The pollution in the environment is one of the most serious problems facing modern society. The activities of biological species cause different types of pollutions. For instance, industrial effluents and domestic sewage containing toxic substances beyond permissible limits are directly discharged into water bodies, causing water pollution. Further, emission from industrial plants, fuel combustion in motor vehicles, homes and buildings and thermal plants contains numerous chemicals that drift in the atmosphere cause air pollution. Once, in the living food chain, the pollutants are accumulated beyond the permissible limits; they adversely affect the living beings. Some of the known effects of pollution on living beings include cancer, lesions, genetic disorder, and development of deformities, abnormal behavior, reproductive failure and death.

Many researchers have studied the effect of one toxicant or simultaneous effects of two or more toxicants on biological species, using mathematical models (Rescigno 1977; Hallam and Clark 1982; Hallam et al. 1983a, b; Hallam and Deluna 1984; Deluna and Hallam 1987; Agrawal 1999; Freedman and Shukla 1991; Shukla and Dubey 1996; Shukla and Agrawal 1999; Agrawal et al. 2000). In particular, Hallam et al. (1983a, b) have studied the effect of a toxicant emitted into the environment on a biological population by assuming that the growth rate of population density depends linearly upon the uptake concentration of this toxicant. But the effect of the

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environmental toxicant on the carrying capacity of the biological species has not been considered. However, Freedman and Shukla (1991) have studied the role of a toxicant on a single species and predator–prey system by considering its effect on both the intrinsic growth rate of the population as well as on the carrying capacity of population. Further, Shukla and Dubey (1996) have proposed a model to study the simultaneous effects of two toxicants, one being more toxic than the other, on a biological population. Shukla and Agrawal (1999) have proposed mathematical models to study the effects of toxicant on biological species by considering different situations such as allelopathy. Agrawal et al. (2000) have proposed and analyzed a nonlinear mathematical model to study the simultaneous effects of n toxicants on a biological population. But in all of these studies, it has been postulated that the toxicant affects each and every individual of the biological population uniformly. However, this may not be a valid assumption in all the cases; for instance, it may happen that the effect of a toxicant on some members of the population is more severe than the others and hence a subclass of such members may exhibit abnormal symptoms such as changes in shape, size or even deformity, etc. This is possible even if all the individuals of the population are exposed to the toxicant for the same duration as in the case of aquatic systems (Hamilton and Saether 1971; Cushman 1984; Warwick et al. 1987; Dickman et al. 1990a, b; Dickman et al. 1992; Woin and Bronmark 1992; Hartwell et al. 1993; Dickman and Rygiel 1996; Sun et al. 2009; Patil and David 2010; Vermeulen 1995; David et al. 2012; Veroli et al. 2012a, b) and in terrestrial ecosystems, affecting leaf size and causing necrotic markings (lesion), etc. in plants (Kozłowski 1975, 1980) and in animals (Semenza et al. 1997; Ronit and Eldad 1998; Veeramachaneni et al. 2001; Miguel et al. 2006). For example, Hartwell et al. (1993) have studied the growth of *Eurytemora affinis* (Copepoda) in flow through chambers at different locations of polluted sites in Chesapeake Bay tributaries and found that growth rate and fecundity may be chosen as indicators of water quality at appropriate locations and in between the locations. Sun et al. (2009) have shown that morphological deformities can be considered as biomarkers in fish from contaminated rivers in Taiwan. Patil and David (2010) have found that fish in toxic media exhibits irregular, random, circular swimming movements, hyperexcitability, loss of equilibrium, and sink to the bottom. Caudal bending is the prime morphological malformation. The behavioral and morphological deformities are due to inhibition of acetylcholinesterase (AChE) activity. Woin and Bronmark (1992) have studied the effect of DDT and MCPA on reproduction of snail collected from eutrophic pond in southern Sweden and showed that these pollutants may have no effect on mortality but have profound effect on the distribution,

abundance and growth of the species through a reduction in the reproductive potential. Vermeulen (1995) has considered chironomid deformities as bioindicators of toxic sediment stress, Veroli et al. (2012a, b) have studied the mouthpart deformities of *Chironomus plumosus* (Insecta Diptera) in Lake Trasimeno in relation with sediment contamination by heavy metals. Dickman and Rygiel (1996) have studied the effects of heavy metals and oily wastes discharged from a stainless steel company in the Niagara River on an invertebrate population of midge (chironomid) larvae and found that 26 % of the chironomids from sites located 10–800 m down stream were deformed. Ronit and Eldad (1998) have studied the effect of colchicine [used for the treatment of various diseases including gouty arthritis, familial Mediterranean fever (FMF) and Behcet's disease] treatment on sperm production and function. Veeramachaneni et al. (2001) have chemicals and found its adverse effect on male reproduction. Semenza et al. (1997) have found that two reproductive toxins namely dibromochloropropane (DBCP) and ethylene dibromide (EDB) play a crucial role in reproductive failure of alligators. Miguel et al. (2006) have performed sublethal exposure test with propiconazole and found that animals in most treatments show a significant increase in intermoult duration compared to those in the control treatment. The shrimps exposed to the sublethal concentrations of the fungicide show morphological deformities, with a significant positive relationship between concentration and deformities of the rostrum, paraeopods, and uropods.

It may be noted here that the study of this very important phenomenon where a subclass of the biological species is severely affected by the toxicant and shows abnormal symptoms such as deformity, etc. using mathematical models is very limited. Agrawal and Shukla (2012) have studied this phenomenon using mathematical model with the assumption that the toxicant is being emitted into the environment by some external source. We have made certain modifications in this model to produce more accurate results and it is also found that Hopf-bifurcation may occur on changing some key parameters. Here, we have also considered a different situation in which toxicant is produced by the population and affects the population.

Mathematical model

Let us consider a logistically growing biological species affected by a toxicant which is emitted in the environment by the species itself. This toxicant affects a subclass of the biological population acutely showing severe symptoms, such as change in shape, size, deformity, necrotic markings, and reduction in reproduction capability, etc. Keeping these aspects in view, we propose a mathematical model

consist of the following system of nonlinear differential equations,

$$\begin{aligned}\frac{dN_A}{dt} &= bN_A - \frac{rN_A N}{K(T)} - r_1 N_A U - dN_A \\ \frac{dN_D}{dt} &= r_1 N_A U - \frac{rN_D N}{K(T)} - \alpha N_D - dN_D \\ \frac{dT}{dt} &= \lambda N - \delta T - \gamma TN + \pi vNU \\ \frac{dU}{dt} &= \gamma TN - \beta U - vNU.\end{aligned}\quad (2.1)$$

Here $N(t)$ is the density of the biological species, which is affected by the toxicant with environmental concentration $T(t)$, $N_A(t)$ is the density of the population without deformities and $N_D(t)$ is the density of the population, which is affected severely by this toxicant and incapable to participate in reproduction. The toxicant is assumed to be emitted into the environment by biological species itself at a rate of discharge of toxicant into the environment λN . $U(t)$ is the uptake concentration of toxicant by the species with density $N(t)$.

All the parameters in the model system (2.1) are positive constants and they are defined as follows:

- b — the birth rate of biological population,
- d — the death rate of biological population,
- r — the intrinsic growth rate of the population without deformity in the environment without pollutant, i.e., $r = (b - d)$
- r_1 — the decreasing rate of the intrinsic growth rate associated with the uptake of the pollutant,
- α — the mortality rate coefficient of the deformed population due to high toxicity,
- λ — the rate of discharge of toxicant by the biological species,
- δ — the natural depletion rate coefficient of $T(t)$,
- β — the natural depletion rate coefficient of $U(t)$,
- γ — the depletion rate coefficient of $U(t)$ due to uptake by the species,
- v — the depletion rate coefficient of $U(t)$ due to decay of some members of N ,
- π — the fraction of the depletion of $U(t)$ due to decay of some members of N which may reenter into the environment.

In model system (2.1), the density of biological population “ N ” is a sum of density of biological population without deformity “ N_A ” and density of biological population with deformity “ N_D ”, i.e., $N = N_A + N_D$.

Using the above fact, model system (2.1) reduces to the following system in terms of N, N_D, T and U :

$$\begin{aligned}\frac{dN}{dt} &= rN - \frac{rN^2}{K(T)} - (\alpha + b)N_D \\ \frac{dN_D}{dt} &= r_1(N - N_D)U - \frac{rN_D N}{K(T)} - (\alpha + d)N_D \\ \frac{dT}{dt} &= \lambda N - \delta T - \gamma TN + \pi vNU \\ \frac{dU}{dt} &= \gamma TN - \beta U - vNU \\ N(0) &\geq 0, N_D(0) \geq 0, T(0) \geq 0, U(0) \geq cN(0), \\ c &> 0, 0 \leq \pi \leq 1\end{aligned}\quad (2.2)$$

The constant “ $c > 0$ ” is a proportionality constant determining the measure of initial toxicant concentration in the population at $t = 0$.

In the model system (2.1), the function “ $K(T)$ ” denotes the maximum population density of biological species “ N ” which the environment can support and it decreases when “ T ” increases, i.e., “ $K(T)$ ” satisfies the following properties:

$$K(0) = K_0 > 0, \quad \frac{dK}{dT} < 0 \quad \text{for } T > 0 \quad (2.3)$$

where K_0 is the toxicant free carrying capacity of biological species.

Equilibrium points of the model

Due to nonlinearity of model system (2.2), it is not possible to find exact solution of the system. Therefore, we investigate the long term behavior of the system. For this purpose, firstly we find out the equilibrium points of the system. An equilibrium point of a dynamical system is the solution that does not change with time. It can be obtained by putting the right hand side of the model system (2.2) equal to zero.

The model system (2.2) exhibits two non-negative equilibria $E_1(0, 0, 0, 0)$ and $E_2(N^*, N_D^*, T^*, U^*)$. The existence of E_1 is obvious, hence omitted. In the following, we show the existence of the equilibrium E_2 .

Existence of E_2 In the equilibrium $E_2(N^*, N_D^*, T^*, U^*)$, the values of N^*, N_D^*, T^* and U^* are the positive solutions of following equations:

$$N = \frac{(r - r_1 U)K(T)}{r}, \quad (3.1)$$

$$N_D = \frac{r_1 U N K(T)}{rN + (r_1 U + \alpha + d)K(T)}, \quad (3.2)$$

$$T = \frac{\lambda N(\beta + vN)}{f(N)} = g(N), \quad (3.3)$$

$$U = \frac{\lambda \gamma N^2}{f(N)} = h(N), \quad (3.4)$$

where,

$$f(N) = \delta\beta + (\delta v + \gamma\beta)N + \gamma v(1 - \pi)N^2. \quad (3.5)$$

From the Eqs. (3.3) and (3.4), we found that the environmental concentration of toxicant, “ T ” and the uptake concentration of toxicants, “ U ” increase as rate of discharge of toxicant by biological species, “ λ ” increases. Since carrying capacity of biological species, “ $K(T)$ ” is a decreasing function of “ T ” therefore “ $K(T)$ ” decreases as “ T ” (or “ λ ” increases).

Let

$$F(N) = rN - (r - r_1 h(N))K(g(N)), \quad (3.6)$$

we then note from Eq. (3.6) that

$$F(0) < 0 \quad \text{and} \quad F(K_0) > 0$$

This guarantees the existence of a root of equation, $F(N) = 0$ for $0 < N < K_0$, say N^* .

Uniqueness of E_2 Further, the root N^* will be unique, if

$$F'(N) = r - (r - r_1 h(N)) \frac{dK}{dT} \frac{dg}{dN} + r_1 K(g(N)) \frac{dh}{dN} > 0. \quad (3.7)$$

From Eq. (2.3), $\frac{dK}{dT} < 0$, therefore Eq. (3.7) is satisfied if, $\frac{dg}{dN} > 0$ and $\frac{dh}{dN} > 0$. On differentiating (3.3) and (3.4), we get

$$\frac{dg}{dN} = \frac{\lambda}{f^2(N)} [\delta\beta^2 + 2\delta\beta vN + (\delta v^2 + \gamma\pi v\beta)N^2] > 0,$$

$$\frac{dh}{dN} = \frac{\lambda\gamma N}{f^2(N)} [2\delta\beta + (\delta v + \gamma\beta)N] > 0.$$

Hence, the condition (3.7) is automatically satisfied and uniqueness of N^* is guaranteed. Thus, Eq. (3.6) always posses a unique positive root in the interval $(0, K_0)$. This implies that the equilibrium density of biological species is always positive and never exceeds the toxicant free carrying capacity K_0 .

Knowing the value of N^* , the values of N_D^* , T^* and U^* can be obtained using the Eqs. (3.2)–(3.5).

Dynamical behavior and Hopf-bifurcation

Local stability analysis

In this section, we perform the local stability analysis of the obtained equilibrium points. This analysis characterizes the behavior of the solutions which start near the equilibrium point. An equilibrium point is said to be locally

asymptotically stable if for all initial starts near the equilibrium point, the solution trajectory approaches to the equilibrium point as $t \rightarrow \infty$. The local stability of an equilibrium can be investigated by determining the sign of the eigenvalues of Jacobian matrix evaluated at the equilibrium point.

Let M_1 be the Jacobian matrix of the model system (2.2) evaluated at the equilibrium point $E_1(0, 0, 0, 0)$, then we have

$$M_1 = \begin{bmatrix} r & -(\alpha + b) & 0 & 0 \\ 0 & -(\alpha + d) & 0 & 0 \\ \lambda & 0 & -\delta & 0 \\ 0 & 0 & 0 & -\beta \end{bmatrix}$$

From the matrix M_1 , we note that its eigenvalues are r , $-(\alpha + d)$, $-\delta$ and $-\beta$. Thus, E_1 is a saddle point with unstable manifold locally in the N direction and stable manifold locally in $N_D - T - U$ space.

Now, we study the local stability behavior of $E_2(N^*, N_D^*, T^*, U^*)$. For this purpose, we linearize the model system (2.2) about E_2 by using the following transformation:

$$N = N^* + n, \quad N_D = N_D^* + n_d, \quad T = T^* + \tau, \quad U = U^* + u$$

where n, n_d, τ, u are small perturbations around E_2 .

The model system (2.2) in terms of new variables and matrix form can be written as,

$$\dot{X} = AX + B \quad (4.1)$$

Here AX & B are the linear and nonlinear parts of the model system (2.2), respectively. Moreover,

$$X = \begin{bmatrix} n \\ n_d \\ \tau \\ u \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix},$$

$$\text{and } B = \begin{bmatrix} l_1 n^2 + 2l_2 n\tau + l_3 n^2 \tau \\ l_1 n n_d + l_4 n\tau + r_1 n u + l_2 n_d \tau - r_1 n_d u + l_3 n n_d \tau \\ -\gamma n \tau + \pi v n u \\ \gamma n \tau - v n u \end{bmatrix}$$

where, $a_{11} = r + 2l_1 N^*$, $a_{12} = -(\alpha + b)$, $a_{13} = l_2 N^*$, $a_{21} = l_1 N_D^* + r_1 U^*$, $a_{22} = -\frac{r_1 N^* U^*}{N_D^*}$, $a_{23} = l_2 N_D^*$, $a_{24} = r_1 (N^* - N_D^*)$, $a_{31} = \lambda - (\gamma T^* - \pi v U^*)$, $a_{33} = -(\delta + \gamma N^*)$, $a_{34} = \pi v N^*$, $a_{41} = (\gamma T^* - v U^*)$, $a_{43} = \gamma N^*$, $a_{44} = -(\beta + v N^*)$, $l_1 = -\frac{r}{K(T^*)}$, $l_2 = \frac{r N^* K'(T^*)}{K^2(T^*)}$, $l_3 = \frac{r K'(T^*)}{K^2(T^*)}$, $l_4 = \frac{r N_D^* K'(T^*)}{K^2(T^*)}$. Here, matrix A is a Jacobian matrix corresponding to the equilibrium points E_2 and characteristic equation of matrix A can be written as:

$$p(x) = x^4 + A_1x^3 + A_2x^2 + A_3x + A_4 \quad (4.2)$$

where,

$$\begin{aligned} A_1 &= -(a_{11} + a_{22} + a_{33} + a_{44}) \\ A_2 &= a_{11}(a_{22} + a_{33} + a_{44}) + a_{22}(a_{33} + a_{44}) + a_{33}a_{44} \\ &\quad - (a_{34}a_{43} + a_{12}a_{21} + a_{13}a_{31}) \\ A_3 &= (a_{33} + a_{44})(a_{12}a_{21} - a_{11}a_{22}) + (a_{11} + a_{22}) \\ &\quad \times (a_{34}a_{43} - a_{33}a_{44}) \\ &\quad - a_{12}(a_{23}a_{31} + a_{24}a_{41}) + a_{13}a_{31}(a_{22} + a_{44}) - a_{13}a_{34}a_{41} \\ A_4 &= (a_{11}a_{22} - a_{12}a_{21})(a_{33}a_{44} - a_{34}a_{43}) + a_{12}a_{24} \\ &\quad \times (a_{41}a_{33} - a_{31}a_{43}) \\ &\quad + (a_{31}a_{44} - a_{34}a_{41})(a_{12}a_{23} - a_{13}a_{22}) \end{aligned}$$

Now, using Routh–Hurwitz Criterion, we can say that all the roots of the polynomial $p(x)$ are either negative or with negative real parts iff

$$A_j > 0 \quad (j = 1, 4) \quad (4.3)$$

$$A_1A_2 > A_3 \quad (4.4)$$

$$A_1A_2A_3 > A_3^2 + A_1^2A_4 \quad (4.5)$$

Thus, we have the following result regarding the local stability of the equilibrium E_2 :

Theorem 4.1 *The positive equilibrium $E_2(N^*, N_D^*, T^*, U^*)$ is locally asymptotically stable under the conditions (4.3)–(4.5).*

Thus, the solutions of the model system (2.2), which are initiated near the equilibrium E_2 , will settle down to E_2 provided the above conditions (4.3)–(4.5) hold.

Hopf-bifurcation analysis

Now, we investigate for the possibility of Hopf-bifurcation (Hassard et al. 1981; Kuznetsov 2004) from the interior equilibrium E_2 by taking λ (i.e., the rate of discharge of toxicant by the biological species) as a bifurcation parameter. The term Hopf-bifurcation refers to development of periodic orbits (self-oscillations) from a stable fixed point, as a parameter crosses a critical value.

Theorem 4.2 *The model system (2.2) undergoes a Hopf-bifurcation from the interior equilibrium E_2 , if λ passes through the critical value $\lambda^* > 0$ such that*

1. $A_j(\lambda^*) > 0 (j = 1, 4)$
2. $A_1(\lambda^*)A_2(\lambda^*) > A_3(\lambda^*)$
3. $A_1(\lambda^*)A_2(\lambda^*)A_3(\lambda^*) - A_3^2(\lambda^*) - A_1^2(\lambda^*)A_4(\lambda^*) = 0$
4. $\left[\frac{dR}{d\lambda} \right]_{\lambda=\lambda^*} = \left[\frac{\frac{d}{d\lambda}(A_1A_2A_3 - A_3^2 - A_1^2A_4)}{2A_1(4A_4 - A_1A_3 - A_2^2)} \right]_{\lambda=\lambda^*} \neq 0$

The proof is given in Appendix 1.

The above theorem tells that if the rate of discharge of toxicant by the biological species “ λ ” crosses a critical value “ λ^* ”, the densities of both the populations (total and deformed populations of biological species) starts oscillation and do not settle to the equilibrium levels.

Now, it is important to know the nature of bifurcating periodic solutions arising through Hopf-bifurcation. We have analyzed the stability and direction of the bifurcating periodic solutions by following the procedure given in Hassard et al. (1981) and get the following result:

Theorem 4.3 *System (2.2) shows a Hopf-bifurcation with respect to parameter λ , with the following properties:*

1. *If $\mu_2 > 0 (\mu_2 < 0)$, the Hopf-bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\lambda > \lambda^* (\lambda < \lambda^*)$,*
2. *If $\beta_2 < 0 (\beta_2 > 0)$, the bifurcating periodic solutions are stable (unstable),*
3. *If $\tau_2 > 0 (\tau_2 < 0)$, the period of bifurcating periodic solutions increases (decreases).*

The proof is given in Appendix 2.

Global stability analysis

In previous sections, we have investigated the nature of the solutions which are initiated near the equilibrium point. Now, we investigate the long term behavior of the solutions, those initiate from a distant state from the equilibrium point (La-Salle and Lefschetz 1961). For, this we make use of Lyapunov’s direct method. The fundamental concept of the Lyapunov’s direct method is that if the total energy of a system is continuously dissipating, then the system will eventually reach to an equilibrium point and remain at that point. Using this method, we show that E_2 is globally asymptotically stable under certain conditions.

The following theorem characterizes the global stability behavior of the equilibrium point E_2 .

Theorem 4.4 *In addition to the assumption (2.3), let $K(T)$ satisfies the following inequalities in Ω :*

$$K_m \leq K(T) \leq K_0, \quad |K'(T)| \leq \kappa$$

where K_m and κ are positive constants. If the following inequalities hold in Ω ,

$$\begin{aligned} &\left[\frac{\alpha + b}{N^*} + \frac{rr_1\lambda K_0^2}{K(T^*)\{r_1\lambda K_0 + (\alpha + d)\delta_m\}} + r_1U^* \right]^2 \\ &< \frac{4r_1U^*N^*}{9} \left\{ \frac{r}{K(T^*)} - \frac{(\alpha + b)r_1\lambda K_0^2}{N^{*2}\{r_1\lambda K_0 + (\alpha + d)\delta_m\}} \right\} \end{aligned} \quad (4.6)$$

$$\left[\lambda + \frac{rK_0}{K_m^2} \kappa + (\gamma + \pi v) \frac{\lambda K_0}{\delta_m} \right]^2 < \frac{4}{9} (\delta + \gamma N^*) \left\{ \frac{r}{K(T^*)} - \frac{(\alpha + b)r_1 \lambda K_0^2}{N^{*2} \{r_1 \lambda K_0 + (\alpha + d)\delta_m\}} \right\} \quad (4.7)$$

$$\left[(\gamma + v) \frac{\lambda K_0}{\delta_m} \right]^2 < \frac{4}{9} (\beta + v N^*) \left\{ \frac{r}{K(T^*)} - \frac{(\alpha + b)r_1 \lambda K_0^2}{N^{*2} \{r_1 \lambda K_0 + (\alpha + d)\delta_m\}} \right\} \quad (4.8)$$

$$\left[\frac{rr_1 \lambda K_0^3 \kappa}{K_m^2 \{r_1 \lambda K_0 + (\alpha + d)\delta_m\}} \right]^2 < \frac{4}{9} \frac{r_1 U^* N^*}{N_D^*} (\delta + \gamma N^*) \quad (4.9)$$

$$[r_1 K_0]^2 < \frac{4}{9} \frac{r_1 U^* N^*}{N_D^*} (\beta + v N^*) \quad (4.10)$$

$$[(\gamma + \pi v) N^*]^2 < \frac{4}{9} (\delta + \gamma N^*) (\beta + v N^*) \quad (4.11)$$

then E_2 is globally stable for all solutions initiating in Ω

For details see Appendix 3.

Numerical simulation

To give a better picture of our analytical results, we present here numerical simulation of the mathematical model (2.2) by assuming

$$K(T) = K_0 - \frac{b_1 T}{1 + b_2 T}$$

and taking the following set of parameter values:

$$\left. \begin{aligned} b &= 0.044, K_0 = 10.0, d = 0.0001, b_1 = 0.01, \\ b_2 &= 1.0, \alpha = 0.0001, r_1 = 0.8, \lambda = 0.0001, \\ \delta &= 0.025, \gamma = 0.0001, \pi = 0.5, v = 0.0001, \beta = 0.01 \end{aligned} \right\}. \quad (5.1)$$

For this set of parameter values, the conditions (4.3–4.5) of local asymptotic stability of interior equilibrium E_2 are satisfied. The Figs. 1 and 2 show the variations in the real and imaginary parts of eigenvalues of Jacobian matrix A on increasing the rate of discharge of toxicant λ by the biological species, keeping other parameters fixed. From these figures, we can see that, for $\lambda < \lambda^* (= 0.01766)$, all the eigenvalues of Jacobian matrix A have negative real parts ($R_1, R_2, R_3, R_4 < 0$); but at $\lambda = \lambda^*$, two eigenvalues of Jacobian matrix have negative real parts ($R_3, R_4 < 0$) while the other two eigenvalues have zero real part and the Jacobian matrix A possesses one conjugate nonzero purely imaginary pair ($R_1, R_2 = 0$ and $I_1, I_2 \neq 0$). Thus, the system (2) is locally asymptotically stable for each value of $\lambda < \lambda^*$ and a Hopf-bifurcation ‘H’ may occur at $\lambda = \lambda^*$.

In “Figs. 3 and 4”, we can see that, when the value of parameter λ is less than the critical value λ^* , the densities of both the populations settle down to their equilibrium values but when value of λ exceeds the critical value λ^* , both densities oscillate and do not settle down to their equilibrium values. In “Figs. 5 and 6”, we have shown the phase plot for total population versus deformed population for different values of bifurcation parameter λ .

For the set of parameter values (5.1), it is found that $\mu_2 = 7.0309 \times 10^{-10} > 0$ and $\beta_2 = -1.5920 \times 10^{-10} < 0$ at the critical point $\lambda = \lambda^*$, which implies that Hopf-bifurcation is supercritical ($\mu_2 > 0$), bifurcating periodic

Fig. 1 Real parts of eigenvalues of Jacobian matrix A respect to the parameter λ

Fig. 2 Imaginary parts of eigenvalues of Jacobian matrix A respect to the parameter λ

Fig. 3 Time graph for total population ' N '

solutions exist for $\lambda > \lambda^*$ and bifurcating periodic solutions are stable ($\beta_2 < 0$), see “Figs. 7 and 8”.

Results and discussion

In this paper, we have proposed and analyzed a nonlinear mathematical model to study the effect of a toxicant on a biological population. It is assumed that the toxicant is emitted in the environment by the species itself and this toxicant affects a subclass of the biological population acutely showing severe symptoms, such as deformity, fecundity (less production capability), etc. It is found that the system exhibits two non negative equilibria E_1 and E_2 .

The equilibrium E_1 is a saddle point and E_2 is locally asymptotically stable under the conditions (4.3–4.5). The global stability analysis of the interior equilibrium E_2 has been performed. This analysis provides the conditions for the stabilization of the densities of both populations. The Hopf-bifurcation analysis has been performed with respect to the parameter λ (rate of discharge of toxicants). The conditions for occurrence of Hopf-bifurcations have been derived. It is found that when the rate of discharge of toxicants by population ' λ ' exceeds a critical value ' λ^* ', the system loses its stability and periodic solutions arise via Hopf-bifurcation. The stability and direction of bifurcation periodic solutions are analyzed in detail. For a particular set of parameter values, it is shown that when the emission

Fig. 4 Time graph for deformed population ' N_D '

Fig. 5 Phase space graph between N and N_D at $\lambda = 0.02$ ($>\lambda^*$)

rate of toxicants by the population ' λ ' increases, the density of total population N decreases and the density of deformed population N_D firstly increases and then decreases; but as λ crosses the critical value $\lambda^* = 0.01766$, both densities start oscillating and gives supercritical Hopf-bifurcation with stable periodic solutions (see Figs. 7, 8).

It is quite well known that the pollutants emitted by human population (in the form of industrial wastes, vehicular exhaust, radioactive wastes, etc.) contain various toxic substances which are harmful to all the populations living on this earth including human population. May be human population does not go under any morphological deformity because of these toxicants but still there are large number of humans which get severely affected by these pollutants and certain types of diseases such as bronchial

asthma, lung cancer, COPD, decreased reproductive capability, etc. Thus, the mortality rate of these human beings is increased. In our model, we have assumed that the toxicant is being emitted into the environment by the population itself, instead, if we would have assumed it to be emitted by some external source with a constant rate, the model can be applied to plant population as well as aquatic population also. If we compare the growth of plants located in polluted site with the growth of plants in normal (pollutant free) site, it can be seen that plants which are located in polluted site suffer decreased growth (Kozłowski 1975, 1980; Saquib et al. 2010). These toxic materials may cause morphological deformity also as in case of aquatic systems (Hamilton and Saether 1971; Cushman 1984; Warwick et al. 1987; Dickman et al. 1990a, b; Woin and Bronmark

Fig. 6 Phase space graph between N & N_D at $\lambda = 0.05$ ($>\lambda^*$)

Fig. 7 Dynamic behavior of density of total population ' N ' with respect to the parameter λ

1992; Hartwell et al. 1993; Dickman and Rygiel 1996; Sun et al. 2009; Patil and David 2010; Vermeulen 1995; Veroli et al. 2012a, b).

Appendix A: Proof of the Theorem 4.2 The necessary and sufficient conditions for the existence of Hopf-bifurcation in system (2.2) are: Eigenvalues ($x_j = R_j + iI_j$, $j = 1, 2, 3, 4$) of the Jacobian matrix A have a pair of purely imaginary eigenvalues and other two eigenvalues have negative real parts (i.e., $R_1, R_2 = 0$, $I_1 = -I_2 \neq 0$ and $R_3, R_4 < 0$) at $\lambda = \lambda^*$ (>0) and $dR_j/d\lambda \neq 0$, $j = 1, 2$ at $\lambda = \lambda^*$. Now, according to the *Liu's criterion* (Liu,

1994), system (2.2) undergoes a Hopf-bifurcation at $\lambda = \lambda^* > 0$, if following conditions hold:

1. $A_j(\lambda^*) > 0$ ($j = 1, 4$)
2. $A_1(\lambda^*)A_2(\lambda^*) > A_3(\lambda^*)$
3. $A_1(\lambda^*)A_2(\lambda^*)A_3(\lambda^*) - A_3^2(\lambda^*) - A_1^2(\lambda^*)A_4(\lambda^*) = 0$
4. The eigenvalues of the Jacobian matrix A should be in the form of $x_j = R_j + iI_j$, where $\frac{dR_j}{d\lambda} \neq 0$, $j = 1, 2$ at $\lambda = \lambda^*$.

Now, we will verify the Hopf-bifurcation condition (iv), putting $x = R + iI$ in “Eq. (4.2)”, we get

Fig. 8 Dynamic behavior of density of deformed population ' N_D ' with respect to the parameter λ

$$(R + iI)^4 + A_1(R + iI)^3 + A_2(R + iI)^2 + A_3(R + iI) + A_4 = 0 \quad (7.1)$$

Separating the real and imaginary parts, we get

$$R^4 + I^4 - 6R^2I^2 + A_1R^3 - 3A_1RI^2 + A_2(R^2 - I^2) + A_3R + A_4 = 0 \quad (7.2)$$

$$-4RI^3 + 4R^3I - A_1I^3 + 3A_1R^2I + 2A_2RI + A_3I = 0. \quad (7.3)$$

Eliminating I between “Eqs. (7.2) and (7.3)”, we have

$$\begin{aligned} &R^4(4R + A_1)^2 + (4R^3 + 3A_1R^2 + 2A_2R + A_3)^2 \\ &+ (A_1R^3 + A_2R^2 + A_3R + A_4)(4R + A_1)^2 \\ &- (4R^3 + 3A_1R^2 + 2A_2R + A_3)(6R^2 + 3A_1R + A_2) \\ &\times (4R + A_1) = 0. \end{aligned} \quad (7.4)$$

Differentiating “Eq. (7.4) with respect to λ at $\lambda = \lambda^*$, we have

$$\left[\frac{dR}{d\lambda} \right]_{\lambda=\lambda^*} = \left[\frac{\frac{d}{d\lambda}(A_1A_2A_3 - A_3^2 - A_1^2A_4)}{2A_1(4A_4 - A_1A_3 - A_2^2)} \right]_{\lambda=\lambda^*} \neq 0. \quad (7.5)$$

Hence the Theorem 4.2 is proved. $\square$

Appendix B: Proof of the Theorem 4.3

At the Hopf-bifurcation point, pair of complex conjugate roots of “Eq. (4.2)” have zero real parts, then the roots of “Eq. (4.2)” are

$$x_{1,2} = \pm iI, \quad x_3 = -J_1, \quad x_4 = -J_2$$

$$\text{where, } I = \sqrt{\frac{A_3}{A_1}}, \quad J_1 = \frac{-A_1^2 - \sqrt{A_1(A_1^3 - 4A_1A_2 + 4A_3)}}{2A_1} \text{ and } J_2 = \frac{-A_1^2 + \sqrt{A_1(A_1^3 - 4A_1A_2 + 4A_3)}}{2A_1}$$

Now, to determine the direction of Hopf-bifurcation and stability of bifurcating periodic solutions (Khare et al. 2011; Misra et al. 2015), we transform the system (2.2) in normal form.

Let $X = PY$ then the system (4.1) becomes

$$\dot{Y} = JY + F, \quad Y = \text{col.}(y_1, y_2, y_3, y_4), \quad (8.1)$$

where

$$\begin{aligned} J = P^{-1}AP &= \begin{bmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & -J_1 & 0 \\ 0 & 0 & 0 & -J_2 \end{bmatrix} \text{ and } F = P^{-1}f \\ &= \begin{bmatrix} F_1(y_1, y_2, y_3, y_4) \\ F_2(y_1, y_2, y_3, y_4) \\ F_3(y_1, y_2, y_3, y_4) \\ F_4(y_1, y_2, y_3, y_4) \end{bmatrix}. \end{aligned} \quad (8.2)$$

The matrix P is defined as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix},$$

where,

$$\begin{aligned}
 P_{1j} &= a_{12}D_j, P_{2j} = E_j, P_{3j} = a_{12}a_{31}B_j, \\
 P_{4j} &= a_{12}a_{31}C_j, (j = 1, 2, 3, 4) \\
 B_1 &= a_{31}a_{44} - a_{34}a_{41}, B_2 = a_{31}I, B_3 = B_1 + J_1a_{31}, \\
 B_4 &= B_1 + J_2a_{31} \\
 C_1 &= a_{33}a_{41} - a_{31}a_{43}, C_2 = a_{41}I, C_3 = C_1 + J_1a_{41}, \\
 C_4 &= C_1 + J_2a_{41} \\
 D_1 &= IB_2 - a_{33}B_1 - a_{34}C_1, D_2 = -(IB_1 + a_{33}B_2 + a_{34}C_2), \\
 D_3 &= -\{(a_{33} + J_1)B_3 + a_{34}C_3\}, D_4 = -\{(a_{33} + J_2)B_4 + a_{34}C_4\}, \\
 E_1 &= ID_2 - a_{11}D_1 - a_{13}a_{31}B_1, E_2 = -(ID_1 + a_{11}D_2 + a_{13}a_{31}B_2), \\
 E_3 &= -\{(a_{11} + J_1)D_3 + a_{13}a_{31}B_3\}, E_4 = -\{(a_{11} + J_2)D_4 + a_{13}a_{31}B_4\}
 \end{aligned}$$

and

$$f = \begin{bmatrix} f_1(y_1, y_2, y_3, y_4) \\ f_2(y_1, y_2, y_3, y_4) \\ f_3(y_1, y_2, y_3, y_4) \\ f_4(y_1, y_2, y_3, y_4) \end{bmatrix},$$

where,

$$\begin{aligned}
 g_{11} &= \frac{1}{4} \left\{ \left(\frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2} \right) + i \left(\frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right) \right\}, \\
 g_{02} &= \frac{1}{4} \left\{ \left(\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} - 2 \frac{\partial^2 F_2}{\partial y_1 \partial y_2} \right) \right. \\
 &\quad \left. + i \left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2 \frac{\partial^2 F_1}{\partial y_1 \partial y_2} \right) \right\}, \\
 g_{20} &= \frac{1}{4} \left\{ \left(\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} + 2 \frac{\partial^2 F_2}{\partial y_1 \partial y_2} \right) \right. \\
 &\quad \left. + i \left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2 \frac{\partial^2 F_1}{\partial y_1 \partial y_2} \right) \right\}, \\
 \text{and } g_{21} &= G_{21} + (2G_{110}^1 w_{11}^1 + G_{101}^1 w_{20}^1 + 2G_{110}^2 w_{11}^2 + G_{101}^2 w_{20}^2)
 \end{aligned}$$

where,

$$\begin{aligned}
 f_1(y_1, y_2, y_3, y_4) &= l_1(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)^2 + 2l_2(P_{11}y_1 + P_{12}y_2 \\
 &\quad + P_{13}y_3 + P_{14}y_4)(P_{31}y_1 + P_{32}y_2 + P_{33}y_3 + P_{34}y_4) \\
 &\quad + l_3(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)^2(P_{31}y_1 + P_{32}y_2 + P_{33}y_3 \\
 &\quad + P_{34}y_4) \\
 f_2(y_1, y_2, y_3, y_4) &= l_1(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)(P_{21}y_1 + P_{22}y_2 + P_{23}y_3 \\
 &\quad + P_{24}y_4) + l_4(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)(P_{31}y_1 + P_{32}y_2 \\
 &\quad + P_{33}y_3 + P_{34}y_4) + r_1(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)(P_{41}y_1 \\
 &\quad + P_{42}y_2 + P_{43}y_3 + P_{44}y_4) + l_2(P_{21}y_1 + P_{22}y_2 + P_{23}y_3 \\
 &\quad + P_{24}y_4)(P_{31}y_1 + P_{32}y_2 + P_{33}y_3 + P_{34}y_4) - r_1(P_{21}y_1 + P_{22}y_2 \\
 &\quad + P_{23}y_3 + P_{24}y_4)(P_{41}y_1 + P_{42}y_2 + P_{43}y_3 + P_{44}y_4) + l_3(P_{11}y_1 \\
 &\quad + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)(P_{21}y_1 + P_{22}y_2 + P_{23}y_3 \\
 &\quad + P_{24}y_4)(P_{31}y_1 + P_{32}y_2 + P_{33}y_3 + P_{34}y_4) \\
 f_3(y_1, y_2, y_3, y_4) &= -\gamma(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)(P_{31}y_1 + P_{32}y_2 + P_{33}y_3 \\
 &\quad + P_{34}y_4) + \pi v(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)(P_{41}y_1 + P_{42}y_2 \\
 &\quad + P_{43}y_3 + P_{44}y_4) \\
 f_4(y_1, y_2, y_3, y_4) &= \gamma(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)(P_{31}y_1 + P_{32}y_2 + P_{33}y_3 \\
 &\quad + P_{34}y_4) - v(P_{11}y_1 + P_{12}y_2 + P_{13}y_3 + P_{14}y_4)(P_{41}y_1 + P_{42}y_2 \\
 &\quad + P_{43}y_3 + P_{44}y_4).
 \end{aligned}$$

Hence, Eq. (8.1) is the normal form of Eq. (4.1). For evaluating the direction of bifurcating solution, we evaluate following quantities at $\lambda = \lambda^*$ and $(y_1, y_2, y_3, y_4) = (0, 0, 0, 0)$.

$$\begin{aligned}
G_{21} &= \frac{1}{8} \left\{ \left(\frac{\partial^3 F_1}{\partial y_1^3} + \frac{\partial^3 F_2}{\partial y_2^3} + \frac{\partial^3 F_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F_1}{\partial y_1 \partial y_2^2} \right) \right. \\
&\quad \left. + i \left(\frac{\partial^3 F_2}{\partial y_1^3} - \frac{\partial^3 F_1}{\partial y_2^3} - \frac{\partial^3 F_1}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F_2}{\partial y_1 \partial y_2^2} \right) \right\}, \\
G_{110}^1 &= \frac{1}{2} \left\{ \left(\frac{\partial^2 F_1}{\partial y_1 y_3} + \frac{\partial^2 F_2}{\partial y_2 y_3} \right) + i \left(\frac{\partial^2 F_2}{\partial y_1 y_3} - \frac{\partial^2 F_1}{\partial y_2 y_3} \right) \right\}, \\
G_{110}^2 &= \frac{1}{2} \left\{ \left(\frac{\partial^2 F_1}{\partial y_1 y_4} + \frac{\partial^2 F_2}{\partial y_2 y_4} \right) + i \left(\frac{\partial^2 F_2}{\partial y_1 y_4} - \frac{\partial^2 F_1}{\partial y_2 y_4} \right) \right\}, \\
G_{101}^1 &= \frac{1}{2} \left\{ \left(\frac{\partial^2 F_1}{\partial y_1 y_3} - \frac{\partial^2 F_2}{\partial y_2 y_3} \right) + i \left(\frac{\partial^2 F_2}{\partial y_1 y_3} + \frac{\partial^2 F_1}{\partial y_2 y_3} \right) \right\}, \\
G_{101}^2 &= \frac{1}{2} \left\{ \left(\frac{\partial^2 F_1}{\partial y_1 y_4} - \frac{\partial^2 F_2}{\partial y_2 y_4} \right) + i \left(\frac{\partial^2 F_2}{\partial y_1 y_4} + \frac{\partial^2 F_1}{\partial y_2 y_4} \right) \right\}, \\
h_{11}^1 &= \frac{1}{4} \left(\frac{\partial^2 F_3}{\partial y_1^2} + \frac{\partial^2 F_3}{\partial y_2^2} \right), h_{11}^2 = \frac{1}{4} \left(\frac{\partial^2 F_4}{\partial y_1^2} + \frac{\partial^2 F_4}{\partial y_2^2} \right), \\
h_{20}^1 &= \frac{1}{4} \left(\frac{\partial^2 F_3}{\partial y_1^2} - \frac{\partial^2 F_3}{\partial y_2^2} - 2i \frac{\partial^2 F_3}{\partial y_1 \partial y_2} \right), \\
h_{20}^2 &= \frac{1}{4} \left(\frac{\partial^2 F_4}{\partial y_1^2} - \frac{\partial^2 F_4}{\partial y_2^2} - 2i \frac{\partial^2 F_4}{\partial y_1 \partial y_2} \right), \\
w_{11}^1 &= \frac{-h_{11}^1}{J_1}, w_{11}^2 = \frac{-h_{11}^2}{J_2}, w_{20}^1 = \frac{-h_{20}^1}{J_1 - 2iI}, w_{20}^2 = \frac{-h_{20}^2}{J_2 - 2iI}
\end{aligned}$$

To examine the qualities of bifurcation periodic solutions, we calculate the following quantities:

$$\begin{cases} C_1(0) = \frac{i}{2I} \left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right] + \frac{1}{2}g_{21}, \mu_2 = -\frac{\operatorname{Re}C_1(0)}{\operatorname{Re}\lambda'(\lambda^*)}, \\ \beta_2 = 2\operatorname{Re}C_1(0), \tau_2 = -\frac{(\operatorname{Im}C_1(0) + \mu_2 \operatorname{Im}\lambda'(\lambda^*))}{I} \end{cases} \quad (8.3)$$

Hence the Theorem 4.3 is proved. $\square$

Appendix 3: Global stability conditions

To establish global stability conditions, firstly we give the following lemma which establishes the region of attraction for E_2 .

Lemma 4.1 *The region $\Omega = \{(N, N_D, T, U) : 0 \leq N \leq K_0, 0 \leq N_D \leq \frac{r_1 \lambda K_0^2}{r_1 \lambda K_0 + (\alpha + d)\delta_m}, 0 \leq T(t) + U(t) \leq \frac{\lambda K_0}{\delta_m}\}$ where $\delta_m = \min(\delta, \beta)$ attracts all solutions initiating in the interior of the positive orthant.*

Proof From the first equation of model system (2.2), we have

$$\frac{dN}{dt} \leq rN - \frac{rN^2}{K_0} = r \left(1 - \frac{N}{K_0} \right) N$$

Thus, $\limsup_{t \rightarrow \infty} N(t) \leq K_0$. From the second equation of model system (2.2), we have

$$\frac{dN_D}{dt} \leq \frac{r_1 \lambda K_0^2}{\delta_m} - \left\{ \frac{r_1 \lambda K_0}{\delta_m} + (\alpha + d) \right\} N_D.$$

Thus, $\limsup_{t \rightarrow \infty} N_D(t) \leq \frac{r_1 \lambda K_0^2}{r_1 \lambda K_0 + (\alpha + d)\delta_m}$. Now, adding the last two equations of model system (2.2), we get

$$\frac{dT}{dt} + \frac{dU}{dt} = \lambda N - \delta T - \beta U - (1 - \pi)vNU \leq \lambda K_0 - \delta_m(T + U).$$

Thus, $\limsup_{t \rightarrow \infty} [T(t) + U(t)] = \frac{\lambda K_0}{\delta_m}$, proving the lemma. $\square$

Proof of Theorem 4.4 We consider the following positive definite function,

$$\begin{aligned} W(N, N_D, T, U) &= \left\{ N - N^* - N^* \log \left(\frac{N}{N^*} \right) \right\} + \frac{1}{2} (N_D \\ &\quad - N_D^*)^2 + \frac{1}{2} (T - T^*)^2 + \frac{1}{2} (U - U^*)^2 \end{aligned}$$

Differentiating W with respect to t along the solutions of system (2.2), we get

$$\begin{aligned} \frac{dW}{dt} &= (N - N^*) \left[r - \frac{rN}{K(T)} - (\alpha + b) \frac{N_D}{N} \right] \\ &\quad + (N_D - N_D^*) \left[r_1 N U - r_1 N_D U - \frac{r N_D N}{K(T)} - (\alpha + d) N_d \right] \\ &\quad + (T - T^*) [\lambda N - \delta T - \gamma T N + \pi v N U] + (U \\ &\quad - U^*) [\gamma T N - \beta U - v N U]. \end{aligned}$$

Using “Eqs. (3.1)–(3.5)”, we get after a little algebraic manipulations

$$\begin{aligned} \frac{dW}{dt} &= - \left[\frac{r}{K(T^*)} + (\alpha + b) N_D \zeta(N) \right] (N - N^*)^2 \\ &\quad - \frac{r_1 U^* N^*}{N_D^*} (N_D - N_D^*)^2 \\ &\quad - (\delta + \gamma N^*) (T - T^*)^2 - (\beta + v N^*) (U - U^*)^2 \\ &\quad + \left[r_1 U^* - \frac{(\alpha + b)}{N^*} - \frac{r N_D}{K(T^*)} \right] (N - N^*) (N_D - N_D^*) \\ &\quad - r N_D N \eta(T) (N_D - N_D^*) (T - T^*) + [\lambda - r N \eta(T) \\ &\quad - \gamma T + \pi v U] (N - N^*) (T - T^*) \\ &\quad + (\gamma T - v U) (N - N^*) (U - U^*) \\ &\quad + [r_1 (N - N_D)] (N_D - N_D^*) (U - U^*) \\ &\quad + [(\gamma + \pi v) N^*] (T - T^*) (U - U^*) \end{aligned}$$

where

$$\eta(T) = \begin{cases} \frac{1}{K(T)} - \frac{1}{K(T^*)}, & T \neq T^* \\ -\frac{K'(T^*)}{K^2(T^*)}, & T = T^* \end{cases} \quad \text{and} \\ \zeta(N) = \begin{cases} \frac{1}{N} - \frac{1}{N^*}, & N \neq N^* \\ -\frac{1}{N^{*2}}, & N = N^* \end{cases}$$

Thus, $\frac{dW}{dt}$ can be written as the sum of the quadratics,

$$\begin{aligned} \frac{dW}{dt} = & -\frac{1}{2}a_{11}(N - N^*)^2 + a_{12}(N - N^*)(N_D - N_D^*) \\ & -\frac{1}{2}a_{22}(N_D - N_D^*)^2 - \frac{1}{2}a_{11}(N - N^*)^2 \\ & + a_{13}(N - N^*)(T - T^*) - \frac{1}{2}a_{33}(T - T^*)^2 \\ & -\frac{1}{2}a_{11}(N - N^*)^2 + a_{14}(N - N^*)(U - U^*) \\ & -\frac{1}{2}a_{44}(U - U^*)^2 - \frac{1}{2}a_{22}(N_D - N_D^*)^2 \\ & + a_{23}(N_D - N_D^*)(T - T^*) - \frac{1}{2}a_{33}(T - T^*)^2 \\ & -\frac{1}{2}a_{22}(N_D - N_D^*)^2 + a_{24}(N_D - N_D^*)(U - U^*) \\ & -\frac{1}{2}a_{44}(U - U^*)^2 - \frac{1}{2}a_{33}(T - T^*)^2 \\ & + a_{34}(T - T^*)(U - U^*) - \frac{1}{2}a_{44}(U - U^*)^2 \end{aligned}$$

where,

$$\begin{aligned} a_{11} &= \frac{2}{3} \left\{ \frac{r}{K(T^*)} + (\alpha + b)N_D\zeta(N) \right\}, a_{22} = \frac{2r_1U^*N^*}{3N_D^*}, \\ a_{33} &= \frac{2}{3}(\delta + \gamma N^*), \\ a_{44} &= \frac{2}{3}(\beta + \nu N^*), a_{12} = \left\{ r_1U^* - \frac{(\alpha + b)}{N^*} - \frac{rN_D}{K(T^*)} \right\}, \\ a_{13} &= \{\lambda - \gamma T + \pi \nu U - rN\eta(T)\}, a_{14} = (\gamma T - \nu U), \\ a_{23} &= \{-rN_DN\eta(T)\}, \\ a_{24} &= r_1(N - N_D), a_{34} = (\gamma + \pi \nu)N^* \end{aligned}$$

Thus $\frac{dW}{dt}$ will be negative definite provided

$$a_{12}^2 < a_{11}a_{22}, \quad (10.1)$$

$$a_{13}^2 < a_{11}a_{33}, \quad (10.2)$$

$$a_{14}^2 < a_{11}a_{44}, \quad (10.3)$$

$$a_{23}^2 < a_{22}a_{33}, \quad (10.4)$$

$$a_{24}^2 < a_{22}a_{44}, \quad (10.5)$$

$$a_{34}^2 < a_{33}a_{44}. \quad (10.6)$$

We note that (10.1) $\Rightarrow$ (4.6), (10.2) $\Rightarrow$ (4.7), (10.3) $\Rightarrow$ (4.8), (10.4) $\Rightarrow$ (4.9), (10.5) $\Rightarrow$ (4.10) and (10.6) $\Rightarrow$ (4.11). Hence, W is a Lyapunov function, whose domain is Ω , under the conditions (4.6)–(4.11) and therefore E_2 is globally asymptotically stable in Ω provided conditions (4.6)–(4.11) hold. Hence the theorem. $\square$

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