A mathematical model for the conservation of forestry biomass with an alternative resource for industrialization: a modified Leslie Gower interaction

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Abstract In this paper we have considered an age structure forestry biomass characterized as pre-mature and mature population stages and industrialization as a state variable. Industries prefer to harvest trees as its favorite raw material to be used, but it may depend on the alternatives in the shortage or low growth of mature trees, based on this assumption a modified Leslie–Gower interaction is considered for industrial growth. We have obtained sufficient conditions for the persistence and global attractivity of the system by applying the differential inequality theory. Theory of differential equation is used to establish the stability of the system. Pontryagin’s principle is applied for the optimal control and found the solution in the interior equilibrium. For the verification of analytic results we have done numerical simulation, sensitivity of parameters involved in interior equilibrium is also examined.

Keywords Age structure model · Modified Leslie–Gower term · Optimal control · Sensitivity analysis

Mathematics Subject Classification 34D · 34H · 90A · 92B

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Introduction

The concept of modeling through rates of change probably originated with Isaac Newton in the 17th Century, and is well known in physics and engineering. The idea behind the generalization is to obtain the system through a suitable number of differential equations with considered state variable, outputs are represented as the functions of current state (Garcia 2013). Biotic and abiotic both are dependent component to each other and their balance is essential for the sustainability and stability of the world. Forests are essential to life on earth, to mitigating and adapting to climate change, ensuring adequate supply of fresh water, enhancing biodiversity and providing sustainable incomes and livelihoods, including food security. But it’s face unprecedented and unrelenting pressures (www.fao.org/home/en). The year 2011 was declared the International year of “Forests” by the United Nations to raise awareness and strengthen the sustainable management, conservation and sustainable development of all types of forests for the benefit of present and future generations (<http://www.un.org/en/events/iyof2011/>). In article (Khaine and Woo 2015) the author reviews both the effect of deforestation on climate change and the impact of climate change on forests. They also mention that the annual rate of deforestation is 0.14 % per year with 2.3 million square kilometers lost between 2000 and 2012. At present forests cover is about 31 % of the world’s area (FAO et al. 2010). Since past four-five decades a lot of work is done for the conservation of fishery based on different harvesting and optimal control strategies by many researchers, the first work on fishery conservation based on optimal control is done by Clark in the year (1976) which is followed by Chaudhuri (1988). As fishery, forestry, conservation is also a key factor which is responsible for the stability of the ecosystem, in my view

very little attention has paid to optimal control for the exploitation and conservation of forestry resources mainly tree biomass.

The dynamics between the forestry biomass and forestry resource based Industrialization is similar to the predator-prey interaction. A Leslie Gower predator prey model is introduced by Leslie (1948) and the formulation of Leslie–Gower is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food, whereas for a modified Leslie Gower functional response in the case of severe scarcity, predator can switch over to other population (or alternative) but its growth will be limited by the fact that its most favorite food prey is not available in abundance (Leslie 1948, 1958; Aziz-Alaoui 2002; Aziz-Alaoui and Okiye 2003; Chen and Chen 2009; Zhang et al. 2011). Later on in Leslie Gower term a positive parameter is added to the denominator, measures the extent to which environment provides protection to prey species and the replaced Leslie Gower term is called modified Leslie Gower term. The present paper is based on the literature of modified Leslie–Gower functional response (Kar and Ghorai 2011; Gupta and Chandra 2013; Yue 2015), forestry population model (Shukla et al. 1988; Dhar 2004; Dubey et al. 2009; Chaudhary and Dhar 2013; Chaudhary et al. 2013; Misra et al. 2014; Misra and Lata 2015), optimal control policy (one applied for the fishery management control policy (Clark 2010; Kar and Ghorai 2011; Devi 2012; Liu and Xing 2013; Gupta and Chandra 2013; Dubey et al. 2014; Ghosh and Kar 2014).

The basic concept behind this research study is to analyze a dynamical model to conserve forestry which is being depleted due to harvesting, increasing forestry based industries, a climatic factor (results natural depletion). Here for this, we have taken an age structured forestry biomass via pre-mature and mature forestry population and it is considered that the industries are restricted, not to harvest pre-mature trees. Further, by taking Modified Leslie–Gower response function is also introduced as the alternative resource for industries when forestry resources are devastated.

Basic assumptions

Modelling is an effective tool to develop decision support to interpret the real word problem. Following are the modeling assumption, consider to portrait system dynamics:

- (i) Forestry population, i.e., trees are categorized via pre-mature (P) and mature trees (M).

- (ii) Premature trees grow logistically with fixed growth rate and carrying capacity.
- (iii) Alternative are provided to industries for the conservation of forest biomass.
- (iv) Industries uses forestry based mature trees and may use alternatives by forestry when there is less forestry biomass available as per their demand.
- (v) The natural washout rate for mature trees is considered, because of natural calamity.
- (vi) Efforts are expanded to harvest mature trees.
- (vii) Total revenue is obtained from the difference of pricing and cost applied to harvest mature forestry trees.

The conceptual model for the schematic diagram is shown in Fig. 1, will be discussed in “Conceptual model: system of differential equation” section and description of parameters for the system is given in table 1:

Conceptual model: system of differential equation

We have proposed an age-structured forestry biomass via premature trees (P) and mature trees (M) and industrialization (I). The basic assumptions of the proposed model are defined in “Basic assumptions” section. Function $f(P)$ is a combination of logistic growth and new plantation for pre-mature trees, defined as $f(P) = rP(1 - \frac{P}{k}) + \gamma P$, similarly $f_1(M, I)$ is the function of modified Leslie Gower term for mature trees and industrialization, defined as $f_1(M, I) = I(\alpha_1 - \frac{\alpha_2 I}{\alpha_3 + M})$. Here for the proposed dynamical system we have taken E as the effort expanded to harvest mature trees. On the basis of the above description and the schematic diagram (see Fig. 1), we formulated the following dynamical system:

Fig. 1 Schematic diagram

Table 1 Description of parameters

Parameter	Description
r	Intrinsic growth rate of premature trees
k	Carrying capacity of forestry biomass
γ	New plantation
β	Transition rate from pre-mature to mature
α_1	Measures the maximum reduction rate of industries can attain
α_2	Maximum value for the reduced rate of industrialization
$\frac{1}{\alpha_3}$	Average rate at which government provides protection to forestry population
c	Cost per unit of effort expended to harvest forestry biomass
q_1	Depletion rates of mature trees
p	Price per unit of harvesting of mature trees
d_1	Natural depletion rate of mature trees
d_2	Decrease in industrialization in the absence of preferred mature trees
E	Harvesting efforts

$$\begin{aligned} \frac{dP}{dt} &= f(P) - \beta P, \\ \frac{dM}{dt} &= \beta P - q_1 EM - d_1 M, \\ \frac{dI}{dt} &= f_1(M, I) - d_2 I, \end{aligned} \quad (1)$$

where $P(0) \geq 0$, $M(0) \geq 0$, $I(0) \geq 0$. Substituting the values of functions $f(P)$ and $f_2(M, I)$, the model equations are as below

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{k} \right) - \beta P + \gamma P, \quad (2)$$

$$\frac{dM}{dt} = \beta P - q_1 EM - d_1 M, \quad (3)$$

$$\frac{dI}{dt} = \left(\alpha_1 - \frac{\alpha_2 I}{\alpha_3 + M} \right) I - d_2 I. \quad (4)$$

In the next section, we will study the positivity and boundedness of the system.

Positivity, system boundedness and permanence of system

Here we will establish the positivity and boundedness of solutions as well as the permanence of the system (2)–(3). We will use the following lemma to prove the boundedness and the permanence of the system (3).

Lemma 1 (Chen 2005) *If $a, b > 0$ and $\frac{dX}{dt} \leq (\geq) X(t)(a - bX(t))$ with $X(0) > 0$, then $\limsup_{t \rightarrow +\infty} X(t) \leq$*

$$\frac{a}{b} \left(\liminf_{t \rightarrow +\infty} X(t) \geq \frac{a}{b} \right).$$

In fact, the above lemma is quantitatively equivalent to the following lemma.

Lemma 2 *If $a, b > 0$ and $\frac{dX}{dt} \leq (\geq) X(t)(a - bX(t))$ with $X(0) > 0$, then for all $t \geq 0$, $X(t) \leq \frac{a}{b - C \exp(-at)}$, with $C = b - \frac{a}{X(0)}$. In particular $X(t) \leq \max\{X(0), \frac{a}{b}\}$ for all $t \geq 0$.*

Positivity, system boundedness

Theorem (a): *All solutions of system (3) with initial conditions are always positive, i.e., $P(t) > 0, M(t) > 0$ and $I(t) > 0$ for all $t \geq 0$.*

Proof (a). Theorem is true since,

$$P(t) = P(0) \exp \int_0^t \left[r \left(1 - \frac{P(s)}{k} \right) - \beta_1 + \gamma \right] ds,$$

$$M(t) = \beta \exp(-q_1 E - d_1 t) \int_0^t (P(s) \exp(q_1 E + d_1 s)) ds$$

and

$$I(t) = I(0) \exp \int_0^t \left[\alpha_1 - d_2 - \frac{\alpha_2 I(s)}{\alpha_3 + M(s)} \right] ds.$$

Before analyzing the model we prove the following lemma corresponding to the region of attraction for solutions of system (3).

Theorem (b): *All solutions of system (3) with initial conditions are always bounded, for all $t \geq 0$. From Eq. (2) we can write*

$$\frac{dP}{dt} \leq rP \left(1 - \frac{P}{k} \right) + \gamma P. \quad (5)$$

From Lemma 2, we have $x(t) \leq \max\{P(0), \frac{(r+\gamma)k}{r}\} \equiv M_1$ for all $t \geq 0$.

Further, from Eq. (3), we have

$$\frac{dM}{dt} = \beta P - q_1 ME - d_1 M \leq \beta M_1 - q_1 ME - d_1 M. \quad (6)$$

Again from Lemma 2, we can write $M(t) \leq \max\{M(0), \frac{\beta M_1}{q_1 E + d_1}\} \equiv M_2$ for all $t \geq 0$.

Further, from Eq. (4), we have

$$\frac{dI}{dt} = I \left(\alpha_1 - \frac{\alpha_2 I}{\alpha_3 + M} \right) \leq I \left(\alpha_1 - \frac{\alpha_2 I}{\alpha_3 + M_2} \right). \quad (7)$$

Again using Lemma 2, we get $I(t) \leq \max\{I(0), \frac{\alpha_1(\alpha_3 + M_2)}{\alpha_2}\} \equiv M_3$ for all $t \geq 0$.

This completes the proof for the boundedness of the system and hence the system under consideration is dissipative.

Permanence

From the definition of permanence (Pal et al. 2011)

Definition System (2)–(4) with initial condition is permanent if there exist positive constant C_1 and C_2 ($0 < C_1 < C_2$) such that each positive solution $(P(t, P_0, M_0, I_0), M(t, P_0, M_0, I_0), I(t, P_0, M_0, I_0))$ of system (2)–(4) with initial condition $(P_0, M_0, I_0) \in \text{Int}(R_+^3)$ satisfies,

$$\begin{aligned} & \min\{\liminf_{t \rightarrow \infty} P(t, P_0, M_0, I_0), \liminf_{t \rightarrow \infty} M(t, P_0, M_0, I_0), \\ & \liminf_{t \rightarrow \infty} I(t, P_0, M_0, I_0)\} \geq C_1 \\ & \max\{\limsup_{t \rightarrow \infty} P(t, P_0, M_0, I_0), \limsup_{t \rightarrow \infty} M(t, P_0, M_0, I_0), \\ & \limsup_{t \rightarrow \infty} I(t, P_0, M_0, I_0)\} \leq C_2. \end{aligned}$$

Theorem (c): The system of Eqs. (2)–(4) with initial condition is permanent if $\limsup_{t \rightarrow \infty} P(t) \leq M_1$, $\limsup_{t \rightarrow \infty} M(t) \leq M_2$ and $\limsup_{t \rightarrow \infty} I(t) \leq M_3$.

Proof c. From Eq. (5) and Lemma 1 it is clear that $0 < P(t) < \frac{(r+\gamma)k}{r}$, from Eq. (6) and Lemma 1 we get $M(t) \leq \frac{\beta M_1}{q_1 E + d_1}$ and from Eq. (7) and Lemma 1 we have $0 < I(t) < \frac{\alpha_1(\alpha_3 + M_2)}{\alpha_2}$ for sufficient large t .

Hence from the Eq. (3), we can write $\frac{dP}{dt} \geq rP(1 - \frac{P}{k} - \beta P)$ or $\frac{(r-\beta)k}{r} = N_1$ and after using lemma 1, we have $\liminf_{t \rightarrow \infty} P(t) \geq N_1$.

Similarly from Eq. (3), we can write $\frac{dM}{dt} \geq \beta N_1 - q_1 M E - d_2 M$, which on using lemma 1 gives $\liminf_{t \rightarrow \infty} M(t) \geq \frac{\beta N_1}{q_1 E + d_1} \equiv N_2$.

Further from Eq. (4) we have $\frac{dI}{dt} = I(\alpha_1 - \frac{\alpha_2}{\alpha_3 + N_2})$, and after using lemma 1 we get $\liminf_{t \rightarrow \infty} I(t) \geq \frac{\alpha_1 - d_2}{\alpha_2}(\alpha_3 + N_2)$.

Also from inequalities (5), (6) and (7), together with lemma 1, we have $\limsup_{t \rightarrow \infty} P(t) \leq M_1$, $\limsup_{t \rightarrow \infty} M(t) \leq M_2$ and $\limsup_{t \rightarrow \infty} I(t) \leq M_3$.

Now choosing $C_1 = \min(N_1, N_2, N_3)$ and $C_2 = \max(M_1, M_2, M_3)$, we get the permanence for the system of Eqs. (5)–(7).

Stability analysis

Equilibrium analysis

The proposed system (3)–(5) has four biologically feasible equilibria:

1. The trivial case, i.e., $E_0 = (0, 0, 0)$, always exist.

2. Non-trivial case $(\hat{P}, \hat{M}, 0)$ when forestry biomass exist and industrialization does not exist, here $\hat{P} = \frac{k(r+\gamma-\beta_1)}{r}$, $\hat{M} = \frac{k\beta_1(r+\gamma-\beta_1)}{(d_1+E)}$ and the axial equilibria exists when $r + \gamma > \beta_1$.
3. Non-trivial case $\tilde{E} = (0, 0, \tilde{I})$ when forestry biomass vanish and industries exist, here $\tilde{I} = \frac{\alpha_3(\alpha_1-d_2)}{\alpha_2}$, it exists when $\alpha_1 > d_2$ i.e., the growth rate of industrialization should always be greater than the depletion of industries.
4. Interior equilibrium $E^* = (P^*, M^*, I^*)$ when all the state variable co-exist, here $P^* = \frac{k(r+\gamma-\beta_1)}{r}$, $M^* = \frac{k\beta_1(r+\gamma-\beta_1)}{(d_1+E)}$ and $I^* = \frac{(\alpha_3+\alpha_1 d_2)(\alpha_1-d_2)}{\alpha_2}$. The interior equilibrium $E^* = (P^*, M^*, I^*)$ exists only when $r + \gamma > \beta_1$ and $\alpha_1 > d_2$.

Local stability

To find the local stability of the system we first find the Jacobian corresponds to the system (2)–(5)

$$J = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix},$$

where $a_{11} = -r + \gamma - \beta_1 - \frac{2Pr}{k}$, $a_{21} = \beta_1$, $a_{22} = -d_1 - E - \beta_2$, $a_{32} = \frac{\alpha_2 I^2}{(M + \alpha_3)^2}$, $a_{33} = -d_2 - \alpha_1 - \frac{2I\alpha_2}{M + \alpha_3}$.

Variational matrix corresponding to the axial equilibrium $(\hat{E})$ is given by

$$J = \begin{bmatrix} -\frac{rP}{k} & 0 & 0 \\ \beta_1 & -d_1 - E & 0 \\ 0 & \frac{\alpha_2 I^2}{(M + \alpha_3)^2} & \alpha_1 - d_3 - \frac{2I\alpha_2}{M + \alpha_3} \end{bmatrix},$$

characteristic equation corresponding to the variational matrix is

$$\left(-\frac{rP}{k} - \lambda\right)(-(d_1 + E) - \lambda)\left(\alpha_1 - d_3 - \frac{2I\alpha_2}{\alpha_3} - \lambda\right) = 0.$$

On solving we have $\lambda = -\frac{rP}{k}$, $\lambda = -(d_1 + E)$ and $\lambda = \alpha_1 - d_3 - \frac{2I\alpha_2}{\alpha_3}$, λ is negative when $\alpha_1 < d_3 + \frac{2I\alpha_2}{\alpha_3}$.

Since all the values of λ are less than zero, hence we may conclude that axial equilibrium $(\hat{E})$ is locally asymptotically stable.

Variational matrix corresponding to the axial equilibrium $(\tilde{E})$ is as below

$$J = \begin{bmatrix} r - \beta + \gamma - \frac{2rP}{k} & 0 & 0 \\ \beta_1 & -d_1 - E & 0 \\ 0 & \frac{\alpha_2 I^2}{(M + \alpha_3)^2} & -\frac{I\alpha_2}{M + \alpha_3} \end{bmatrix},$$

characteristic equation corresponding to the variational matrix is

$$(r - \beta + \gamma - \frac{2rP}{k} - \lambda)(-d_1 - E - \lambda)(-\frac{I\alpha_2}{M + \alpha_3} - \lambda) = 0,$$

which on solving gives $\lambda = r - \beta + \gamma - \frac{2rP}{k}$, $\lambda = -(d_1 + E)\lambda = -\frac{I\alpha_2}{M + \alpha_3}$.

All the values of λ are negative, only when $r + \gamma < \beta + \frac{2rP}{k}$. Variational matrix corresponding to the interior equilibrium (E^*) is given by

$$J = \begin{bmatrix} -\frac{rP}{k} & 0 & 0 \\ \beta_1 & -d_1 - E & 0 \\ 0 & \frac{\alpha_2 I^2}{(M + \alpha_3)^2} & -\frac{I\alpha_2}{M + \alpha_3} \end{bmatrix},$$

characteristic equation corresponding to the variational matrix is

$$\lambda^3 + A_{11}\lambda^2 + A_{22}\lambda + A_{33} = 0,$$

where $A_{11} = \frac{rP}{k} + d_1 + E + \frac{\alpha_2 I}{\alpha_3 + M}$, $A_{22} = \frac{rd_1 P}{k} + \frac{rPE}{k} + \frac{EI\alpha_2}{M + \alpha_3} + \frac{r\alpha_2 PI}{k(M + \alpha_3)} + \frac{Id_1 \alpha_2}{M + \alpha_3}$, $A_{33} = \frac{rEIP\alpha_2}{k(M + \alpha_3)} + \frac{IPrd_1 \alpha_2}{k(M + \alpha_3)}$ and on applying Routh–Hurwitz criteria, the system is locally asymptotically stable as all the co-efficient $A_{11} > 0$, $A_{22} > 0$ and $A_{33} > 0$, of the characteristic equation are positive.

$$A_{11}A_{22} - A_{33} > 0.$$

Thus the interior equilibrium is locally asymptotically stable.

Bionomic equilibria

Let c be the constant harvesting costs per unit effort for the mature tree population and p is the price per unit of forest biomass. The economic rent at any time is given by $\Pi(P, M, I, E, t) \equiv (q_1 p M - c)E$, where p is the price per unit forestry mature biomass and c is the cost per unit of utilized forestry mature biomass is c . The bioeconomic equilibrium ($P_\infty, M_\infty, I_\infty, E_\infty$) is given by the following simultaneous equations

$$rP \left(1 - \frac{P}{k}\right) - \beta P + \gamma P = 0 \quad (8)$$

$$\beta P - q_1 M E - d_1 M = 0, \quad (9)$$

$$\alpha_1 - \frac{\alpha_2 I}{\alpha_3 + M} - d_2 = 0, \quad (10)$$

$$\Pi = (q_1 p M - c)E = 0. \quad (11)$$

on solving, we get

$$P_\infty = \frac{k(r - \beta_1 + \gamma)}{r}, \quad (12)$$

when $(r + \gamma) > \beta_1$.

$$M_\infty = \frac{c}{q_1 p}, \quad (13)$$

$$I_\infty = \frac{(\alpha_1 - d_3)(\alpha_3 q_1 p + c)}{\alpha_2 q_1 p}, \quad (14)$$

when $\alpha_1 > d_3$.

$$E_\infty = \frac{\beta_1 k p (r - \beta_1 + \gamma)}{rc} - \frac{d_1}{q_1}. \quad (15)$$

Considering above equation, we can say that

$$E_\infty > 0 \text{ when } \frac{\beta_1 k p (r - \beta_1 + \gamma)}{rc} > \frac{d_1}{q_1}. \quad (16)$$

Hence, we can state that the bionomic equilibrium exists when washout of mature population and increasing effort should always be less than the product of the bioeconomic equilibrium of premature trees population and ratio of price and cost per unit of forest biomass. We have also calculated the sensitivity of bioeconomic equilibrium and is described in ‘‘Sensitivity analysis’’ section.

Optimal harvesting policy

Here in this section, we will find the optimal level of equilibrium using Pontryagin’s maximum principle and the associated Hamiltonian function is given as

$$\pi(P, M, I, \tau, t) = (q_1 p - c)M. \quad (17)$$

Our objective is to maximize the present value of Q , for a continuous time stream of revenues given by

$$Q = \int_0^\infty e^{-\delta t} \pi(P, M, I, \tau, t) dt, \quad (18)$$

with δ as the instantaneous rate of annual discount. Thus, our objective is to maximize Q subject to the equations (2)–(4) and the control constraints

$$\tau_{min} \leq \tau \leq \tau_{max}. \quad (19)$$

Associated Hamiltonian for the control problem is defined as:

$$H = e^{-\delta t} [q_1 p M - c] E + \lambda_1(t) \left[r P \left(1 - \frac{P}{k} \right) - \beta_1 P + \gamma P \right] + \lambda_2(t) [\beta_1 P - q_1 M E - d_1 M] + \lambda_3(t) \left[\left(\alpha_1 - \frac{\alpha_2}{\alpha_3 + M} \right) I - d_2 I \right]. \quad (20)$$

Here λ_1 , λ_2 and λ_3 are the adjoint variables. For the singular solution to be optimal, $\sigma(t) = e^{-\delta t} (q_1 p M - c) - \lambda_2 q_1 M$, also known as switching function. For $E = E_{max}$, $\sigma(t) > 0$, $\lambda_2 e^{\delta t} < p - \frac{c}{q_1 M}$. When $E = 0$, $\sigma(t) < 0$, $\lambda_2 e^{\delta t} > p - \frac{c}{q_1 M}$, and for $\sigma(t) = 0$, we have

$$\lambda_2 = e^{-\delta t} \left(p - \frac{c}{q_1 M} \right). \quad (21)$$

Now, as per maximum principle, the adjoint variable must satisfy

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial P}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial M}, \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I}. \quad (22)$$

Considering above equation and the system of equation (3)–(5), we get

$$\frac{d\lambda_1}{dt} = - \left[r \left(1 - \frac{2P}{k} \right) - \beta_1 + \gamma \right] - \beta_1 \lambda_2.$$

or

$$\frac{d\lambda_1}{dt} = \frac{rP\lambda_1}{k} - \beta_1 \lambda_2, \quad (23)$$

$$\frac{d\lambda_2}{dt} = -e^{-\delta t} (q_1 p E) + (d_1 + q_1 E) \lambda_2 - \lambda_3 \frac{\alpha_1 \alpha_3 I^2}{\alpha_2 + \alpha_3 M}, \quad (24)$$

$$\frac{d\lambda_3}{dt} = - \left(\alpha_1 - \frac{2\alpha_1 I}{\alpha_2 + \alpha_3 M} - d_2 \right) \lambda_3, \quad (25)$$

since $\lambda_2 = -e^{-\delta t} \left(p - \frac{c}{q_1 M} \right)$, on differentiating, we have

$$\frac{d\lambda_2}{dt} = -\delta e^{-\delta t} \left(p - \frac{c}{q_1 M} \right). \quad (26)$$

Now from Eq. (22) we have

$$\frac{d\lambda_1}{dt} - \frac{rP\lambda_1}{k} = -\beta_1 e^{-\delta t} \left(p - \frac{c}{q_1 M} \right),$$

or

$$\lambda_1 = e^{-\delta t} \frac{b_1}{a_1 + \delta}, \quad (27)$$

where $a_1 = \frac{rP}{k}$ and $b_1 = -\beta_1 \left(p - \frac{c}{q_1 M} \right)$.

Using Eqs. (25) and (26), we get

$$\lambda_3 = e^{-\delta t} \frac{(\alpha_3 + M)^2}{\alpha_1 \alpha_2 I^2} \left(p d_1 - \frac{c d_1}{q_1 M} - \frac{c E}{M} - \frac{c \delta}{q_M} + p \delta \right), \quad (28)$$

Substituting the value of λ_3 in Eq. (24) we can calculate λ_2 and is given as below

$$\lambda_2 = e^{-\delta t} \frac{b_{22}}{a_{22} + \delta}, \quad (29)$$

where $a_{22} = (d_1 + q_1 E)$ and $b_{22} = p q_1 E + \delta p - \frac{\delta c}{q_1 M} + p d_1 - \frac{c d_1}{q_1 M} - \frac{c E}{M}$. On comparing Eqs. (21) and (29) we get

$$\left(p - \frac{c}{q_1 M} \right) = \frac{b_{22}}{a_{22} + \delta}, \quad (30)$$

we will get the singular path by substituting the values of a_{22} and b_{22} and is defined as below

$$E = \frac{(p q_1 M - c)(d_1 + \delta) M}{p q_1 M - c - p q_1^2 M^2 - c q_1 M}. \quad (31)$$

Hence, on solving the model Eqs. (4) and (5) with the help of above equation we get optimal solution (M_δ and I_δ) and the optimal harvesting effort of forestry biomass as $E = E_\delta$. Numerical values of optimal solution M_δ , I_δ and $E = E_\delta$ are defined in numerical simulation (see “Numerical simulation” section.).

Numerical simulation

Numerical simulation is done to check system (2)–(4) dynamics using following set of parametric values

$$r = 1.5, k = 10, \beta = 0.5, \gamma = 0.03, a_1 = 0.4, a_2 = 2.2, a_3 = 4, d_1 = 0.1, d_2 = 0.1, E = 2.5, q_1 = 1, p = 0.1, c = 0.001. \quad (32)$$

For the above set of assumed parameters described in (32), conditions for the existence of equilibrium are satisfied and the values for non-trivial (interior and axial equilibrium) case are E^* (6.867, 1.321, 0.726), $\hat{E}$ (0, 0, 0.5457) when intrinsic growth $r = 0.2$, $\tilde{E}$ (6.8666, 5.7223, 0) when $d_2 = 0.25, a_1 = 0.4$ and $E = 0.5$, all feasible equilibria's are plotted in Figs. 2, 3, 4 and 5 using a set of parameters described in (32) respectively. From Fig. 3 we can admit that the system is stable as all the populations coexist, Fig. 4 shows the axial equilibrium when there is the extinction of premature and mature population and from Fig. 5 extinction of industrialization is shown. Next, we have also presented the graph shows the decrease in mature tree population and industrialization with respect to effort in Fig. 6. Numerical values for the optimal equilibrium levels are $M_\delta = 0.01020626$, $I_\delta = 0.273423$ and $E_\delta = 336.311$.

Sensitivity analysis

Sensitivity of the interior equilibrium (E^*) and bioeconomic equilibrium (E_∞) is calculated using the normalized forward sensitivity method as discussed in Chitnis

Fig. 2 Population distributions with respect to time

Fig. 3 Population extinction with respect to time

et al. (2008), Sahu and Dhar (2015). Sensitivity of interior and bioeconomic equilibrium is defined in Table 2 and is calculated using the parametric values defined in Eq. (32).

Description of interior and bio-economic equilibrium with respective sensitive parameter:

Sensitivity of P^* : for the parameters r and β_1 , P^* is moderately sensitive to E^* ,

Sensitivity of M^* : k is the parameter which is positively sensitive to interior equilibrium and parameters r , E , β_1 are moderately sensitive to E^* ,

Sensitivity of I^* : parameter a_1 and a_2 are positively and negatively highly sensitive respectively and β_1 , d_2 and a_3 , k and E are moderately sensitive, to interior equilibrium,

Sensitivity of BE : We can conclude that r , β_1 are moderately sensitive, k , p are positively highly sensitive and c is negatively sensitive parameters to E^* .

Sensitivity of the parameters r , d_3 , τ , p , a_1 , a_2 , and k , increase or decrease the values of semi-mature trees and mature trees, while pre-mature trees remains same. Hence we can say, significant change in E^* by small changes in these parameters.

Fig. 4 Extinction of pre mature and mature trees population (axial equilibrium $\hat{E}$)

Fig. 5 Extinction of industrialization (axial equilibrium $\hat{E}$)

Fig. 6 Effect on mature and industrial population with increasing effort

Table 2 Sensitivity of parameters

Parameter (y_j)	$\gamma_{y_j}^{P^*}$	$\gamma_{y_j}^{M^*}$	$\gamma_{y_j}^{I^*}$	$\gamma_{y_j}^{E_\infty}$
r	0.103753	0.103753	0.0314802	0.104213
k	1	1	0.303416	1.00443
γ	0.00662252	0.00662252	0.00200938	0.00665188
β_1	-0.110375	-0.110375	0.269926	0.89357
q_1	0	0	0	0.00443459
a_1	0	0	1.33333	0
a_2	0	0	-1	0
a_3	0	0	0.696584	0
c	0	0	0	-1.00443
p	0	0	0	1.00443
d_1	0	-0.0384615	-0.0116698	-0.00443459
d_2	0	0	-0.333333	0
E	0	-0.961538	-0.291746	0

Fig. 7 Effect of maximum reduction rate a_2 on industrialization

Conclusion

In this study, we have considered an age structured forestry based biomass together with industrialization as a state variable. From the literature review, we conclude that Leslie–Gower and Modified-Leslie Gower functional response works well for the problems which we have discussed in this study. From the state space approach, we are able to distinguish state variable with respect to time at different stages. The local stability conditions for all the feasible equilibria are obtained. We noticed that the dynamic equilibrium exists under condition defined in Eq. (16). Numerical simulation is also done to verify the aforesaid analytic outcomes and we have also shown graph (Figs. 2, 3, 4, 5) of all the feasible equilibrium by means of

parameters defined in Eq. (32). Decrease in both the mature forestry biomass and industrialization is observed with increasing effort see Fig. 6, effect of maximum reduction rate a_2 on industrialization is shown in Fig. 7. Pontryagin’s maximum principle is used to find the optimum equilibrium level. Further from the sensitivity analysis, we came across the parameter which are sensitive to the interior and bio-economic equilibrium and is discussed in the numerical simulation (see “Numerical simulation” section).

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References

- Aziz-Alaoui MA (2002) Study of a Leslie–Gower-type tritrophic population model. *Chaos Solitons Fractals* 14(8):1275–1293
- Aziz-Alaoui M-A, Okiye M-D (2003) Boundedness and global stability for a predator-prey model with modified Leslie–Gower and holling-type ii schemes. *Appl Math Lett* 16(7):1069–1075
- Chaudhary M, Dhar J, Sahu GP (2013) Mathematical model of depletion of forestry resource: effect of synthetic based industries. *Int J Biol Vet Agric Food Eng* 7(4):130–134
- Chaudhary M, Dhar J (2013) Forestry biomass conservation with synthetic industry: a mathematical model. In: Engineering (NUICONe), 2013 Nirma University international conference on IEEE, pp 1–5
- Chaudhuri K (1988) Dynamic optimization of combined harvesting of a twospecies fishery. *Ecol Model* 41(1):17–25
- Chen F (2005) On a nonlinear nonautonomous predator-prey model with diffusion and distributed delay. *J Comput Appl Math* 180(1):33–49
- Chen L-J, Chen F (2009) Global stability of a Leslie–Gower predator-prey model with feedback controls. *Appl Math Lett* 22(9):1330–1334
- Chitnis N, Hyman J-M, Cushing J-M (2008) Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. *Bull Math Biol* 70(5):1272–1296
- Clark CW (1976) *Mathematical bioeconomics: the optimal management of renewable resources*. Wiley, Hoboken
- Clark CW (2010) *Mathematical bioeconomics: the mathematics of conservation*, vol 91. Wiley, Hoboken
- Devi S (2012) Nonconstant prey harvesting in ratio-dependent predator-prey system incorporating a constant prey refuge. *Int J Biomath* 5(02):1250021
- Dhar J, Singh H (2004) Modelling the depletion of forestry resource by wholly dependent industrialization in two adjoining habitat. *Kobe J Math* 21(1):1–13
- Dubey B, Sharma S, Sinha P, Shukla J (2009) Modelling the depletion of forestry resources by population and population pressure augmented industrialization. *Appl Math Model* 33(7):3002–3014
- Dubey B, Patra A, Sahani S (2014) Modelling the dynamics of a renewable resource under harvesting with taxation as a control variable. *Appl Appl Math* 9(2):592–621
- Garcia O (2013) Forest stands as dynamical systems: an introduction. *Modern Appl Sci* 7(5):32
- Ghosh B, Kar T (2014) Sustainable use of prey species in a prey-predator system: jointly determined ecological thresholds and economic tradeoffs. *Ecol Model* 272:49–58
- Gupta R, Chandra P (2013) Bifurcation analysis of modified Leslie–Gower predator-prey model with michaelis-menten type prey harvesting. *J Math Anal Appl* 398(1):278–295
- Kar T, Ghorai A (2011) Dynamic behaviour of a delayed predator-prey model with harvesting. *Appl Math Comput* 217(22):9085–9104
- Khaine I, Woo SY (2015) An overview of interrelationship between climate change and forests. *For Sci Technol* 11(1):11–18
- Leslie P (1948) Some further notes on the use of matrices in population mathematics. *Biometrika* 35(3–4):213–245
- Leslie P (1958) A stochastic model for studying the properties of certain biological systems by numerical methods. *Biometrika* 16–31
- Liu X, Xing Y (2013) Bifurcations of a ratio-dependent holling-tanner system with refuge and constant harvesting. In: *Abstract and applied analysis*, vol 2013. Hindawi Publishing Corporation, Cairo
- Misra A, Lata K, Shukla J (2014) A mathematical model for the depletion of forestry resources due to population and population pressure augmented industrialization. *Int J Model Simul Sci Comput* 5(01):1350022
- Misra A, Lata K (2015) Depletion and conservation of forestry resources: a mathematical model. *Differ Equ Dyn Syst* 23(1):25–41
- Pal PJ, Sarwardi S, Saha T, Mandal PK (2011) Mean square stability in a modified leslie-gower and holling-type ii predator-prey model. *J Appl Math Inf* 29:781–802
- Sahu GP, Dhar J (2015) Dynamics of an seqihrs epidemic model with media coverage, quarantine and isolation in a community with pre-existing immunity. *J Math Anal Appl* 421(2):1651–1672
- Shukla J, Misra O, Agarwal M, Shukla A (1988) Effect of pollution and industrial development on degraation of biomass-resource: a mathematical model with reference to doon valley. *Math Comput Model* 11:910–913
- UFAO et al (2010) *Global forest resource assessment*, UN Food and Agriculture Organisation, Rome
- Yue Q (2015) Permanence for a modified leslie-gower predator-prey model with beddington-deangelis functional response and feedback controls. *Adv Differ Equ* 2015(1):1–10
- Zhang N, Chen F, Su Q, Wu T (2011) Dynamic behaviors of a harvesting Leslie–Gower predator-prey model. *Dis Dyn Nat Soc* 18