Mathematical study of a Leslie–Gower-type tritrophic population model in a polluted environment

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Abstract In this paper, the effect of toxicant on the dynamics of Leslie–Gower tritrophic food chain population model is studied. Two models have been studied to visualize the dynamical behaviour of prey and predator populations under toxicant stress. All the feasible equilibria of the systems are obtained and using stability theory the conditions are derived for the survival or extinction of species. From the analysis of the models it is observed that the effect of toxicant on predator population increases the equilibrium density of prey population. Finally, we support our analytical findings with numerical simulations.

Keywords Leslie–Gower-type · Food-chain · Toxicant · Equilibrium · Stability · Variational matrix

Mathematics Subject Classification 93A30 · 92D40 · 34D20

Introduction

The rapid development of modern industry and agriculture is polluting our environment and habitats due to which many species are on the verge of extinction. Organisms exposed to polluted environment take up toxicants and if the level of toxicant concentration exceeds the threshold value then undesirable consequences such as morbidity or

mortality in the exposed population may occur. So controlling the environmental pollution vis-a-vis the conservation of biodiversity is now the concern of almost all the countries. Hence, it is important to study the effects of toxicants on populations using mathematical models in order to find a theoretical threshold value of the organismal concentration of toxicant, so that morbidity or mortality in the population may be avoided.

Simple prey predator food chain models often display rich nonlinear mathematical behavior, including varying numbers and stability of equilibrium states and limit cycles (Jana et al. 2014; Haque et al. 2013; Holt and Barfield 2013; Gwaltney et al. 2007; Kooi and Hanegraaf 2001; Upadhyaya and Iyengar 2005; Xu et al. 2005). A number of studies have been carried out in recent years to study the dynamics of three trophic-level food chains comprising of prey, specialist predator and generalist predators, using mathematical models based on modified version of Leslie–Gower scheme (Aziz-Alaoui 2002; Rai and Upadhyay 2004; Hanski et al. 1991; Bandyopadhyay et al. 2008; Kumari 2013; Naji et al. 2010; Sharma and Samanta 2015; Upadhyay 2009, 2003). It is observed in the marine ecosystem that the top predator is generalist predator in the food chain system consisting of *phytoplankton* → *zooplankton* → *molluscs*. Also another example can be seen in the marine ecosystem where the top predator is specialist predator in the food chain system consisting of *phytoplankton* → *zooplankton* → *fish*. In the terrestrial ecosystem owl is specialist predator where as cat and fox are generalist predators in the food chain comprising of *rodent* → *owl* → *cat/fox*. Generally, the effects of toxicants are to decrease the growth rate of biological species as well as its carrying capacity. Several investigators have studied the effects of toxicants on biological species using mathematical models (Freedman and Shukla 1991; Hamilton 2013; Huang et al. 2013; Misra et al.

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2003; Dubey et al. 2003; Chattopadhyay et al. 2003; Khare et al. 2011).

Previously, some research work has been carried out on tri-trophic food-chain systems including toxicant effects on the survival or extinction of species using mathematical models (Misra and Raveendra Babu 2014, 2016; Hallam and Luna 1984; Thomann et al. 1974; Thomann and Connolly 1984). In Misra and Raveendra Babu (2014) the authors have studied the effects of toxicant in a three species food chain system with “food-limited” growth of prey population. In the model, the authors have assumed that the carrying capacity and growth rate of prey is affected by environmental toxicant which is being transferred to intermediate and top predator populations through food chain pathways. All the feasible equilibria of the system are obtained and the criteria for the survival or extinction of species under the effect of toxicant are derived. In Misra and Raveendra Babu (2016) the authors have considered a three-species food-chain with Holling type-II functional responses and discussed the system level effects of a toxicant. They have assumed that the presence of top predator reduces the predatory ability of the intermediate predator and also the growth equation for the prey population in the absence of predator is assumed to be governed by a modified smith-type differential equation. The stability analysis of the models are carried out and the sufficient conditions for the existence and extinction of the populations under the stress of toxicant are obtained. In Hallam and Luna (1984) the authors have proposed a mathematical model of a toxicant-population interaction and investigated their behavior. The population is assumed to be governed by a modification of Smith’s model. This modelling investigation discusses system level effects of a toxicant on a population when exposure is via both environmental and food chain pathways. The main analytical results yield conditions sufficient for persistence and conditions sufficient for extinction of the toxicant stressed population. The model in the paper Thomann et al. (1974) demonstrated the increase in concentration of potentially toxic substances, such as cadmium, as one proceeds up the food chain. The food chain model also illustrated how interactive modeling between complex non-linear and linear compartment model can be accomplished. The authors have observed when the toxicant (cadmium) is released into the water environment, the trophic level structure of the model indicates the general behavior. In Thomann and Connolly (1984) the authors have studied the age dependent food-chain species system under the exposure of PCB (poly chlorinated biphenyl) concentration in the Lake Michigan, and their model was capable of reproducing the age-dependent trends and magnitude of PCB contamination observed in Lake Michigan alewife and lake trout in 1971.

In view of the earlier mentioned biological examples, a leslie gower type three species food-chain model with specialist and generalist predators is proposed in this paper to study the survival or extinction of biological populations in a polluted environment. The model is analyzed qualitatively and the results are supported by numerical calculations.

The mathematical model

In this paper, the effects of toxicant on the dynamics of Leslie–Gower type tritrophic population model (Aziz-Alaoui 2002) is studied by considering the adverse impact of toxicant, first on prey population and then on intermediate (*specialist*) predator population. The main purpose of this paper is to compare the results of the system studied by Aziz-Alaoui (2002) when the toxicant is considered in the prey population and also in the case when the toxicant is present in the specialist predator. The underlying system consists of the state variables $X(t)$, the density of the prey population; $Y(t)$, the density of the intermediate (*specialist*) predator population; $Z(t)$, the density of the top (*generalist*) predator population; $C_O(t)$, the concentration of toxicant in the organism at time t ; $C_E(t)$, the concentration of toxicant in the environment of the population at time t . It may be noted that the environmental concentration $C_E(t)$ acts as the exogenous forcing for the concentration in the organism; $C_O(t)$ on account of uptake by the population (Hallam et al. 1983).

Model-A: (toxic effect in prey)

$$\begin{aligned}\frac{dX}{dt} &= a_0X - b_0X^2 - i_1XC_O - \frac{v_0XY}{d_0 + X} \\ \frac{dY}{dt} &= \frac{v_1XY}{d_1 + X} - a_1Y - \frac{v_2YZ}{d_2 + Y} \\ \frac{dZ}{dt} &= c_3Z^2 - \frac{v_3Z^2}{d_3 + Y} \\ \frac{dC_O}{dt} &= c_1XC_E - s_1C_O - s_2XC_O \\ \frac{dC_E}{dt} &= q_0 - s_3C_E - c_1XC_E + s_2XC_O\end{aligned}\quad (1)$$

with $X(0) = X_0 > 0, Y(0) = Y_0 > 0, Z(0) = Z_0 > 0, C_O(0) = 0, C_E(0) = C_{E0} > 0$.

The meaning of the parameters of the Model-A are given as follows: a_0 is the growth rate of prey, b_0 measures the strength of competition among individuals of species prey, v_0 is the maximum value which per capita reduction rate of prey can attain, d_0 measures the extent to which environment provides protection to prey, v_1 and d_1 have a similar meaning as v_0 and d_0 , a_1 represents the rate at which intermediate predator will die out when there is no prey, v_2 and v_3 have a similar biological connotation as that

of v_0 and v_1 , d_2 is the value of intermediate predator at which the per capita removal rate of intermediate predator becomes $v_2/2$, c_3 describes the growth rate of top predator, assuming that the number of males and females is equal, d_3 represents the residual loss in species top predator due to severe scarcity of its favorite food intermediate predator; the second term on the right-hand side in the third equation of (1), depicts the loss in the top-predator population. q_0 is the external input of toxicant into environment. i_1 is the death rate of prey due to organismal toxicant concentration. The toxicant are assumed to wash out from the environment as well as from the organism at rate s_3 and s_1 , respectively. c_1 is the removal rate of toxicant from environment due to uptake by the prey species. s_2 is the removal rate of toxicant from the prey species due to its death. It is assumed that the growth rate of uptake concentration of organism $C_O(t)$ increases by the same amount as the depletion rate of toxicant concentration $C_E(t)$ in the environment and is assumed to be proportional to the prey population as well as the environmental concentration of the toxicant (i.e., c_1XC_E). Also it is assumed that the uptake concentration of organism may decrease with rate coefficient $s_2 \geq 0$ due to decaying of prey population and which (i.e., s_2XC_O) may reenter into the environment (Shukla and Dubey 1997).

We can reduce the number of parameters in the system from 12 to 8 for analytical analysis by using the following transformations, but for our numerical analysis, we will continue to use the original system:

$$\begin{aligned} X &= \frac{a_0}{b_0}x, & Y &= \frac{a_0^2}{b_0v_0}y, & Z &= \frac{a_0^3}{b_0v_0v_2}z, & C_O &= \frac{a_0}{s_2}c_o, \\ C_E &= \frac{a_0}{c_1}c_e, & T &= \frac{t}{a_0}, \\ a &= \frac{b_0d_0}{a_0}, & b &= \frac{a_1}{a_0}, & c &= \frac{v_1}{a_0}, & d &= \frac{v_0b_0d_2}{a_0^2}, \\ p &= \frac{c_3a_0^2}{v_0b_0v_2}, & q &= \frac{v_3}{v_2}, \\ r &= \frac{v_0b_0d_3}{a_0^2}, & p_0 &= \frac{cq_0}{a_0^2}, & o_1 &= \frac{s_1}{a_0}, & o_2 &= \frac{s_3}{a_0}, \\ t_1 &= \frac{s_2}{b_0}, & u_1 &= \frac{c_1}{b_0}, & \beta_1 &= \frac{i_1}{s_2}. \end{aligned}$$

All these parameters, of course, assume only positive values. We hence forth assume, without loss of generality, that the environment provides an equal protection for species X and Y. That is, $d_0 = d_1$. Thus, system (1) after re-scaling becomes as follows:

Model 1:

$$\frac{dx}{dt} = x(1-x) - \beta_1xc_o - \frac{xy}{a+x} \quad (2)$$

$$\frac{dy}{dt} = \frac{cxy}{a+x} - by - \frac{yz}{d+y} \quad (3)$$

$$\frac{dz}{dt} = pz^2 - \frac{qz^2}{r+y} \quad (4)$$

$$\frac{dc_o}{dt} = t_1xc_e - o_1c_o - t_1xc_o \quad (5)$$

$$\frac{dc_e}{dt} = p_0 - o_2c_e - u_1xc_e + u_1xc_o \quad (6)$$

with $x(0) = x_0 > 0, y(0) = y_0 > 0, z(0) = z_0 > 0, c_o(0) = 0, c_e(0) = c_{e0} > 0$.

Model-B: (toxic effect in specialist predator)

$$\begin{aligned} \frac{dX}{dT} &= a_0X - b_0X^2 - \frac{v_0XY}{d_0+X} \\ \frac{dY}{dT} &= \frac{v_1XY}{d_1+X} - a_1Y - i_2YC_O - \frac{v_2YZ}{d_2+Y} \\ \frac{dZ}{dT} &= c_3Z^2 - \frac{v_3Z^2}{d_3+Y} \\ \frac{dC_O}{dT} &= c_1XC_E - s_1C_O - s_2XC_O \\ \frac{dC_E}{dT} &= q_0 - s_3C_E - c_1XC_E + s_2XC_O \end{aligned} \quad (7)$$

The state variables and parameters of the *Model-B* are the same as defined in the *Model-A*, except i_2 which defines the death rate of intermediate predator due to organismal toxicant concentration.

We can reduce the number of parameters in the system, even if, for our numerical tests, we will continue to use the original system:

$$\beta_2 = \frac{i_2}{s_2}, \quad t_2 = \frac{s_2a_0}{b_0v_0}, \quad u_2 = \frac{c_1a_0}{b_0v_0}, \quad p_1 = \frac{c_1q_0}{a_0^2}$$

Rest of the parameters are the same as defined for the *Model-A*. All these parameters, of course, assume only positive values. Thus, system (7) after re-scaling becomes as follows:

Model 2:

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{a+x} \quad (8)$$

$$\frac{dy}{dt} = \frac{cxy}{a+x} - \beta_2yc_o - by - \frac{yz}{d+y} \quad (9)$$

$$\frac{dz}{dt} = pz^2 - \frac{qz^2}{r+y} \quad (10)$$

$$\frac{dc_o}{dt} = t_2yc_e - o_1c_o - t_2yc_o \quad (11)$$

$$\frac{dc_e}{dt} = p_1 - o_2c_e - u_2yc_e + u_2yc_o \quad (12)$$

with $x(0) = x_0 > 0, y(0) = y_0 > 0, z(0) = z_0 > 0, c_o(0) = 0, c_e(0) = c_{e0} > 0$.

Analysis of Model 1

Boundedness of the Model 1

Lemma 3.1 *The positive cone $\text{Int}(R_+^5)$ is invariant for Model 1.*

Lemma 3.2 *Let ψ be an absolutely continuous function satisfying the differential inequality:*

$$\frac{d\psi(t)}{dt} + m_1\psi(t) \leq m_2, \quad t \geq 0 \quad (13)$$

where $(m_1, m_2) \in R^2$, $m_1 \neq 0$. Then, for all $t \geq T \geq 0$,

$$\psi(t) \leq \frac{m_2}{m_1} - \left(\frac{m_2}{m_1} - \psi(T) \right) e^{-m_1(t-T)}.$$

Theorem 3.1 *Let us assume*

$$c + \frac{c}{4b} + r < \frac{q}{p}, \quad (14)$$

and let Ω_1 be the set defined by:

$$\Omega_1 = \left\{ (x, y, z, c_o, c_e) \in R_+^5 : 0 \leq x \leq 1, 0 \leq x + \frac{y}{c} \leq 1 + \frac{1}{4b}, \right. \\ \left. x + \frac{y}{c} + \alpha z \leq 1 + \frac{1}{4b} + \frac{M}{b}, c_o + c_e \leq \frac{1 + p_0 t_1}{\theta_1} \right\},$$

where $\alpha = \frac{1}{b^2(c + \frac{c}{4b} + r)}$, $M = \frac{1}{4(q - (c + \frac{c}{4b} + r)p)}$ and $\theta_1 = \min\{o_1 u_1, t_1 o_2\}$.

Then

- (i) Ω_1 is positively invariant,
- (2i) all non-negative solutions (i.e., solutions initiating in R_+^5) of (2)–(6) are uniformly bounded forward in time, (thus they exist for all positive times), they eventually enter the attracting set Ω_1 .
- (3i) system (2)–(6) is dissipative.

Equilibria of Model 1

The Model 1 has following four non-negative equilibria in x, y, z, c_o and c_e space namely, $E_{10} = (0, 0, 0, 0, p_0)$, $E_{11} = (\bar{x}, 0, 0, \bar{c}_o, \bar{c}_e)$, $E_{12} = (\bar{x}, \bar{y}, 0, \bar{c}_o, \bar{c}_e)$ and $E_{13} = (\bar{x}, \bar{y}, \bar{z}, \bar{c}_o, \bar{c}_e)$. The existence of E_{10} is obvious. We prove the existence of E_{11} , E_{12} and E_{13} as follows:

– Existence of $E_{11}(\bar{x}, 0, 0, \bar{c}_o, \bar{c}_e)$, from (2),

$$\bar{c}_o = (1 - \bar{x})/\beta_1 \quad (15)$$

$\bar{c}_o > 0$ if $1 > \bar{x}$ from (5) and (6),

$$\bar{c}_e = [p_0 t_1 \beta_1 - o_1 u_1 (1 - \bar{x})]/(\beta_1 t_1 o_2) \quad (16)$$

$\bar{c}_e > 0$ if $p_0 t_1 \beta_1 > o_1 u_1 (1 - \bar{x})$. Using (15) and (16) in (5), we get

$$\bar{x}^2(o_1 u_1 + o_2 t_1) + \bar{x}[(p_0 t_1 \beta_1 + o_1 o_2) - (o_1 u_1 + o_2 t_1)] - o_1 o_2 = 0 \quad (17)$$

The Eq. (17) will have a positive root $\bar{x}$ always.

– Existence of $E_{12}(\bar{x}, \bar{y}, 0, \bar{c}_o, \bar{c}_e)$, from (3),

$$\bar{x} = ab/(c - b) \quad (18)$$

$\bar{x} > 0$ if $c > b$ from (2),

$$\bar{c}_o = \frac{1}{\beta_1} \left(1 - \bar{x} - \frac{\bar{y}(c - b)}{ac} \right) \quad (19)$$

$\bar{c}_o > 0$ if $1 > (\bar{x} + \frac{\bar{y}(c - b)}{ac})$, from (5) and (6),

$$\bar{c}_e = \frac{1}{t_1 o_2} \left[p_0 t_1 - \frac{o_1 u_1}{\beta_1} \left(1 - \bar{x} - \frac{\bar{y}(c - b)}{ac} \right) \right] \quad (20)$$

$\bar{c}_e > 0$ if $p_0 t_1 > o_1 u_1 \bar{c}_o$, from (5),

$$\bar{y} = (\bar{x} + a) \left[\frac{(o_1 u_1 \bar{x} + o_2(o_1 + \beta_1 \bar{x}))(1 - \bar{x}) - p_0 t_1 \beta_1 \bar{x}}{o_1 u_1 \bar{x} + o_2(o_1 + \beta_1 \bar{x})} \right] \quad (21)$$

$\bar{y} > 0$ if $(o_1 u_1 \bar{x} + o_2(o_1 + \beta_1 \bar{x}))(1 - \bar{x}) > p_0 t_1 \beta_1 \bar{x}$ and $1 > \bar{x}$.

– Existence of $E_{13}(\bar{x}, \bar{y}, \bar{z}, \bar{c}_o, \bar{c}_e)$, from (4),

$$\bar{y} = \frac{q}{p} - r \quad (22)$$

$\bar{y} > 0$ if $q > rp$, from (2),

$$c_o = \frac{1}{\beta_1} \left[1 - x - \frac{\bar{y}}{x - a} \right] = h_1(x) \quad (23)$$

from (5) and (6),

$$c_e = \frac{1}{t_1 o_2} [p_0 t_1 - o_1 u_1 h_1(x)] = h_2(x) \quad (24)$$

from (3),

$$z = (\bar{y} + d) \left[\frac{cx}{x + a} - b \right] = h_3(x) \quad (25)$$

Let

$$H(x) = p_0 - (o_2 + u_1 x)h_2(x) + u_1 x h_1(x) \quad (26)$$

Then we note that

$$H(0) = \frac{o_1 u_1}{t_1 \beta_1} \left(1 + \frac{\bar{y}}{a} \right) > 0 \quad (27)$$

and

$$H(k_0) = \frac{1}{\beta_1} \left[\frac{o_1 u_1}{t_1} \left(1 + \frac{k_0 u_1}{o_2} \right) + u_1 k_0 \right] \\ \left(1 - k_0 - \frac{\bar{y}}{k_0 - a} \right) - \frac{p_0 k_0 u_1}{o_2} < 0 \quad (28)$$

This guarantees the existence of a root of $H(x) = 0$ for $0 < x < k_0$, say $\bar{x}$. Further, this root will be unique provided

$$H'(x) = \frac{u_1}{\beta_1} \left(1 - x - \frac{\bar{y}}{x - a} \right) \left(1 - \frac{o_1 u_1}{t_1 o_2} \right) - \frac{u_1}{\beta_1} \left(1 - \frac{\bar{y}}{(x - a)^2} \right) \left(x + \frac{o_1(u_1 x + o_2)}{t_1 o_2} \right) - \frac{u_1 p_0}{o_2} < 0$$

Knowing the value of $\bar{x}$, the values of c_o , c_e and z can be computed from equations (23) to (25) respectively.

Stability of equilibria of Model 1

The local stability of equilibrium of Model 1 is determined by computing the eigenvalues of the variational matrix about the equilibrium points E_{10} , E_{11} , E_{12} and E_{13} . The following results are derived:

- The equilibrium point E_{10} is unstable.
- The equilibrium point E_{11} is locally stable if the following conditions are hold:

$$\ddot{x} < \frac{b_0 d_0}{v_1 - a_1} \quad (29)$$

and using Routh–Hurwitz criterion in Eq. (30), it is observed that E_{11} is locally stable provided that K_1 , K_2 and K_3 are positive and $K_1 K_2 > K_3$.

$$\lambda^3 + \lambda^2 K_1 + \lambda K_2 + K_3 = 0 \quad (30)$$

where, $K_1 = (o_1 + o_2) + \ddot{x}(1 + t_1 + u_1) > 0$, $K_2 = \ddot{x}(o_1 + o_2 + u_1 \ddot{x} + t_1 \ddot{x}) + (o_1 o_2 + o_1 u_1 \ddot{x} + o_2 t_1 \ddot{x}) + \ddot{x} \beta_1 t_1 (\ddot{c}_e - \ddot{c}_o) > 0$, $K_3 = \ddot{x}(o_1 o_2 + o_1 u_1 \ddot{x} + t_1 o_2 \ddot{x}) + \ddot{x} \beta_1 t_1 o_2 (\ddot{c}_e - \ddot{c}_o) > 0$.

Remark 1 From the stability of E_{11} , it may be concluded that only prey population will survive and predators will tend to extinction in the presence of toxicant.

- For the equilibrium point E_{12} , one eigenvalue of the characteristic equation is zero and the other four eigenvalues are given by the roots of the following equation

$$\lambda^4 + \lambda^3 Q_1 + \lambda^2 Q_2 + \lambda Q_3 + Q_4 = 0 \quad (31)$$

where, $Q_1 = (o_1 + o_2 + t_1 \ddot{x} + u_1 \ddot{x}) + \ddot{x}(1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) > 0$, $Q_2 = (o_1 o_2 + o_1 u_1 \ddot{x} + t_1 o_2 \ddot{x}) + \ddot{x} \beta_1 t_1 (\ddot{c}_e - \ddot{c}_o) + \frac{ac \ddot{y} \ddot{y}}{(\ddot{x} + a)^3} +$

$$\ddot{x}(o_1 + o_2 + t_1 \ddot{x} + u_1 \ddot{x})(1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) > 0, \quad Q_3 = \ddot{x}(o_1 o_2 + o_1 u_1 \ddot{x} + t_1 o_2 \ddot{x})(1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) + \ddot{x} \beta_1 t_1 o_2 (\ddot{c}_e - \ddot{c}_o) + (o_1 + o_2 + t_1 \ddot{x} + u_1 \ddot{x}) \frac{ac \ddot{y} \ddot{y}}{(\ddot{x} + a)^3} > 0,$$

$$Q_4 = (o_1 o_2 + o_1 u_1 \ddot{x} + t_1 o_2 \ddot{x}) \frac{ac \ddot{y} \ddot{y}}{(\ddot{x} + a)^3} > 0.$$

Using Routh–Hurwitz criterion in Eq. (31), it is observed that E_{12} is locally asymptotically stable provided that

$$\ddot{y} < (\ddot{x} + a)^2 \quad (32)$$

where $a = b_0 d_0 / a_0$, and $Q_1 Q_2 Q_3 > Q_3^2 + Q_1^2 Q_4$.

Remark 2 From (32), it may be noted that if the equilibria of intermediate predator is less than the square of the equilibria of prey and the ratio of the product of prey and the environment provides protection to prey and the growth of prey, then only prey and intermediate predator population will survive and the top predator will tend to extinction in the presence of toxicant.

- The characteristic equation about the equilibrium point E_{13} , given by

$$\lambda^5 + \lambda^4 U_1 + \lambda^3 U_2 + \lambda^2 U_3 + \lambda U_4 + U_5 = 0 \quad (33)$$

where, $U_1 = (o_1 + o_2 + t_1 \ddot{x} + u_1 \ddot{x}) + \ddot{x}(1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) - \frac{\bar{y} \ddot{z}}{(\ddot{y} + d)^2}$, $U_2 = (o_1 o_2 + o_1 u_1 \ddot{x} + t_1 o_2 \ddot{x}) - (o_1 + o_2 + t_1 \ddot{x} + u_1 \ddot{x}) [\ddot{x}(1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) + \frac{\bar{y} \ddot{z}}{(\ddot{y} + d)^2}] + \ddot{x} \beta_1 t_1 (\ddot{c}_e - \ddot{c}_o) + \frac{ac \ddot{y} \ddot{y}}{(\ddot{x} + a)^3} + \frac{\bar{y} \ddot{z}}{\ddot{y} + d} [\frac{\ddot{x}}{\ddot{y} + d} (1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) + \frac{q \ddot{z}}{(\ddot{y} + r)^2}]$, $U_3 = (o_1 + o_2 + t_1 \ddot{x} + u_1 \ddot{x}) [\frac{ac \ddot{y} \ddot{y}}{(\ddot{x} + a)^3} + \frac{\bar{y} \ddot{z}}{(\ddot{y} + d)^2} (\frac{q \ddot{z}(\ddot{y} + d)}{(\ddot{y} + r)^2} - \ddot{x}(1 - \frac{\bar{y}}{(\ddot{x} + a)^2}))] + (o_1 o_2 + o_1 u_1 \ddot{x} + t_1 o_2 \ddot{x}) [\ddot{x}(1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) - \frac{\bar{y} \ddot{z}}{(\ddot{y} + d)^2}] + \frac{q \ddot{y} \ddot{z}^2}{(\ddot{y} + d)(\ddot{y} + r)^2} (1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) + \ddot{x} \beta_1 t_1 (\ddot{c}_e - \ddot{c}_o) [o_2 - \frac{\bar{y} \ddot{z}}{(\ddot{y} + d)^2}]$, $U_4 = (o_1 + o_2 + t_1 \ddot{x} + u_1 \ddot{x}) \frac{q \ddot{y} \ddot{z}^2}{(\ddot{y} + d)(\ddot{y} + r)^2} (1 - \frac{\bar{y}}{(\ddot{x} + a)^2}) - \frac{\bar{y} \ddot{z} \beta_1 t_1 (\ddot{c}_e - \ddot{c}_o)}{(\ddot{y} + d)^2} [o_2 - \frac{q \ddot{z}(\ddot{y} + d)}{(\ddot{y} + r)^2}] + (o_1 o_2 + o_1 u_1 \ddot{x} + t_1 o_2 \ddot{x}) [\frac{ac \ddot{y} \ddot{y}}{(\ddot{x} + a)^3} + \frac{\bar{y} \ddot{z}}{(\ddot{y} + d)^2} (\frac{q \ddot{z}(\ddot{y} + d)}{(\ddot{y} + r)^2} - \ddot{x}(1 - \frac{\bar{y}}{(\ddot{x} + a)^2}))]$, $U_5 = [t_1 \beta_1 (\ddot{c}_e - \ddot{c}_o) + (o_1 o_2 + o_1 u_1 \ddot{x} + t_1 o_2 \ddot{x})(1 - \frac{\bar{y}}{(\ddot{x} + a)^2})] \frac{q \ddot{y} \ddot{z}^2}{(\ddot{y} + d)(\ddot{y} + r)^2}$. According to the Routh–Hurwitz criterion,

E_{13} is locally asymptotically stable provided that

$$U_i > 0, i = 1, 2, 3, 4, 5, U_1 U_2 U_3 > U_3^2 + U_1^2 U_4, \text{ and } (U_1 U_4 - U_5)(U_1 U_2 U_3 - U_3^2 - U_1^2 U_4) > U_5(U_1 U_2 - U_3)^2 + U_1 U_5^2$$

are satisfied.

Analysis of Model 2

Boundedness of the Model 2

It is observed that *Model 2* has same boundedness criteria like *Model 1*.

Theorem 4.1 *Let us assume*

$$c + \frac{c}{4b} + r < \frac{q}{p}, \quad (34)$$

and let Ω_2 be the set defined by:

$$\Omega_2 = \left\{ (x, y, z, c_o, c_e) \in R_+^5 : 0 \leq x \leq 1, 0 \leq x + \frac{y}{c} \leq 1 + \frac{1}{4b}, \right. \\ \left. x + \frac{y}{c} + \alpha z \leq 1 + \frac{1}{4b} + \frac{M}{b}, c_o + c_e \leq \frac{1 + p_0 t_1}{\theta_2} \right\},$$

where $\alpha = \frac{1}{b^2(c + \frac{c}{4b} + r)}$, $M = \frac{1}{4(q - (c + \frac{c}{4b} + r)p)}$ and $\theta_2 = \min\{o_1 u_1, t_1 o_2\}$.

Then

- (i) Ω_2 is positively invariant,
- (2i) all non-negative solutions (i.e., solutions initiating in R_+^5) of (8)–(12) are uniformly bounded forward in time, (thus they exist for all positive times), they eventually enter the attracting set Ω_2 .
- (3i) system (8)–(12) is dissipative.

Equilibria of Model 2

The *Model 2* has following four non-negative equilibria in x, y, z, c_o and c_e space namely, $E_{20}(0, 0, 0, 0, p_1)$, $E_{21}(\ddot{x}, 0, 0, \ddot{c}_o, \ddot{c}_e)$, $E_{22}(\tilde{x}, \tilde{y}, 0, \tilde{c}_o, \tilde{c}_e)$ and $E_{23}(\bar{x}, \bar{y}, \bar{z}, \bar{c}_o, \bar{c}_e)$. The existence of E_{20} is obvious. We prove the existence of E_{21} , E_{22} and E_{23} as follows:

- Existence of $E_{21}(\ddot{x}, 0, 0, \ddot{c}_o, \ddot{c}_e)$, from (8), (11) and (12), we obtain $\ddot{x} = 1$, $\ddot{c}_o = 0$ and $\ddot{c}_e = \frac{p_0}{o_2}$.

- Existence of $E_{22}(\tilde{x}, \tilde{y}, 0, \tilde{c}_o, \tilde{c}_e)$, from (9),

$$c_o = \frac{1}{\beta_2} \left[\frac{cx}{x+a} - b \right] = g_1(x) \quad (35)$$

from (8),

$$y = (1-x)(x+a) = g_2(x) \quad (36)$$

from (11) and (12),

$$c_e = \frac{1}{t_1 o_2} [t_1 p_0 - o_1 u_1 g_1(x)] = g_3(x) \quad (37)$$

Let,

$$G(x) = t_1 g_2(x) g_3(x) - (o_1 + t_1 g_2(x)) g_1(x) \quad (38)$$

Then we note that $G(0) = at_1 \left(\frac{p_0 t_1 \beta_2 + o_1 u_1 b}{t_1 o_2 \beta_2} \right) + (o_1 + t_1 a) \frac{b}{\beta_2} > 0$ and $G(k_0) = (1 - k_0)(a + k_0) \frac{t_1 p_0}{o_2} - \frac{1}{\beta_2} \left(\frac{ck_0}{a+k_0} - b \right) [o_1 + (1 - k_0)(a + k_0)(t_1 + \frac{o_1 u_1}{o_2})] < 0$. This guarantees the existence of a root of $G(x) = 0$ for $0 < x < k_0$, say $\tilde{x}$. Further, this root will be unique provided

$$G'(x) = \frac{aco_1 u_1}{o_2 \beta_2 (x+a)^2} [(1+a) + (x-1)(x+a)] \\ + \frac{t_1(1+a)}{\beta_2} \left(\frac{cx}{x+a} - b \right) \\ - \frac{ac[o_1 + t_1(1-x)(a+x)]}{\beta_2 (x+a)^2} < 0$$

Knowing the value of $\tilde{x}$, the values of c_o, y and c_e can be computed from equations (35) to (37) respectively.

- Existence of $E_{23}(\bar{x}, \bar{y}, \bar{z}, \bar{c}_o, \bar{c}_e)$, from (10),

$$\bar{y} = \frac{q}{p} - r = j_1(\text{say}) > 0 \quad (39)$$

$\bar{y} > 0$ if $\frac{q}{p} > r$, from (8),

$$\bar{x}^2 + \bar{x}(a-1) - (a-j_1) = 0 \quad (40)$$

root of (40) is j_2 (say), always positive. from (9),

$$c_o = \frac{1}{\beta_2} \left(\frac{cj_2}{a+j_2} - b - \frac{z}{d+j_1} \right) = f_1(z) \quad (41)$$

from (11) and (12),

$$c_e = \frac{1}{t_1 o_2} (p_0 t_1 - o_1 u_1 f_1(z)) = f_2(z) \quad (42)$$

Let

$$F(z) = t_1 j_1 f_2(z) - (o_1 + t_1 j_1) f_1(z) \quad (43)$$

Then we note that

$$F(0) = \frac{p_0 t_1 j_1}{o_2} - \frac{1}{\beta_2} \left(\frac{cj_2}{a+j_2} - b \right) \\ \left(o_1 + t_1 j_1 + \frac{o_1 u_1 j_1}{o_2} \right) > 0$$

and

$$F(k_0) = \frac{p_0 t_1 j_1}{o_2} - \frac{1}{\beta_2} \left(\frac{cj_2}{a+j_2} - b - \frac{k_0}{d+j_1} \right) \\ \left(o_1 + t_1 j_1 + \frac{o_1 u_1 j_1}{o_2} \right) < 0$$

This guarantees the existence of a root of $F(z) = 0$ for $0 < z < k_0$, say $\bar{z}$. Further, this root will be unique provided

$$F'(z) = \frac{o_1 u_1 j_1}{o_2 \beta_2 (d+j_1)} + \frac{(o_1 + t_1 j_1)}{\beta_2 (d+j_1)} > 0$$

Knowing the value of $\bar{z}$, the values of c_o and c_e can be computed from Eqs. (41) to (42) respectively.

Stability of equilibria of Model 2

The local stability of equilibrium of Model 2 is determined by computing the eigenvalues of the Jacobian matrix about the equilibrium points E_{20}, E_{21}, E_{22} and E_{23} . The following results are obtained:

- The equilibrium point E_{20} is unstable.
- For the equilibrium point E_{21} , the eigenvalues of the characteristic equation are $0, -1, \frac{c}{1+a} - b, -o_1$ and $-o_2$, which shows E_{21} is locally stable if $c < b(1+a)$ which implies

$$a_0 v_1 < a_1(a_0 + b_0 d_0) \quad (44)$$

hold good.

Remark 3 From the stability of E_{21} , it may be concluded that only prey population will survive and the predators will tend to extinction.

- For the equilibrium point E_{22} , one eigenvalue of the characteristic equation is zero and the other four eigenvalues are given by the roots of the following equation

$$\lambda^4 + \lambda^3 V_1 + \lambda^2 V_2 + \lambda V_3 + V_4 = 0 \quad (45)$$

where, $V_1 = (o_1 + o_2 + t_1 \bar{y} + u_1 \bar{y}) + \bar{x}(1 - \frac{\bar{y}}{(\bar{x}+a)}) > 0$, $V_2 = (o_1 o_2 + u_1 o_1 \bar{y} + t_1 o_2 \bar{y}) + \bar{x}(o_1 + o_2 + t_1 \bar{y} + u_1 \bar{y})(1 - \frac{\bar{y}}{(\bar{x}+a)}) + t_1 \beta_2 \bar{y}(\bar{c}_e - \bar{c}_o) + \frac{ac \bar{y}}{(\bar{x}+a)^3} > 0$, $V_3 = o_2 t_1 \beta_2 \bar{y}(\bar{c}_e - \bar{c}_o) + (o_1 + o_2 + t_1 \bar{y} + u_1 \bar{y}) \frac{ac \bar{y}}{(\bar{x}+a)^3} + \bar{x}(o_1 o_2 + u_1 o_1 \bar{y} + t_1 o_2 \bar{y})(1 - \frac{\bar{y}}{(\bar{x}+a)}) + \bar{x} t_1 \beta_2 (\bar{c}_e - \bar{c}_o)(1 - \frac{\bar{y}}{(\bar{x}+a)}) > 0$, $V_4 = \bar{x} \bar{y} t_1 o_2 \beta_2 (\bar{c}_e - \bar{c}_o)(1 - \frac{\bar{y}}{(\bar{x}+a)}) + (o_1 o_2 + u_1 o_1 \bar{y} + t_1 o_2 \bar{y}) \frac{ac \bar{y}}{(\bar{x}+a)^3} > 0$. Using Routh–Hurwitz criterion in Eq. (45), it

is observed that E_{22} is locally asymptotically stable provided that

$$\bar{y} < (\bar{x} + a)^2, \quad (46)$$

where $a = b_0 d_0 / a_0$, and $V_1 V_2 V_3 > V_3^2 + V_1^2 V_4$.

Remark 4 From (46), the similar result can be stated here as mentioned in Remark 2.

- For the characteristic equation of the equilibrium point E_{23} , is written as:

$$\lambda^5 + \lambda^4 R_1 + \lambda^3 R_2 + \lambda^2 R_3 + \lambda R_4 + R_5 = 0 \quad (47)$$

where, $R_1 = (o_1 + o_2 + t_1 \bar{y} + u_1 \bar{y}) + \bar{x}(1 - \frac{\bar{y}}{(\bar{x}+a)}) - \frac{\bar{y} \bar{z}}{(\bar{y}+d)^2}$, $R_2 = (o_1 o_2 + o_1 u_1 \bar{y} + t_1 o_2 \bar{y}) + t_1 \beta_2 (\bar{c}_e - \bar{c}_o) + \frac{ac \bar{x}}{(\bar{x}+a)^3} + \bar{x}(o_1 + o_2 + t_1 \bar{y} + u_1 \bar{y})(1 - \frac{\bar{y}}{(\bar{x}+a)}) - (o_1 + o_2 + t_1 \bar{y} + u_1 \bar{y}) \frac{\bar{y} \bar{z}}{(\bar{y}+d)^2} + \frac{\bar{y} \bar{z}}{\bar{y}+d} [\frac{q \bar{z}}{(\bar{y}+r)^2} - \frac{\bar{x}}{\bar{y}+d} (1 - \frac{\bar{y}}{(\bar{x}+a)})]$, $R_3 = (o_1 + o_2 + t_1 \bar{y} + u_1 \bar{y}) [\frac{q \bar{y} \bar{z}^2}{(\bar{y}+d)(\bar{y}+r)^2} + \frac{ac \bar{x}}{(\bar{x}+a)^3} - \bar{x}(1 - \frac{\bar{y}}{(\bar{x}+a)})] + (o_1 o_2 + o_1 u_1 \bar{y} + t_1 o_2 \bar{y}) [\bar{x}(1 - \frac{\bar{y}}{(\bar{x}+a)}) - \frac{\bar{y} \bar{z}}{(\bar{y}+d)^2}] + t_1 \beta_2 \bar{y}(\bar{c}_e - \bar{c}_o) [o_2 - \bar{x}(1 - \frac{\bar{y}}{(\bar{x}+a)})] + \frac{q \bar{x} \bar{y} \bar{z}^2}{(\bar{y}+d)(\bar{y}+r)^2} (1 - \frac{\bar{y}}{(\bar{x}+a)})$, $R_4 = \bar{x} \bar{y} (1 - \frac{\bar{y}}{(\bar{x}+a)}) [t_1 \beta_2 o_2 (\bar{c}_e - \bar{c}_o) + (o_1 + o_2 + t_1 \bar{y} + u_1 \bar{y}) \frac{q \bar{z}^2}{(\bar{y}+d)(\bar{y}+r)^2}] + \bar{y} (o_1 o_2 + o_1 u_1 \bar{y} + t_1 o_2 \bar{y}) [\frac{ac \bar{x}}{(\bar{x}+a)^3} + \frac{q \bar{z}^2}{(\bar{y}+d)(\bar{y}+r)^2} - \frac{\bar{x} \bar{z}}{(\bar{y}+d)^2} (1 - \frac{\bar{y}}{(\bar{x}+a)})]$, $R_5 = (o_1 o_2 + o_1 u_1 \bar{y} + t_1 o_2 \bar{y}) (1 - \frac{\bar{y}}{(\bar{x}+a)}) \frac{q \bar{x} \bar{y} \bar{z}^2}{(\bar{y}+d)(\bar{y}+r)^2}$.

According to the Routh–Hurwitz criterion, E_{23} is locally asymptotically stable provided that

$$R_i > 0, i = 1, 2, 3, 4, 5, R_1 R_2 R_3 > R_3^2 + R_1^2 R_4, \text{ and } (R_1 R_4 - R_5)(R_1 R_2 R_3 - R_3^2 - R_1^2 R_4) > R_5(R_1 R_2 - R_3)^2 + R_1 R_5^2$$

are satisfied.

Numerical simulations

In this section, we demonstrate the dynamical behaviour of the effect of toxicant on a Leslie–Gower-type three species food chain, when the effect of toxicant on prey and when the effect of toxicant on intermediate (*specialist*) predator with the help of numerical simulations to facilitate the interpretation of our mathematical findings. The figures illustrate the stability behaviour of all the equilibrium points of the models for the given sets of parameters and the graphs have been plotted with the help of MATLAB.

Numerical simulation for Model 1

We choose the following values of parameters for E_{11} :

$$\begin{aligned} a_0 &= 0.062, & b_0 &= 0.063, & a_1 &= 1.0, & c_3 &= 0.038, \\ v_0 &= 1.0, & v_1 &= 2.0, \\ v_2 &= 0.405, & v_3 &= 1.0, & d_0 &= 10, & d_1 &= 10, \\ d_2 &= 10, & d_3 &= 0.01, \\ t_1 &= 0.85, & t_1 &= 0.45, & o_1 &= 3.938, & u_1 &= 0.305, \\ o_2 &= 0.72, & p_0 &= 0.4. \end{aligned}$$

Fig. 1 Time series graph for the Model 1 around the equilibrium point $E_{11}^{\ddot{}}$ showing the stability behavior

It is found that under the above set of parameters, the equilibrium point $E_{11}^{\ddot{}}$

$$\ddot{x} = 0.597, \quad \ddot{y} = 0.0000, \quad \ddot{z} = 0.0000, \\ \ddot{c}_o = 0.0287, \quad \ddot{c}_e = 0.4492$$

is locally asymptotically stable (see Fig. 1).

We choose the following values of parameters for $E_{12}^{\ddot{}}$:

$$a_0 = 0.42, \quad v_0 = 2.5, \quad v_1 = 4.5, \\ v_3 = 4.8, \quad d_3 = 0.001.$$

With the above values of parameters and taking the remaining parameters to be the same as considered for $E_{11}^{\ddot{}}$, it is found that under the above set of parameters the equilibrium point $E_{12}^{\ddot{}}$

$$\ddot{x} = 2.8571, \quad \ddot{y} = 0.9658, \quad \ddot{z} = 0.0000, \\ \ddot{c}_o = 0.0715, \quad \ddot{c}_e = 0.2905$$

is locally asymptotically stable (see Fig. 2).

We choose the following values of parameters for $E_{13}^{\ddot{}}$:

$$a_0 = 0.97, \quad c_3 = 0.04, \quad d_3 = 20.$$

With the above values of parameters and taking the remaining parameters to be the same as considered for $E_{11}^{\ddot{}}$, it is found that under the above set of parameters, the equilibrium point $E_{13}^{\ddot{}}$

$$\ddot{x} = 10.7196, \quad \ddot{y} = 5.0000, \quad \ddot{z} = 1.2863, \\ \ddot{c}_o = 0.1006, \quad \ddot{c}_e = 0.1827$$

is locally asymptotically stable (see Fig. 3).

Fig. 2 Time series graph for the Model 1 around the equilibrium point $E_{12}^{\ddot{}}$ showing the stability behavior

Fig. 3 Time series graph for the Model 1 around the equilibrium point $E_{13}^{\ddot{}}$ showing the stability behavior

Numerical simulation for Model 2

We choose the following values of parameters for $E_{21}^{\ddot{}}$:

$$a_0 = 0.062, \quad b_0 = 0.06, \quad a_1 = 1.0, \quad c_3 = 0.038, \\ v_0 = 1.0, \quad v_1 = 2.0, \\ v_2 = 0.405, \quad v_3 = 1.0, \quad d_0 = 10, \quad d_1 = 10, \quad d_2 = 10, \\ d_3 = 0.01, \\ t_2 = 0.85, \quad t_1 = 0.45, \quad o_1 = 3.938, \quad u_1 = 0.305, \\ o_2 = 0.72, \quad p_0 = 0.4.$$

It is found that under the above set of parameters, the equilibrium point $E_{21}^{\ddot{}}$

Fig. 4 Time series graph for the Model 2 around the equilibrium point E_{21} showing the stability behavior

$$\ddot{x} = 1.0000, \quad \ddot{y} = 0.0000, \quad \ddot{z} = 0.0000, \quad \ddot{c}_o = 0.0422, \\ \ddot{c}_e = 0.3992,$$

is locally asymptotically stable (see Fig. 4).

We choose the following values of parameters for E_{22} :

$$a_0 = 0.42, \quad v_0 = 2.5, \quad v_1 = 4.5, \\ v_3 = 4.8, \quad d_3 = 0.001.$$

With the above values of parameters and taking the remaining parameters to be the same as considered for E_{21} , it is found that under the above set of parameters, the equilibrium point E_{22}

$$\tilde{x} = 3.0915, \quad \tilde{y} = 1.2280, \quad \tilde{z} = 0.0000, \quad \tilde{c}_o = 0.0737, \\ \tilde{c}_e = 0.2823,$$

is locally asymptotically stable (see Fig. 5).

We choose the following values of parameters for E_{23} :

$$a_0 = 0.97, \quad c_3 = 0.0403, \quad d_3 = 20.$$

With the above values of parameters and taking the remaining parameters to be the same as considered for E_{21} , it is found that under the above set of parameters, the equilibrium point E_{23}

$$\bar{x} = 12.6197, \quad \bar{y} = 4.8139, \quad \bar{z} = 1.0379, \\ \bar{c}_o = 0.1030, \quad \bar{c}_e = 0.1741,$$

is locally asymptotically stable (see Fig. 6).

Fig. 5 Time series graph for the Model 2 around the equilibrium point E_{22} showing the stability behavior

Fig. 6 Time series graph for the Model 2 around the equilibrium point E_{23} showing the stability behavior

We choose the following values of parameters for without toxicant case:

$$a_0 = 0.97, \quad b_0 = 0.06, \quad a_1 = 1.0, \quad c_3 = 0.0398, \\ v_0 = 1.0, \quad v_1 = 2.0, \\ v_2 = 0.405, \quad v_3 = 1.0, \quad d_0 = 10, \quad d_1 = 10, \\ d_2 = 10, \quad d_3 = 20,$$

It is found that under the above set of parameters, the equilibrium point for without toxicant case:

$$x = 12.3433, \quad y = 5.1258, \quad z = 3.9168,$$

is locally asymptotically stable (see Fig. 7).

Fig. 7 Time graph of models with and without toxicant for the equilibrium points E_{13}^- and E_{23}^- respectively, showing the stability behavior

Conclusion

In this paper we have proposed and analyzed a nonlinear mathematical model for a Leslie-Gower type three species food-chain system with specialist and generalist predators under the stress of toxicant. The effects of toxicant on the dynamics of Leslie-Gower type tritrophic population model (Aziz-Alaoui 2002) is studied by considering the adverse impact of toxicant, first on prey population and then on intermediate (*specialist*) predator population. The main purpose of this paper is to compare the results of the system studied by Aziz-Alaoui (2002) when the toxicant is considered in the prey population and also in the case when the toxicant is present in the specialist predator.

The results obtained in the present paper: From the stability analysis of E_{11}^- and E_{21}^- , it is concluded that only prey population will survive and predator populations would tend to extinction (see Figs. 1, 4). From the stability of E_{12}^- and E_{22}^- , it is concluded that the only prey and intermediate predator populations will survive in the presence of toxicant, and the top predator population would tend to extinction, but their equilibrium levels decrease due

to toxicant (see Figs. 2, 5). In the case of equilibrium point E_{22}^- , it is observed that the prey population increases due to the presence of toxicant in intermediate predator population (see Fig. 5). The stability of E_{13}^- of Model 1 and E_{23}^- of Model 2 ensures the coexistence of all the three populations in the presence of toxicant, however their equilibrium levels decrease on account of polluted environment (see Figs. 3, 6, 7).

From Fig. 7a, it is observed that the equilibrium level of prey population without the effect of toxicant is less than the equilibrium level of prey when the effect of toxicant is considered in intermediate predator. This shows that the effect of toxicant on predator population increases the equilibrium density of prey population. From Fig. 7b, it is observed that the equilibrium level of intermediate predator population without the effect of toxicant is more than the equilibrium level of intermediate predator when the effect of toxicant is considered in prey population. This shows that the effect of toxicant on prey population decreases the equilibrium density of intermediate predator population. From Fig. 7c, it is observed that the equilibrium level of top predator population without the effect of toxicant is more

than the equilibrium level of top predator when the effect of toxicant is considered in prey population. This shows that the effect of toxicant on prey population decreases the equilibrium density of top predator population.

The following points may be noted when we compare our results with the results of the paper by Aziz-Alaoui (2002): (1) similar observations are noted from this paper and the paper of Aziz-Alaoui (2002) regarding the stability of zero density equilibrium point with and without toxicant. (2) In the paper of Aziz-Alaoui (2002) it is shown that the equilibrium point having positive density of prey and zero densities of predators is unstable but whereas in this paper it is shown that the same equilibrium point (positive prey density and zero predator densities) becomes stable after the introduction of toxicant in the original system (Aziz-Alaoui 2002). (3) The equilibrium point of the model given in Aziz-Alaoui (2002) and the equilibrium points of the models considered in this paper, where the prey and intermediate predator densities are taken to be non zero and top predator density is taken as zero are shown to exhibit same stability behaviour.

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