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Conceptualization of the Continuum, an Educational Challenge for Undergraduate Students

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Abstract The continuum is one of the most difficult mathematical concepts for undergraduate students. We hypothesize that among the difficulties they face in relation with this notion, the graphical evidence provided by the number line fosters the idea of a dichotomy between discreteness and continuity, hiding the property of density-in-itself, i.e. the intrinsic density with respect to order in a totally ordered set. In this paper, we first provide evidences of the weakness of fresh university students' knowledge about real numbers. Then, we briefly present Dedekind's construction of real numbers, which relies on the intuitive idea of the continuous line. Finally, we present a didactical situation aimed at fostering the understanding of the relationships between discreteness, density-in-itself and continuity for an ordered set of numbers.

Keywords The continuum · Density-in-itself · Discreteness · Number line · Didactical situations

Introduction

Recalling briefly a classical trajectory in the teaching of the continuum at university, going from the naïve idea of real numbers as 'all numbers' to a construction of real numbers as cuts or as Cauchy sequences, Bergé (2010) wonders: "What do students understand and misunderstand about R and completeness along that path?" Relying on the results from a test submitted to students having courses in Calculus and Analysis, she writes:

In memoriam Claude Tisseron

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> Understanding completeness as a property or an axiom that settles a critical mathematical issue requires a reflection that does not seem to appear spontaneously as a result of solving the given exercises. For most of the students, doing typical exercises involving the supremum does not lead to the understanding that R is the set that contains all the suprema of its bounded above subsets. (p. 226)

In our work, we assume that developing a clear understanding of the mathematical continuum requires being aware that the set of finite decimal numbers (i.e. real numbers with finite decimal expansions) and the set of rational numbers, although they are dense-in-themselves (i.e. intrinsically dense with respect to the order relation, and hence not discrete), are discontinuous.

We hypothesize that the graphical evidence provided by the number line supports the idea of a dichotomy between discreteness and continuity, hiding the property of density-in-itself. Indeed, there is no appropriate representation of the set of decimal numbers, or of the set of rational numbers on the number line: we can produce a sequence of points to suggest discreteness; a line to represent continuity; but there is nothing in between. This hypothesis is supported by the results from an experiment conducted in the setting of "MATh.en.JEANS"[1] in 1993–1994. A team of the IREM de Lyon[2] proposed to students of grade 11 a *fixed-point problem* (Pontille et al. 1996). In addition, we hypothesize that this *fixed-point problem* is a good candidate for the design of a *didactical situation* (in the sense of Brousseau 1997) aiming at fostering the understanding, by undergraduate students, of the relationships between *discreteness*, *density-in-itself* and *continuity*.

The main objective of this paper is to support this claim about the above-mentioned fixed-point problem, i.e. that it is suitable to have students questioning this *implicit model* of a dichotomy between discreteness and continuity. First we will present briefly our theoretical framing and our research questions. Then, we provide a brief outline on numbers in the French curriculum, completed by some recent experimental results from answers to a test showing the weakness of secondary level and undergraduate students' knowledge about real numbers, in line with the relevant literature (Bronner 1997; Bloch 2009; Birebent 2006). In a third section, we first present the main features of the construction of real numbers by Dedekind (1872), who proposed a formalization of the intuition of continuity as it is suggested by the number line. We then recall the principles of the elaboration of real numbers through decimal expansions. In the last part of the paper, we present the fixed-point problem of which we give some *a priori* analysis elements, and we provide qualitative results from the experiment reported in Pontille et al. (1996), completed with analyses of non published transcripts[3]; in addition, we briefly present the work on the same problem done by graduate students following a mathematical teacher-training program. In conclusion, we summarize our results and open on research perspectives.

[1] The MATh.en.JEANS Association impulses and coordinates research workshops designed for schools, from primary school to university: http://www.mathenjeans.fr/

[2] Research Institute for the Teaching of Mathematics in Lyon, France.

[3] With the authorization of the *IREM de Lyon*, legal owner of the rights.

Theoretical Framework and Research Questions

Our on-going research is framed in the theory of didactical situations (TDS, Brousseau 1997).

> […] in TDS, the fundamental object is the notion of *Situation,* which is defined as the ideal model of the system of relationships between students, a teacher, and a *milieu*. Students' learning is seen as the results of interactions taking place within such systems, and is highly dependent of characteristics of these systems. (González-Martín et al. 2014, p. 119)

The design of a didactical situation entails the search for mathematical problems that have potentialities for implementing a given piece of knowledge. The first step consists in the preliminary epistemological, cognitive and didactical analyses that allow the identification of didactical obstacles (if any), and of the didactical variables opening choices for the elaboration of the didactical situation. A didactical obstacle can be of an ontogenetic, or didactical, or epistemological origin (Brousseau 1997, p. 86). Determining the possible origin(s) of a given obstacle is not easy. In the case of the continuum, we hypothesize that the three aspects contribute to strengthen the *implicit model* of a dichotomy between discreteness and continuity. (In order to discuss this hypothesis, more research is needed; this exceeds the scope of our paper).

There is a dialectical relationship between obstacles and the problems aimed at overcoming them:

> The overcoming of an obstacle very often involves a complete restructuring of models of action, language and proof system. But the didactician can precipitate these breakdowns by favoring the multiplication and alternation of specific dialectics. (Brousseau 1997, p. 90)

A main concern of the TDS is the elaboration of *a milieu* allowing students to assume responsibility for their learning of the mathematical contents at stake, and this corresponds to the so-called *adidactical dimension* of the situation. In order to design *didactical situation*, the researcher should identify *didactical variables* that may influence the way students would solve the problem, and make the appropriate choices regarding these variables so to favor the *adidactical dimension* of the situation (González-Martín et al. 2014, p. 121). The *a priori* analysis consists in determining the relevant variables and identifying their influence on the learning process.

In this paper, we present some elements pertaining to the first steps of the on-going design of a didactical situation around a fixed-point problem, and we address the following questions:

1. Is the knowledge on real numbers acquired by students at the secondary level consistent with the axiomatic definition of the real number system, as a complete ordered field, that they will meet and use at university level in Analysis?
2. In what respect the fixed-point problem may be used to elaborate a *didactical situation* aimed at questioning and overcoming the *implicit model* of a dichotomy between discreteness and continuity, in a way that clarifies and fosters the understanding of the relationship between discreteness, density-in-itself and continuity.

In next section, we present results from a test designed for evaluating students' knowledge about real numbers, at the transition between secondary and tertiary levels.

Students' Knowledge About Real Numbers, at the Transition Between Secondary and Tertiary Levels

This section is aimed at providing evidence that unless a specific classroom work devoted to these issues is being conducted, many students do not elaborate a knowledge of real numbers that is consistent with the way these are defined and used in university mathematics. We first present a brief outline of numbers as a topic in the French curriculum; then we present results from a test submitted to high school students and undergraduates.

A Brief Outline of the Treatment of Numbers in the French Curriculum

In the primary schooling, the elaboration of the concept of natural number is expected from the syllabus, relying on finite discrete collections and on one-to-one correspondences between finite discrete collections and initial sequences of counting numbers names. Rational numbers (fractions) and finite decimal expansions (that from now on we will call *decimal numbers*) are introduced in the context of measurement of continuous magnitudes, along with an arithmetic treatment. The number line plays an important role. At middle school, students go on developing competences about natural numbers, decimal numbers and fractions. They meet irrational numbers through the square root of natural numbers that are not perfect square, such as $\sqrt{2}$. The letter π is introduced in the formula for the circumference of a circle and the area of a disk, but students use mostly its decimal approximation 3,14. At high school, students deal with approximations, mainly with calculators. In grade 12, they learn the *mean value theorem* without a proof, and without a discussion on the fact that this theorem holds in the set of real numbers, but no longer in the sets of decimal numbers or of rational numbers. Consequently, students acceding to university have in general no idea of the differences and interplay between finite decimal numbers, rational numbers and non-terminating decimal expansions, and thus are not prepared for what they will be taught at university. Indeed, in many French universities, in first-year mathematical courses, an axiomatic definition of the set of real numbers is given, most often via "the supremum property", without any explicit construction. In some cases, the representation of real numbers as non-terminating decimal expansions and the corresponding characterization of the type of numbers are introduced, and improper expansions such as $0,\overline{9}$ are discussed with students (e.g. Njomgang-Ngansop and Durand-Guerrier 2013; Vivier 2015).

Context of the Collect of Data

In the frame of his didactic study about real numbers, Bronner (1997) administered a test aimed at evaluating knowledge about finite decimal numbers and non-terminating decimal expansions (that he called *idecimal* numbers) to 71 French preservice teachers, with a Bachelor in mathematics or an equivalent degree. From the results, he concluded that the knowledge on decimal expansions is not adequate for solving non-elementary problems[4] (pp. 210–213). Due to the current French curriculum on numbers (see above), and recurring empirical observations in teacher training sessions or in undergraduate tutorials, we considered this test as being relevant to evaluate nowadays students' knowledge about real numbers at the transition between secondary and tertiary levels, and we used it with three populations in 2013 and 2014. The test is in two parts: in the first students have to choose among different proposals concerning the type of three numbers; in the second, they are asked to give definitions for decimal, rational, irrational and real numbers, the question about *real numbers* being the last one.

The test was first submitted to two classes in the scientific track of French high-school (Grade 12) in spring 2013; 52 students filled in the test. Then, 152 fresh university students following a course called "Foundations of Analysis" in the second semester of the academic year 2013–2014 answered the test. The test was passed during a tutorial session. During the first semester 2013–2014, the set of real numbers has been introduced through one of its axiomatic definitions, the specific axiom of continuity being stated as the existence of a supremum for each non-empty upper-bounded subset. This definition has been used by the lecturers to prove classical theorems on the convergence of sequences of real numbers and of functions. However, students had few opportunities to use it by themselves, and so we did not expect finding answers referring to this axiomatic definition. In November 2014, we submitted anew the test to first-year scientific students, before the first lectures on real numbers that were planned in the program; we collected 225 answers. The two tests given at Université de Montpellier were anonymous, in order to encourage students to answer, even if they were not sure of their answers. In the three cases, students had the possibility to use a calculator; the case being, they were asked to tick a box. In the reports about the students' answers, we did not consider this aspect, due to the fact that it is difficult to evaluate in what respect it influenced the answers, or not.

About the Type of Numbers

The first part of the test was aimed at testing the hypothesis according to which usually, students entering university "do not assign the right meaning to inscriptions such as $\sqrt{2}$, π (…)" (González-Martín et al. 2014, p. 123). In this part of the test, we were more generally concerned with the students' capability to identify that there is no strict correspondence between the type of a number and the way it is represented (e.g. $\sqrt{1,44}$ is not an irrational number; it is a finite decimal that can be represented by 1,2). We asked students to choose among different options concerning the type (finite decimal, rational, irrational, other) of three numbers, namely $\sqrt{2}$, e and $\sqrt{13,21}$, the same that were given by Bronner (1997); for the third number, we asked for a justification. The three proposed numbers are irrational. Our aim was to test if students recognize this type or not, depending on the way the numbers are represented, with the requirement of a justification for the third one.

[4] For example: fixed-point problems with irrational solutions that are not « well-known » from students.

Five main variables are likely to have an influence on students' answers, depending on their value:

- V1: the number is already known from students, or not.
- V2: the type (rational or irrational) should be known from students, or not.
- V3: a decimal approximation of the number has been used at the secondary level, or not.
- V4: the number is represented by a digital term with a comma separation, or not.
- V5: the number is represented by a digital term with a radical sign, or not.

The first number $\sqrt{2}$ is known from grade 8 were it is first met; it is written with a radical sign but without comma separation in the digital term under the radical; it is the main example of irrational number in French secondary schools, so it should be known to be irrational by many students; however, some students are likely to consider it as a finite decimal number due to the fact that it is often being used through one of its decimal approximation (e.g. 1,414), or by using a calculator. Thus, the values of the variables for $\sqrt{2}$ are: "Yes" for V1, V2, V3, V5, and "no" for V4.

In the French curriculum, the number e is introduced in grade 12 as a notation for the image of 1 under the *exponential function*, which is defined as the unique function f, differentiable on R, such that $f' = f$ and $f(0) = 1$ (MEN 2011, p. 6). It is less familiar to students than $\sqrt{2}$. The discussion about the type of this number (rational or irrational) is not an objective of the program in grade 12. As for $\sqrt{2}$, and for similar reasons, some students may consider it to be a finite decimal number. Moreover, the image of an element x is commonly noted by e^x, so it is possible that some students consider e as a notation for the exponential function. Thus the values of the variables for the second number e are: "Yes" for V1, V3 and "No" for V2, V4, and V5.

For the third number, i.e. $\sqrt{13,21}$, the main point is that the number under the radical sign is not a perfect square of a finite decimal number; it is *a priori* not familiar to students and they shouldn't know its type; the representation of the number is a digital term with both a comma separation and a radical sign. In addition, a decimal approximation is not easily available unless a calculator is being used. Thus, the value of the variables for this number is "No" for V1, V2, V3, and "Yes" for V4, V5. The choice for the values given to V4 and V5 aims at testing the hypothesis that some students identify, at least partly, the type of a number mostly from its representation: we will consider that it is the case for students answering that $\sqrt{13,21}$ is a finite decimal number (it uses finite digital terms with a comma separation) *and* an irrational number (it is written under a radical sign). At the opposite, we hypothesize that those students with a clear understanding of what determines the type of a number should recognize the necessity of engaging into a proof to determine this type. A possible complete justification is to use the rewriting $\sqrt{13,21} = \left(\sqrt{1321}\right) \times 10^{-1}$, and to prove that 1321 is not a perfect square (indeed it lays strictly between the squares of the two consecutive whole numbers 36 and 37). To conclude, it is necessary to know that the square root of an integer N is rational if and only if N is a perfect square. Some students may have learned it at secondary school but it is not a requirement of the program.

We give below the results of the experiment in the first year of university in 2013–2014 (Table 1).

Table 1 Students answers to the first item

The following numbers are	$\sqrt{2}$	e	$\sqrt{13,21}$
Finite decimal	18 %	20 %	29 %
Rational	14 %	14 %	12 %
Irrational	78 %	65 %	45 %
Finite decimal and irrational	8 %	6 %	5 %
Other	4 %	9 %	9 %
No answer	0 %	11 %	3 %

For the three numbers (and not only for the third one), some students answered that they were both *finite decimal* numbers and *irrational* numbers. In addition, some students also add the answer ‘other’ to another answer. This explains that the sum in the columns exceeds 100 %. For the third number, for which we asked students to provide a justification for their answer, we got less than 30 % (43 among 153 students justified their answer). Here are some *examples of typical justifications* (direct quotations translated from French by the author).

- 13,21 is a finite decimal number; moreover under the root, it is irrational (answer: *finite decimal and irrational*)

  We consider this answer as an indication that this student identifies the type of a number considering the way it is written, leading him to attribute contradictory properties to the given number.
- $\sqrt{13,21}$ is a finite decimal number for even if the number has a comma under a radical sign, the real value is a real number with a comma; it is a number that exists (answer: *finite decimal*).

  This student seems to consider that the definition of finite decimal (in French *décimal*) is “being written with a comma.” It is difficult to interpret the sentence “it is a number that exists”; this would have need interviews that we have not done (as already said, the test given at university was anonymous).
- It is the square root of a finite decimal number, thus it is a finite decimal number (answer: *finite decimal*)

  This answer could be considered as an improper reversal of the property according to which the square of a finite decimal number is a finite decimal number.
- I think that a number as 13,21 cannot have a rational root. Being more precise, but without being able to assert it, I think that only an integer may have a rational root (answer: *irrational*).

  Although the answer is correct, the argument is not. The student expresses a lack of knowledge and does not engage a reasoning to control his conjecture.

The answers to this first part of the test show that nearly one student among five considers that $\sqrt{2}$ and e are finite decimals, some of them considering that they are both *finite decimal* and *irrational*. Concerning the third number, the qualitative analysis of the results supports our hypothesis that for some students, the type of numbers is at least partly identified through the way they are written.

About the Definition of a Real Number

In the second part of the test, we asked students to provide a definition of a real number. In the French secondary curriculum, there is no explicit definition of real numbers; the students' knowledge of these numbers is mainly based on the mathematical practice developed in numerical activities, and since 2009, the chapter on the type of numbers which was taught previously in grade 10 has disappeared. So, we did not expect a formal definition for real numbers, we wished to identify what *knowledge* about real numbers students had developed through their mathematical practice at the secondary level. We have classified the answers in *a priori* categories, presented in the Table 2 below. These categories have been elaborated in reference to what is being mentioned in the French secondary curriculum and in the lecture notes of the mathematics program followed by students in their first semester, and in interviews with secondary teachers. These interviews were conducted by M. Vergnac in the setting of her Master dissertation, under the supervision of the author of this paper, and are presented in Vergnac and Durand-Guerrier (2014). We hypothesize that these categories are representative of what students dealt with at secondary school.

Categories "All Numbers"

The interviews conducted by M. Vergnac during the school year 2012–2013 with seven secondary teachers (grade 10 to 12) showed that with the French curriculum having been modified, most of them were left to provide very few possible explanations about the real numbers in their teaching. They were thus led to say, in grade 10 and 11 that real numbers are "all the numbers",[5] "all the numbers they know" ('they' referring to the students being taught), this point of view being reconsidered in grade 12 where complex numbers are introduced as new numbers, with the set of real numbers as a subset (Vergnac and Durand-Guerrier 2014).

This supports the three first categories: C1: *all numbers*—C2: *all numbers except complex numbers*—C3: *Complex numbers with an imaginary part equal to zero*.

Category "Real Axis"

The category *real axis* (C4) is supported by the fact that it is widely used in secondary school, in particular through graphic representations of functions. One of the interviewed teachers declared that she introduces the real numbers as the "abscissa of points on the line".

Category "Interval"

The *interval* $]-\infty;+\infty[$ (category C5) is generally introduced for representing the set of real numbers (of all the numbers) when solving equations or inequalities, or to identify the domain of a function defined on the set of real numbers.

[5] Bergé (2010) points out that this is the point of view adopted in the course « Calculus » in the first year of Buenos Aires University.

Category "Realistic"

The category C6, *realistic*, is introduced in reference with the casual use of the term "real", meaning "which exists in the world (ordinary or mathematical)".

Category "Infinite Decimal"

The category C7 has been introduced taking into consideration that some students might have attended and failed the course *Foundations of Analysis* (in which the real numbers are introduced through their infinite decimal expansions) in the previous year.

Categories "Partition"

The category C8, partition between rational and irrational numbers, is likely to appear: indeed, the introduction of real numbers coincides in secondary school with the introduction of irrational numbers, so that some students may have correctly identified the set of real numbers as the disjoint union of these two subsets, even if it is not formally planned to being taught at that level. Besides, the repeating students met this partition in the previous year.

The category C9, named *"incorrect partition"*, encompasses answers expressing that whole, (finite) decimal or rational numbers are not real numbers; for example, answers such as "*the real numbers are all the numbers excluding whole numbers".*

In the category C10, *"Reformulation"*, we consider 'circular' answers such as: "*a real number is an element of the set of real numbers*".

We have introduced a category *others* (C11) and a category *no answer* (C12). The category *others* encompasses a great variety of answers such as, for example, "*a real number is a number without a comma, for example,* 3, –5, π."

A few answers have been classified in two different categories such as "*All numbers in the interval*] –∞; +∞[*that do not involve i* ", that we classified in both C5 and C2.

In Table 2, we summarize the results of the answers from the three populations having passed the test (TS refers to the last grade of secondary school in the scientific track; L1 refers to the first year at University). The results are given as percentages, rounded down to their integer parts.

The results above show a great variety in the answers; all the categories we had identified *a priori* are found in the students' answers. The very low number of answers in the category "the real axis" was at first rather surprising, due to the wide use of this reference in mathematical activity at secondary level. We hypothesize that this is a consequence of the lack of explicit work in class on the relationship between the set of real numbers and the number line, so that it remains implicit, whereas it could play a role in the conceptualization of the continuum. We consider that the weakness of students' knowledge observed among the involved populations supports the need for a specific work with university students on the construction of the real number system. The discrepancy between the widespread use of the number line as an intuitive reference for real numbers in secondary schools and the lack of it being mentioned by students supports the interest of designing didactical situations to help students in moving from an intuitive notion of continuity, relying on geometrical experiences with the number line, towards the mathematical definition of the continuum. In next section, we examine the epistemological consistency of this didactical research objective.

Table 2 What is a real number? (TS—2012–2013; L1—2013–2014 and 2014–2015)

Categories	TS—2012–2013	L1—2013–2014	L1—2014–2015
C1: All numbers	13	8	10
C2: All numbers except complex numbers	0	16	13
C3: Complex numbers with zero imaginary part	14	4	11
C4: The real axis	7	1	0
C5: Interval]−∞; +∞[	11	14	18
C6: Realistic	7	4	3
C7: Infinite decimal writings	0	2	2
C8: Partition rational/irrational	4	5	7
C9: Incorrect partition	14	22	14
C10: Reformulation	13	18	8
C11: Others	13	13	14
C12: No answer	16	5	8

The table presents percentages

Intuition and Formalization of the Continuum in Mathematics

In a paper entitled "The mathematical continuum: from intuition to logic", Longo (1999) claims that:

> This [pencil on a sheet] is the most common experience of the continuum: no one entertains discourse or conscious reflection of the continuum before having drawn lines on pieces of paper thousands of times. […] The points are collected in the trace, which makes their individuality disappear. These points become evident again, as isolated points, when two lines cross each other. (p. 403)

This is exactly the experience of students from primary to secondary school, so that we can consider that fresh university students have elaborated an *implicit model for action*, available in situation involving the continuum, without needing a formal definition. Longo reminds us that Cauchy himself, in his first proof of the *mean value theorem*, refers to this intuition. It is only in the second half of the 19th century that:

> Cantor and Dedekind have proposed a precise mathematical formalization of the intuitive continuum with at least three points of contact with our intuitive demands: the invariance of scale, the absence of jump and of holes. (p. 407)

Of these two mathematicians, only Dedekind proposed a formalization relying explicitly to the geometrical intuition of continuity conveyed by the number line. We present below the main features of this elaboration, keeping in mind the idea that it could serve as an epistemological reference for the design of a didactical situation aimed at fostering the conceptualization of the continuum. In addition, we give a brief outline of the construction of real numbers as decimal expansions.

Continuity and Irrational Numbers (Dedekind's Cuts)

In this section, we present the main features of the construction of the 'system' of real numbers developed by Dedekind (1872, 1963). In this *pamphlet*, Dedekind proposes a completion of the set of rational numbers through the notion of cuts, which appears as a *formalization of the intuition of the continuum.*

In his Preface (Dedekind 1963, p. 1), Dedekind indicates that when having to lecture for the first time on Differential Calculus, he felt the lack of a truly scientific foundation for arithmetic, due to the fact that although geometric intuition is useful from a didactic standpoint, and indispensable for not spending too much time, it cannot be claimed as being scientific. Considering the theorem that "every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value", Dedekind agrees that it "can be regarded in a certain way as a sufficient basis for infinitesimal analysis" (p. 2). He adds:

> It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity. (p. 2)

The construction of Dedekind relies on the relationship between the set of rational numbers and the number line with an origin. In section III, *Continuity of the straight line,* he emphasizes "the fact that in the straight line *L* there are infinitely many points which correspond to no rational numbers" (p. 8). The argument is that, as the ancient Greeks already knew, there are lengths incommensurable with a unit length and that moreover " (…) it can be easily shown that there are infinitely many lengths which are incommensurable with the unit of length" (p. 9). Then, his aim is to create " (…) new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same continuity as the straight line" (p. 9). In order to achieve this goal, Dedekind considered that the essence of continuity (completeness, absence of gaps) lies in the following principle:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions. (p. 11)

Dedekind points out that "The assumption of this property of the line is nothing else than an axiom, by which we attribute to the line its continuity" (p. 12).

In section IV, *Creation of irrational numbers*, Dedekind defines cuts in the set of rational numbers and proves that there are infinitely many such cuts that are not produced by a rational number, in what "consists the incompleteness or discontinuity of the domain R of all rational numbers" (p. 15). This is the last step for *the creation of irrational numbers:*

> Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new, an *irrational* number *a*, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number *a* corresponds to this cut, or that it produces this cut. (p. 15).

As a consequence, from this point on, at every cut of the set of rational numbers corresponds a definite rational or an irrational number. Then, in section V, *Continuity of* *the domain of real numbers*, Dedekind proves that the system $\mathfrak{R}$ of all the real numbers (consisting in rational and irrational numbers) forms a well-ordered domain of one-dimension on the one hand, and that the system is continuous on the other hand, i.e. that for every cut of $\mathfrak{R}$ there exists one and only one real number which produces this cut (pp. 19–21). The section VI is devoted to *Operation with real numbers* while in section VII, Dedekind gives examples in order to explain the connection between the principle of continuity and infinitesimal analysis.[6]

The construction of Dedekind is clearly focused on a formalization of the intuition of the continuity conveyed by the number line, and it links closely *continuity* and *order*, putting emphasis on the fact that the motivation for completing the domain of rational numbers may arise from our intuition of the continuity of the number line. In this respect, it is first necessary to acknowledge that once a unit is chosen on the number line, there are infinitely many points that are not associated to a decimal or even a rational number.

Continuity and Decimal Expansions

Some authors as Lelong-Ferrand (1964) recommended the introduction of real numbers in secondary school by the means of decimal expansions. One of the main arguments is that this construction relies on the finite decimal numbers that are familiar to students from primary school, and that it is closely linked to decimal approximations, useful in many activities. In this presentation, it can be proved that repeating decimal expansions are in bijection with rational numbers, and this leads to characterize the type of a number by the property of its decimal expansion. This position had influenced the Reform of Modern Mathematics in France (1968–1978), and gave a role to the notion of *idecimality* in the presentation of real numbers (Bronner 1997, pp. 126–128). In this introduction, the finite decimal expansions play a central role. The set of decimal numbers with a fixed number of decimal places are also of great importance. As the set of integers, these sets are discrete sets, while the set of finite decimal expansions in which the number of digits is not limited is dense-in-itself. It is rather easy to show that the square of a finite decimal is an integer if and only if this finite decimal is an integer. This perspective was the one adopted in the course "Foundations of Analysis" followed by the students who answered to the test in 2013–2014. It is worth recalling that in this introduction of real numbers, the consideration of unlimited decimal expansions raises questions related to infinity (e.g. Fischbein 2001; Njomgang-Ngansop and Durand-Guerrier 2013).

A Didactical Situation Fostering the Conceptualization of the Continuum

The aim of this section is to provide sound arguments supporting our hypothesis, that the fixed-point problem presented in Pontille et al. (1996) is a good candidate for designing a didactical situation aimed at fostering the understanding of the relationships between *discreteness*, *density-in-itself* and *continuity* for an ordered set of numbers at the undergraduate level. The problem was given to high school students (grade 11) in

[6] We do not develop this point here; in further research we intend to investigate the didactic viability of these relationships.

the setting of "MATh.en.JEANS"[7] in 1993–1994. As stressed by Pontille et al. (1996), the students were not in an ordinary teaching context. All along school year, voluntary students (grade 11) had worked on the fixed-point problem that was given by a mathematician; they had regular working sessions outside the teaching slots and took part into seminars, where they presented their work to other students involved in the MATh.en.JEANS project. The aim of the research was not the achievement of a well-identified knowledge:

> The main objective of the researcher who gave the problem was to make students work on infinity and ordered set. A major difficulty had been anticipated: viewing a real number as the supremum of an upper-bounded set. (p. 14)[8]

However, the authors of the paper claim that their observations pointed out potentialities for the learning of calculus at secondary level, in particular concerning finite decimal numbers and their relationships with real numbers. In their paper, they focused on two main points:

– the students' views on numbers and functions, their formulation in various mathematical frames and the related emerging obstacles and conflicts, and their evolution over a long time (a school year),
– the testing of their knowledge and their know-how in this problem, for both mathematical objects and methods. (p. 13)[9]

We present below the problem and some *a priori* analysis elements.

The Problem Given to Students

Let us consider a function f of $\{1, 2,\ldots n\}$ into $\{1, 2, \ldots n\}$, where n is a non zero natural number, and where f is supposed to be increasing (including non-strictly increasing functions); show that there exists an integer k such that $f(k) = k$; k is named a fixed-point. Then, study possible generalizations in the following cases, with f an increasing function.

1. $f : \mathrm{D}\cap[0; 1] \rightarrow \mathrm{D}\cap[0; 1]$, where D is the set of finite decimal numbers,
2. $f : \mathrm{Q}\cap[0; 1] \rightarrow \mathrm{Q}\cap[0; 1]$, where Q is the set of rational numbers,
3. $f : [0; 1] \rightarrow [0; 1]$,

or any other generalization.

The generalization does not work in case 1 and 2, and works in case 3. The preliminary case and the case 3 are particular cases of the theorem of Knaster – Tarski (Fig. 1):

[7] http://www.mathenjeans.fr/
[8] Our translation from French.
[9] Our translation from French.

Let (i) $U = (A, \leq)$ be a complete lattice[10] ; (ii) f be an increasing function from A to A ; (iii) P be the set of all fixed points of f. Then the set P is not empty and the system $(P, \leq)$ is a complete lattice (Tarski, 1955, 286-287).

Fig. 1 The Knaster-Tarski theorem

Indeed, a finite segment of the set of natural numbers is a discrete set and a complete lattice, and since the set of real numbers is ordered-complete, the interval [0; 1] is a complete lattice. From grade 11, students are *a priori* able to solve the first problem using *reductio ad absurdum*, or a proof by induction with the explicit property that "every whole number has a successor". A classical proof for case 3, very similar to the proof by Tarski (1955), consists in considering the subset $A = \{x \in [0;\ 1] | f(x) \geq x\}$ and proving that its supremum is a fixed-point of f (see proof in the Appendix). This proof enlightens the close connection between this problem and the property of an ordered set to be complete. It allows anticipating that it is possible to find a counterexample with well-chosen functions in each of the two cases 1 and 2, in D and Q respectively, for which the domain (which is also the co-domain) is dense-in-itself but is not a complete lattice.

Another way to get engaged in the three generalizations is to move from the preliminary problem to the first generalization in D∩[0; 1]. This is the way the students had worked in the experimentation reported in Pontille et al. (1996). We provide in the next session *a priori* analysis elements of the problem in the subset D∩[0; 1], considering that it is the first generalization to be studied.

A Priori Analysis Elements of the Problem in the Subset D∩[0; 1]

We consider at this point that students (either high school students or undergraduates) have solved the first problem either by a proof by *reductio ad absurdum*, or by a proof by induction. In both cases, the decisive argument for the proof is that for each natural number there exists a successor. At the considered level (grade 11), generalization from ℕ to D is likely to confront students with several questions.

In case students try to adapt their proof of the first problem, they should be facing the following question: does every finite decimal number admit a successor? Of course the answer is "no" and this is a consequence that D is dense-in-itself, since as an ordered set it satisfies the property that between two elements, a third one, distinct from the first two, can be inserted. It is noticeable that the answer is "yes" in every subset D_n of D (the set of finite decimal numbers with a given number n of decimal places); so that the transition from discreteness to density-in-itself raises clue questions about infinity, that are not considered in this paper but will be taken into consideration in the next step of the research.

As we have already said, in general, density-in-itself is a property of ordered sets such as D and Q that remains implicit in the secondary schooling. As in the example below from Bergé (2010), some university students consider this property as a characteristic of the set of real numbers:

> Some answers refer to the real-line: 'Real numbers cover the line, to each point corresponds a real number' 'real numbers complete the line, between two real numbers there is another, one cannot take the former number' […]. (p. 225)

To move forward in the problem, students must give up adapting the proof from the preliminary problem and (at least implicitly) assume the density-in-itself of D. It is likely that questions concerning infinity will be raised: there is potentially an infinity of elements in D∩[0; 1]; this may lead to a new question: is it possible to get all the points on the number line with finite decimal numbers? At secondary schooling in France, finite decimal numbers are widely used for the graduation of the number line, and recognizing the property of density-in-itself may incline students to conjecture that this allows reaching every point on the number line. Nevertheless, students have already encountered decimal expansions with infinitely many non-zero decimals, and hence a contradiction may emerge. At this point, a new object—a *decimal line*—may appear and lead to discussions on how to represent points on it. This is closely related with the fact that the set of finite decimal numbers is not continuous, as is the case with the *rational line*, this last point motivating Dedekind in his construction. A main difficulty considering this issue relies on the fact that thinking a number line with "holes" without thickness is counter-intuitive and in contradiction with graphical evidences. In other words, it is difficult to conceive of "lacunar lines without (visible) holes" that can cross each other without having a common point.

The next step is to make a conjecture concerning the existence, or not, of a fixed point. A first claim could be that, since it is not possible to adapt the very first proof, there is no fixed point. However this argument could be rejected as unconvincing for some students. Another possibility is to move towards a graphical representation. In this case, there could be visual evidences that there is necessarily a fixed point, in particular if the continuity of the considered function is implicitly assumed. This could lead to an attempt to proving. Dependent on the academic level of the students, different paths may be followed. University students have met in grade 12 the mean-value theorem (without a proof) and they have used it for solving problems, in particular problems involving monotonic functions (MEN 2011, p. 5); so they could try to use it in order to elaborate a proof. But in grade 11, students have no theoretical elements that could help them in a proof. So, students may try to look for examples and/or counterexamples, more likely with usual increasing continuous functions, for which it is visually evident that there will be a fixed-point in the set of real numbers. Then, looking for a counterexample leads, at least implicitly, to the following question: is it possible that the abscissa of the intersection point between the curve and the bisecting line $y = x$ could not be a finite decimal number? A well chosen affine function (e.g. $f(x) = \frac{1}{3} + \frac{x}{2}$) allows asserting that it is the case, thus providing a counterexample to the conjecture that there is necessarily a fixed point in the case of the subset D∩[0; 1]. The case of the set of rational number is theoretically identical, but no affine function can provide a counterexample; it is possible to find a relevant quadratic function, for instance $f(x) = \frac{x^2}{2} + \frac{1}{3}$.

Secondary Students' Encounter with the Issue of the Continuum

In this section, we present first a brief summary of the students' work as presented in Pontille et al. (1996); this enlightens the fact that the students faced sound epistemological questions that are linked to the concept of continuum. We then develop the analysis of the students' work relying on the collective notebook in which they wrote down their results and questions at the end of each session, and on unpublished transcripts of two collective sessions that complete the excerpts presented in Pontille et al. (1996).

Short Summary of the Presentation in Pontille et al. (1996)

The coauthors of Pontille et al. (1996) collected the data during the academic year 1993–1994. The students worked from October 1993 till June 1994. They managed to prove the preliminary result in November by *reductio ad absurdum* using (without formulating it) the property of $\mathbb{N}$: "$\forall p \in \mathbb{N}, \forall q \in \mathbb{N}, (p > q \Rightarrow p \geq q + 1)$", which leads to a contradiction. In the following sessions, the students worked on the first generalization: $f: D \cap [0; 1] \rightarrow D \cap [0; 1]$, trying to adapt their proof, which relies strongly on the fact that every integer has a successor. This leads them to encounter significant epistemological questions:

1. Is there a fixed minimal distance between neighboring finite decimal numbers?
2. Given any positive finite decimal, is it possible to find a smaller positive finite decimal?
3. Is there a relation between questions 1 and 2?

Finally, the students concluded that the answer to both questions 1 and 2 were NO. Then a new question appeared: "Is it possible that the graph of an increasing function from the set D of finite decimal numbers to itself crosses the first bisector outside D?" Through drawings and discussion, the students developed various representations of the numbers at stake and finally concluded that it is possible that the crossing be on a « hole ». Coming back to the initial question, they looked for a counterexample *with simple and concrete functions (their wording)*. They finally found counterexamples of the type given above, thus proving that there is not necessarily a fixed point in both generalizations 1 and 2.

Discreteness, Density-in-Itself and Continuity

In this paragraph, we present and comment excerpts of the students' notebook and of the transcripts of two consecutive sessions in January 1994. Our aim is to complete and deepen the analysis of the same data provided by Pontille et al. (1996), enlightening the relationships between discreteness, density-in-itself and continuity in the students' work.

At the end of the session from December the 8th of 1993, the students wrote in their notebook how they first tried to adapt the proof from $\mathbb{N}$ to D (Fig. 2).

"We see immediately that some finite decimal numbers are "missing", between 0 and 10^{-1000} and 2×10^{-1000}, etc. Thus, it is not possible to pass from a function in ℕ to a function in D. We must try something else!" We then try to use the proof made in ℕ, letting "*e* be the least finite decimal". We could write down the finite decimals as follow: 0, *e*, 2*e*, 3*e*, etc. In that case, we could use anew exactly the proof from ℕ to D by replacing "1" by "*e*." The problem is this number *e*. Indeed, does a " smallest finite decimal" actually exist?

Fig. 2 Excerpt of the students' notebook—8/12/1993. This was partly reproduced in Pontille et al. (1996, p. 21)

At this point, the students have moved from adapting the function to working on the domain of the function. They proposed a discrete model relying on a copy of $\mathbb{N}$ in which the successor property holds; this would be the case if they were working with numbers with a given number of digits after the comma (i.e. the set D_n of decimal numbers with a fixed number n of decimal places). The last question they raise (on the existence of a least non-zero finite decimal number) is at the heart of the following session (the 5th one) on January 12th 1994, as attested by the following excerpts (the two students are named E1 and E2).

1-E2: If there is not a smallest finite decimal number, …
2-E1: That means that it is a line,
3-E2: That all the finite decimal numbers are aligned, they form a line,
4-E1: We will find it …. Let *e* be the smallest finite decimal number
5-E2: To find *e*… it is an interval —*he draws an interval that he notes* [0, *e*)—if you have *a* and *b* two consecutive finite decimals;
6-E1: Does it exist?

This excerpt shows that the two students questioned the possible existence of a least finite decimal number. This question led them to introduce two new ideas: the line in case of non-existence; the idea of consecutive numbers in case of existence. At this moment, the possibility that D could be a non-discrete set is implicitly introduced. We may suspect that the reference to the line is implicitly conveying the idea of *continuity*.

A student E3, working on another problem in another group, intervened to answer the last question, saying: "there is not". E2 asked: "Why, aren't 1,1 and 1,2 consecutive?"; E3 replied: "there is 1,15 in between". E1 and E2 agreed that two numbers *a* and *b* being given, it is always possible to place $a + b$ divided by 2 in between. Later they explicitly formulated the pending question of the equivalence between the existence of a least finite decimal number and the existence of two consecutive finite decimal numbers.

17-E2: I wonder if it is the same thing: there exists a least finite decimal number and there exits two consecutive finite decimal numbers. It is always possible to look for the distance *b* minus *a*; by dividing by 2…, *b* minus *a* divided by two gives a finite decimal number; it is always possible to divide by 2.
18-E1: There exists always a distance, but it is not measurable. There does not exist a least finite decimal number *e*; would it be so, 0 and *e* would be consecutive.

E1 and E2 went on discussing about this question, looking for a proof that there is a least finite decimal number. It seems that E2 was convinced by the argument that it is always possible to divide by two while E1 remained puzzled, but agreed to continue. They began to get a representation of a function with domain $D\cap[0; 1]$ and values in $D\cap[0; 1]$. They represented the finite decimal numbers as successive dashes on the axis, on the first bisector and on the graph of the function, and they wondered if it is possible that the lines cross outside the dotted line. E1 remained unconvinced as it appears on the report in the notebook (the researchers indicated that the following text was written by E1) (Fig. 3).

In spite of this definition (and we will have probably to come back to it) we will consider "to continue" that "there does not exist a smallest finite decimal number" (*underlined by the student*). If we arrive at a deadlock, we will come back to this definition. Is it possible that the graph of the function *f* "crosses" the graph of "$y = x$" without "passing" through a point of "$y = x$" with finite decimal coordinates? (I think that [the answer is] no).

Fig. 3 Excerpt of the students' notebook (written by E1)—12.01/1994

During the 6th session on 19/01/1994, E1 and E2 discussed again about the equivalence between "the existence of a smaller finite decimal number" and "the existence of two consecutive finite decimal numbers".

44-E1: we cannot really speak of a gap
45-E2: If there is no gap, then it is like a straight line
46-E1: hey no
47-E2 (*laughs*): ah no, I do not see, if there is no gap, all the points are connected.
53-E1: and you can already find a finite decimal between two finite decimal numbers, but you can always find a gap between two decimal numbers also.
55-E1: Yes look you have two points – you can put a point there, but you can put also a gap.

We are at the beginning of the session. It seems that for E1 there is no contradiction between the property of density-in-itself (there is always a finite decimal number between two finite decimals) and the existence of gap between two decimals, while for E2 it is not compatible:

66-E2: Yeah, yeah, because there is a slight conflict of theory.
67-E3: no
68-E2: yeah but, I am sorry, if there is not a smaller finite decimal, it means that there is no gap.
69-E3: we draw points anyway (laughs)
70-E2: yeah but points that are so close that it makes a straight line.
73-E1: but it depends on the scale.

We interpret this exchange as an indicator that E1 has a discrete vision of the decimal line, while E2 has a continuous vision of this line, both of them relying on perceptual aspects. This echoes with a discussion, in the history of mathematics, on the possibility to make or not the line with points (Gilbert et al. 1994). The students went discussing this point for a long while, expressing several times the density-in-itself without explicating it in a formal way, this being intermingled with other aspects (such as infinite, infinitely small, etc.) but finding difficulties in overcoming the contradiction; they also moved to graphical representation and explored the idea of making a zoom. Finally a pending question remained: "is it possible to put enough points to have a curve" (turn 340).

Are There Holes in the Decimal Line?

Once the students formulated the question mentioned above, the teacher introduced the idea of considering the set of finite decimals as a subset of the set of real numbers, and reformulated the students' question:

344-P (*teacher*): Is there any jumps or not, is there any holes or not
345-E2: Well, that's it
346-E1: in fact it depends, it depends, it depends on … the axis, in fact it depends on what we put on the axis.
347-P: the holes, they depend on the axis?
348-E1: well it depends on what we put on the axis yeah
352-E1: well if we say we put finite decimal numbers, there will be no holes. If we say that we put real numbers, there will be holes normally.
353-E2: but normally, normally, I don't know the axis, they are real numbers, it is the set of real numbers.

Here E1 opened the possibility of considering the axis as a finite decimal line, while E2 assumed that the axis is necessarily the real line (which is the tacit assumption in classical mathematical activities in school). Finally, the students agreed that there are holes in the decimal line and a new conjecture directly related to the fixed-point problem emerges:

496-E1: Okay – this [the decimal line] it has holes.
497-E2: there are holes; then we have to show does the straight // the only worst case is if the line passes through something other than a finite decimal, what, when it is an abscissa other than a finite decimal number, for example one third?
498-E1: but your line, it is not a line
499-E2: no the curve should not pass through two third
500-E1: it is not, it is not a curve.
504-E1: yeah it is a curve with missing points.

This excerpt shows that at this point, the students were aware that there are gaps in the decimal line. In other words, they were ready to reconsider the initial implicit model that there is nothing in between discreteness and continuity. This opened the possibility of finding a counterexample.

Discussing the Existence of Counterexamples

Later, E2 expressed clearly that the coordinates are both finite decimal numbers (turn 618); then students discussed the possibility of the non-existence of a fixed-point (the possibility of a hole, the lack of intersection point on the curve). However, at that moment, the two students were thinking that it is not possible:

680-E2: we must prove that this never occurs, that this counterexample is false, that if you have this counterexample there will be a finite decimal number without an image or something like this, without a finite decimal image.

683-E1: Yeah, but, yeah, in fact we must prove that it is not possible, that it is something that never occurs but it's not easy.

At this point of their research, students had moved from the dichotomy between the discrete line and the continuous line to the possibility of considering a decimal line, and they tried to find a graphical representation, something difficult (Pontille et al. 1996, p. 24). They were using the concept of density-in-itself, for the set of finite decimal numbers, as an implicit model. At the end of the session, E1 seemed to consider the possibility of a counterexample. Finally, he will come back at the following session with a counterexample for the case of finite decimal numbers and for the case of rational numbers (homework).

In this account of the students' work, we focused on the specific aspects directly related to the relationships between discreteness, density-in-itself and continuity.[10] It seems that for E2, the intuitive model of the continuum is dominant, while E1 tries to overcome the contradiction between density-in-itself, quickly recognized, and the possibility of gap (or holes) that lingered all along the two sessions we analyzed. The production of counterexamples in the cases of finite decimal numbers and of rational numbers is related to the incompleteness of D and Q, which is a first step to motivate the completion of Q in Dedekind's manner. Students began to work on case 3 in February 1994; they conjecture the existence of a fixed point and tried to construct sequences converging to fixed points but did not succeed; in March 1994, the mathematician who gave the problem suggested to consider the subset $A = \{x \in [0; 1], f(x) > x\}$; students managed to solve the problem when subset A has a maximum (Pontille et al. 1996, pp. 18–19).

Brief Account of the Work of Graduate Students Engaged in a Teacher Training Program

In 2012–2013, we proposed this problem to graduate students in a teacher-training program, after having presented the construction of the set of real numbers by Dedekind. Students were then asked to provide a report on their work. Their reports confirmed that this problem is likely to rise up sound epistemological questions about discreteness, density-in-itself and continuity, even with advanced mathematics students (with at least a Bachelor or an equivalent degree), as we can see in the following excerpts.

The students had worked for about an hour; they first worked individually, and then worked in small groups of 2 or 3. Some of them made an individual report. We received six reports. Students first solved the case for integers, either by induction of by *reductio ad absurdum*. We provide and comment briefly some excerpts of their reports (Fig. 4).

K. and C. are brilliant students; we could expect that at this level, the fact that the set of finite decimal numbers is not discrete would be immediately available; but it has not been the case. Indeed, nearly all students first tried to adapt their proof from the integers, and for some of them, it took time before they realized that it is not possible.

[10] For complements and other aspects, see Pontille et al. (1996), and for potentialities of this problem for fostering proof and proving competencies, see Durand-Guerrier et al. (2012, pp. 360–361).

K. (*About the case with the set of finite decimal numbers*): I first thought it was possible to adapt the proof; then influenced by some of my classmates, I identified that the property of the existence of a successor does not hold in the set of finite decimal numbers.

C. (*About the case with the set of finite decimal numbers*): I supposed the conjecture for f: D∩[0; 1] → D∩[0; 1] is true, thinking that I would generalize the proof in ℕ. Since D is countable, one can create a bijection from D∩[0; 1] to ℕ. However, we cannot apply the results shown before because there is no successor in D. As D is not discrete, our proof no longer works.

Fig. 4 Excerpt of the report of K and excerpt of the report of C

Some students tried to prove the conjecture in the case of finite decimal numbers by the theorem of monotonic convergence for sequences. This was the case of J., who wrote in his conclusion (Fig. 5):

Identification of First Didactical Variables

The analysis of the students' work in the MATh.en.JEANS setting in 1993–1994 and the reports of the teachers' trainees in 2012–2013 support our hypothesis that the fixed-point problem is a good candidate for designing a didactical situation aimed at fostering the understanding of the relationships between discreteness, density-in-itself and continuity. It allows us to identify at least three didactical variables to take into consideration in designing the didactical situation, which we intend to experiment with university students:

1) Asking the students to solve the preliminary problem on whole numbers prior to consider the case with the subsets D∩[0; 1] and Q∩[0; 1], or not.

Indeed, solving the preliminary problem introduces in the *milieu* the explicit characterization of discreteness, with the existence of a successor for each integer, which usually remains at an implicit level. So the choice of the value "yes" for this variable is likely to influence the behavior of students, favoring an attempt to adapt the proof, while the answer "no" could favor (from grade 12 on in France) the recourse to the *Intermediate value theorem,* with a focus on continuous functions.

2) Suggesting the use of graphical representations, or not.

This is motivated by the fact that graphical representations are likely to appear as a tool for solving the problem. For this variable, the level of the students (secondary or tertiary) would be crucial. At secondary level, the number line is commonly used for solving equations or conjecturing properties of function, while it is less present at

J. (*Conclusion of the report*) "Moreover, completeness of the set of real numbers is the key to these problems. They point at this property, and enlighten the fact that a subset of the set of real numbers is not necessarily complete. The set of finite decimal numbers is dense-in-itself, but is not complete. The theorem of 'monotonic convergence' cannot hold here. Eventually, these problems lead us to reflect on the specific characteristic of the real numbers (and also to 'reconsider' the definition of its various subsets); the set of real numbers is complete, which is not so obvious and visible that it could seem. It is even less easily constructible at first ..."

Fig. 5 Excerpt of the report of J

university level, and even rejected by some students as not relevant. In the designing of didactical situations, we should take this point into consideration.

3) The least-upper-bound axiom as a characterization of continuity for the set of real numbers is available for students, or not.

The possible value of this variable is depending on the level. In France, in the current secondary schooling, the value is necessarily no. At the university level, the axiom is being introduced in lectures; however, generally speaking, students don't have the opportunity to use it by themselves in proving activities. At this level, the value of this variable is depending on the kind of work students were exposed to, in lecture or tutorial.

Examining the effect, on students' work, of the choice for the values of these variables is part of our on-coming research in the frame of Didactical Engineering (DE) (Artigue 1990), consisting in the design, the regulation and the controlled observations of didactical situations based on the fixed-point problem presented in this paper.

Conclusion

In this paper, we presented the first elements of an on-going research aimed at facing the didactical challenge of making students aware that an ordered set being dense-in-itself does not mean that this set is continuous. For this purpose, along with epistemological studies and interviews, we will elaborate and implement in experimental condition didactical situations aiming to allow an adequate appropriation of concepts of real analysis. We presented the main results from a test showing the weakness of fresh university students' knowledge about real numbers. These results support our hypothesis that unless a specific work is conducted with university students on the construction of the real number system, many students don't develop an adequate understanding of the continuum. A noticeable result is the quasi-absence of reference to the number line in students' answers, while the continuity of number line is playing an important role at the secondary schooling level through graphical representations.

Then we presented the main features of Dedekind's construction of the set of real numbers, relying on the intuition of the continuum conveyed by the number line. We claim that this elaboration supports the idea of the relevance of considering the number line as an intuitive reference for elaborating didactical situations, aimed at fostering the conceptualization of the continuum. This would be the case provided that through these situations, students are being confronted with the fact that the number line is not an appropriate representation for the set of rational (respectively finite decimal) numbers, as evidenced by Dedekind.

We also emphasized the potentiality of the *fixed-point problem* for questioning with students the relationships between discreteness, density-in-itself and continuity. Indeed, the analyzes of the students' work, in the experiment conducted in Lyon during the school year 1993–1994, show that the density-in-itself property emerges as an implicit model. In addition, the brief account on the work of pre-service mathematics teachers on the same problem supports its relevance for university students. In our ongoing research, we rely on these preliminary results to design a didactical situation whose core will be the fixed-point problem. We will implement this situation in undergraduate contexts on the one hand, in pre-service or in-service teachers training contexts on the other hand, to test its viability. Doing this, the questions raised by the role of infinity concerning the relationships between discreteness, density-in-itself and continuity for an ordered set of numbers will be considered.

A next step will be to investigate in what respect a better appropriation by students of the concept of continuum may support a better understanding of the concept of convergence, and favor the transition between Calculus and Analysis.

Appendix

Proof of the existence of a fixed-point for an increasing function f from [0; 1] to [0; 1].

Let A be the subset given by $A = \{x \in [0; 1] | f(x) \geq x\}$.

A is a non-empty subset of the set of real numbers ($0 \in A$) and is bounded above; hence, it has a supremum α.

Let us prove that α is a fixed-point for f.

Let x be an element in A. Since $x \leq \alpha$ and f is increasing, $f(x) \leq f(\alpha)$; hence $f(\alpha)$ is an upper-bound for A; since α is the supremum of A, $\alpha \leq f(\alpha)$ (1).

Since f is increasing, from (1) we infer that $f(\alpha) \leq f(f(\alpha))$. This means that $f(\alpha)$ is an element of A; as α is the supremum of A, $f(\alpha) \leq \alpha$ (2).

From (1) and (2), we conclude that $f(\alpha) = \alpha$, e.g. α is a fixed-point for f.

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