Conceptual Structure of the Accumulation Function in an Interactive and Multiple-Linked Representational Environment

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Abstract In light of the recent growing interest in conceptual learning and teaching of calculus, and especially with the focus on using technological environments, our study was designed to explore the learning processes and the role played by multiple-linked representations and by interactive technological environment in objectifying the accumulation function. The study focuses on 13 pairs of 17-year-old students familiar with the concept of differentiation but not of integration. The study was inspired and guided by semiotic mediation theory, which considers artefacts to be fundamental to cognition and views learning as a process of becoming aware of the knowledge that exists within a cultural context. Students were asked to explain the possible connection between multiple-linked representations: the function graph, the accumulation function graph, and the table of values of the accumulation function. We include three types of analysis representing the mathematical elements involved in the students' learning path of the accumulation function, and the students' interactions with the artefact. The data analysis identified four phases in the processes of objectification. We describe a conceptual structure that typifies the learning of the accumulation function concept in an interactive, multiple-linked representation environment.

Keywords Integral · Accumulation · Interactive · Multi-representation · Semiotic mediation · Objectification

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Introduction

Learning mathematical concepts with multiple-representation and interactive artefacts has been a focus of the research community in the last two decades. Researchers have studied how students learn and understand a variety of mathematical concepts with multi-representational artefacts, including the use of technological artefacts as facilitating resources, to illustrate the ideas underlying the derivative or the integral concept (Thompson 1994; Robutti 2003; Tall 2009), and the examination of the difficulties in sketching a derivative function graph from function graphs using technological artefacts (for example, Ubuz 2007). Interpretation of the relationship between a function and its derivative has been studied in a variety of contexts using iconic artefacts (e.g., Yerushalmy 1997; Shternberg and Yerushalmy 2004). Sealey (2014) investigated how calculus students develop the structure of the Riemann sum through engagement with physical context problems, and he suggested a four-layer framework to characterize student understanding of Riemann sums and definite integrals. Sealey examined in-depth the first three layers, but not the function layer. The central object in this layer is the accumulation function that is represented as a sum of products.

Few studies have examined the learning of the accumulation function concept (Thompson and Silverman 2008; Thompson et al. 2013), especially among secondary school students (Kouropatov and Dreyfus 2013). Some studies have examined the role of graphic and numeric artefacts in learning the accumulation function. In these studies, technology was used mostly in attempts to relate a function with its accumulation function in a simulated-physical situation such that of time graphs involving velocity and position (e.g., Noble et al. 2001; Berry and Nyman 2003).

In the present study we introduce the concept of accumulation as the cognitive base for learning the idea of integration. Learning the concept of integral based on the accumulation function may create an opportunity for the student to learn integration meaningfully and to apply it significantly. For this to happen, students need (a) to see the area under a curve as representing the accumulation of quantity and not necessarily an area, and (b) to make the connection between the derivative and the integral (Thompson et al. 2013; Thompson and Silverman 2008). The former requires students to become aware that the covered area is created by accruing bits that are formed multiplicatively. Therefore, the focus on the accumulation function approach is to measure the accumulation of quantities by summing the value of $f(c)\Delta x$. By contrast, the focus of the traditional approach is on the calculation of a number representing the area bound by the curve. Thompson and Silverman (2008) argued that the traditional approach is difficult to apply to quantities other than areas, but the accumulation function approach requires the understanding that $f(c)$ and Δx represent quantity, whereas $f(c)\Delta x$ represents the conjugated quantity. In the accumulation function approach, making the connection between the derivative and the integral is directly connected to the idea of accumulation, because when something changes something is accumulated, and when something is accumulated it accumulates at some rate.

In view of the recent growing interest in conceptual learning and teaching of calculus, especially learning in technological environments, we designed a learning environment that enables students to learn the concept of integration based on the idea

of accumulation in a multi-representational and interactive setting. In a previous study (Yerushalmy and Swidan 2012) we examined the ways used by one pair of students to conceptualize the accumulation function while learning with an artefact designed to support exploration through experimentation with interactive multiple-linked representations of the function and of its accumulation function. The artefact supports interactive changes of parameters, providing immediate feedback, and direct manipulation of mathematical objects in a graphic presentation. In that study, we identified the role played by the exploration of the functions of the lower limit in the definition of the accumulation graph (Yerushalmy and Swidan 2012). To deepen our understanding of the ways in which students use interactive and multiple-linked representation artefacts to conceptualize the accumulation function, we broadened our research sample and observed how students objectify the accumulation function when they vary the upper limit but keep the other parameters invariant. To better understand the role played by the interactive and multi-representational artefact in learning the accumulation function, we explore the ways in which high school students learn the accumulation function with an interactive and multiple-linked representation artefact, and examine the role played by the artefactual design features in objectifying the idea of accumulation.

The article is divided into five parts: in the first part we describe the artefact used in the study; in the second part we summarize the theoretical framework and the cognitive challenges involved in learning the accumulation concept; in the third part we discuss methodological issues; in the fourth part we use empirical findings to present the mathematical elements involved in the objectification of the accumulation function, the learning paths of the accumulation function, and the students' interaction with the artefact. We discuss the findings and pedagogical implications in the last part.

The Artefact Used in the Study

The artefact in our study, shaped by pedagogical considerations and culturally accepted mathematical meaning, is computer software, Calculus UnLimited (CUL, Fig. 1) (Schwartz and Yerushalmy 1996). We considered the elements in CUL (objects such as Cartesian graphs, value tables, symbolic icons, etc.) to be the signs that carry meaning accepted in the mathematical culture. As a multi-representational artefact, CUL contains different types of signs that we grouped into four categories.

Graphing Tools Two vertically aligned Cartesian systems, coordinated vertically. The trajectory in the upper Cartesian system signifies a function. The function is defined symbolically by the free input of a single variable expression. The trajectory in the bottom system presents the values of Riemann sums $\sum_{i=1}^n f(x_i) \cdot \Delta x_i$.

Numeric Tools The associated table of values contains three columns. The left column presents the upper limit values, the middle column displays the delta x values, and the right column shows the accumulated values. The user can show numerically the accumulated point coordinates by right-clicking on a point (Fig. 2).

Fig. 1 CUL interface: reflection of varying the upper limit parameter on the accumulation function graph of the function $f(x) = x^2$

Accumulation Tools Five methods of accumulation are available by clicking an icon or making a menu selection (right, left, or middle rectangles, trapezoids, and continuously accumulating areas (see the icons in Fig. 1)). Because our study focused on the computation of right rectangles, the rectangles that appear on request in the upper Cartesian system (bounded between the function and the x-axis) represent the product of Δx_i and $f(x_i + \Delta x_i)$. Rectangles are color-coded to reflect the product sign (positive or negative) (Fig. 3). Each point along the trajectory in the lower Cartesian system (the

Fig. 2 Showing the accumulated point coordinate values using the pointer

Fig. 3 Color-coded rectangles

accumulation function) is a representation of a Riemann sum (sum of products) expressed mathematically as $\sum_{i=1}^n f(x_i) \cdot \Delta x_i$.

Tools for Value Control After the function f and the method of accumulation are specified, three parameters determine the value of the accumulation at a given point: the lower and upper boundaries of a bounded region and the width of each interval into which the region should be divided (Fig. 1). The design of CUL attempts to direct the attention of the user to these parameters and emphasizes the control over the upper limit value with immediate visual feedback showing the value of the accumulation points. Students control the bounded area by using the arrow keys to move the marker in intervals of Δx to the left or to the right of the lower limit (Δx is an absolute measure of the interval, always achieving a positive value both right and left of the lower limit, as shown in Fig. 1). The upper limit value is represented in the accumulation function graph by a marked colored point, and in the table of values by a framed accumulated value cell (Fig. 1). Varying the lower limit causes the accumulation function graph to transform vertically, as shown in Fig. 4.

Theory of Semiotic Mediation

The study was carried out using the theoretical framework of semiotic mediation, as developed by Bartolini Bussi and Mariotti (2008). The theory of semiotic mediation models the learning process by taking advantage of the potential of artefacts. The model aims to describe how meanings related to the use of a certain artefact can evolve into meanings recognizable as mathematical. The theory of semiotic mediation assumes that social interaction and semiotic processes play a key role in learning, particularly in

Fig. 4 Reflection of varying the lower limit value on the accumulation function graph of the function $f(x)=x^2$ –4. **a** Lower limit value=–3; **b** lower limit value=1

situations in which learners are encouraged to use the artefact in order to solve a given task. The theory of semiotic mediation considers learning to be an alignment between the personal meanings arising from the use of a certain artefact for the accomplishment of a task and the mathematical meanings that are deployed in the artefact. The Stanford Encyclopedia of Philosophy defines an artefact as an object that has been intentionally made or produced for a certain purpose. We used the term in a more restricted sense to refer to hand-made objects that have meanings within a certain culture. The semiotic mediation considers artefacts of any kind to be central to cognition and to play a fundamental role in it. It has been assumed that the relation between artefact and knowledge is expressed by culturally determined signs and that the relation between the artefact and the learners in the course of accomplishing a specific task is expressed by signs such as gestures, speech, and drawings. The semiotic mediation approach makes possible the analysis of the evolution of personal meaning into mathematical meaning by distinguishing between different kinds of signs.

Consistent with the shared claim of semiotic systems in general (Radford 2003; Arzarello 2006), we use the term “sign” in a broad sense to include oral and written language, gestures, and interaction with the artefact. Signs in general, and mathematical signs in particular (e.g., graphs and tables of values in our example) play two roles. Radford et al. (2005) defined these roles as “social objects in that they are bearers of culturally objective facts in the world that transcends the will of the individual. They are subjective products in that in using them, the individual expresses subjective and personal intentions” (p. 117). Berger (2004), who studied the functional use of mathematical signs, suggested a dual interpretation of the meaning of signs and objects: personal meanings, “to refer to a state in which a learner believes/feels/thinks (tacitly or explicitly) that he has grasped the cultural meaning of an object (whether he has or has not),” and a mathematical meaning, “to the extent that its usage is congruent with its usage by the mathematical community” (p. 83). In the context of using artefacts, the semiotic mediation theory describes the relations between personal meanings and

mathematical meaning as a double semiotic relationship. On one hand, we concentrate on the use of the artefact for accomplishing a task, recognizing the construction of knowledge within the solution of the task. On the other hand, we analyze the use of the artefact, distinguishing between the constructed personal meanings arising in individuals from their use of the artefact in accomplishing the task (top part of Fig. 5) and meanings that an expert recognizes as mathematical (bottom part of Fig. 5) when observing the students' use of the artefact in completing the task (left triangle in Fig. 5). We refer to this double semiotic relationship as the *semiotic potential of an artifact*. The semiotic mediation theory assumes that any artefact can have valuable *semiotic potential* with respect to particular educational goals (Bartolini Bussi and Mariotti 2008). We assume that the potential is defined with respect to a particular design and set of pedagogical goals. Therefore, determining the semiotic potential constitutes a basic element in the design of any pedagogical plan based on the use of a given artefact.

From a pragmatic point of view, Radford suggests a semiotic tool to analyze the dynamically evolving relationship between personal and mathematical meanings. The basic components of the semiotic tool are the students' progressive attention and awareness of the mathematical object. Varieties of semiotic means of objectification that have a representational function attract the students' attention to mathematical objects. Furthermore, the properties of the artefact can help students attend to the mathematical objects related to the activity under consideration. Paying attention to the necessary aspects of the mathematical phenomenon and using various semiotic means of objectification, students become aware of the attributes of mathematical objects within that phenomenon. Being aware, students attain the objectification of the mathematical objects, which then become apparent to them through various devices and signs.

Different kinds of signs are produced in the practical activity with the artefact. The semiotic mediation theory distinguishes between three kinds of signs: artefactual, pivot, and mathematical. *Artefactual signs* refer to the artefact and its use. This type of sign is produced through the social use of the artefact (upper right vertex in Fig. 5). These shared signs, generated through the social use of the artefact, may evolve into mathematical signs that refer to the mathematics context. The mathematical signs are related to the mathematical meanings shared in the institution to which the classroom belongs (right side of Fig. 5). Through a complex process of evolution of the artefact sign into a mathematical sign, other types of signs, which Bartolini Bussi and Mariotti called *pivot signs*, play a crucial role. The authors suggest that the characteristic of these signs is

Fig. 5 Semiotic mediation model (from Bartolini Bussi and Mariotti (2008))

their shared polysemy, that is, they may refer to the activity with the artefact as well as to natural language and to the mathematical domain.

Based on the definition of artefact as a hand-made object that has meanings within a certain culture, we consider CUL to be artefact because it has been designed as a hand-made object that represents culturally accepted meanings of relationships between a function and its accumulation function. The linked graphs, the table of values, and the value control tool, which are the focus of the present study, are culturally determined signs that convey cultural knowledge: the first graph signifies the function and the other its accumulation function. The table of values is also considered a culturally determined sign that signifies the accumulation function, and the dynamic tools signify the various parameters of the accumulation function. Exploiting the semiotic potential of the artefact enables teachers to use it as a tool of semiotic mediation, taking advantage of the possibility of guiding students to connecting the personal meanings that arise from the use of the artefact with those that an expert recognizes as mathematical meanings. To exploit the semiotic potential of the artefact, we focus on the social interactions between the students and the artefact and eliminate the teacher's intervention in the process of becoming aware of the accumulation function as it is represented in the artefact. According to our theoretical assumption, through social interaction with the artefact, students employ different types of signs to align their individual perspectives with the cultural knowledge deposited in the artefact. To achieve our aim, we explore the ways in which students become aware of the accumulation function as it is represented in the artefact. We propose to answer the following research questions:

- 1) What are the mathematical elements used by the students in the process of becoming aware of the mathematical meanings of the accumulation function when it is learned with interactive and multiple-linked representational artefacts?
- 2) How do the features of the interactive and multiple-linked representational artefact assist in the evolution of the personal meanings that arise in the course of accomplishing a task into mathematically accepted meanings?
 - a) How does the interaction with the interactive and multiple-linked representational artefact help students become aware of the mathematical meanings built into in the artefact?
 - b) How do the artefactual signs promote the evolution of personal meanings into cultural meanings?

Cognitive Challenges in Learning the Accumulation Function

The integral is a complex concept because it can be interpreted in different ways: an anti-derivative process leading to the indefinite integral; the area bounded by a graph and the x -axis; the Riemann sum representing length, area, or volume (a view leading to the definite integral); and the accumulation function, where the upper limit " X " is a variable and the lower limit is a fixed parameter. Interpreting the meanings of the different variables participating in the accumulation function is challenging. The accumulation function relates to at least three dynamic elements: the lower limit is a fixed parameter, its value indicating the beginning of the accumulation of an area,

bounded between the function and the x -axis; the interval Δx , which determines the pace of accumulation and the accuracy of the area computation; and the independent upper limit variable, which determines the value that bounds the accumulation. Each point in the accumulation graph represents the accumulated area (often computed as the product of the dimensions of rectangles) up to the x -value of this point. Zero values in the accumulation graph represent either the value of the accumulation at the lower limit or negative and positive products that accumulated to zero. Understanding the accumulation function requires awareness that each specific value assigned as a lower limit determines a single accumulation function in a family of accumulation functions. It also requires grasping the meaning of the variables x , $f(x)$, and a , in the symbolic representation

$(x, f(x), \int_a^x f(u)du)$, and the dummy variable u that appears in a calculation only

as a placeholder and disappears completely in the final result. Thompson (1994) found the visual understanding of this simultaneous triple change to be a challenge even for students who already learned calculus. Thompson (1994), and Thompson and Silverman (2008) studied and articulated two challenges in learning the accumulation function and grasping its idea in relation to other concepts in calculus: understanding that accumulation is a function and understanding that accumulation occurs at some rate. To grasp the idea of the accumulation function, the learner “sees the accumulation and its rate of change as two sides of the same coin” (2008, p. 51).

The tension between the mathematical meaning and the natural language meaning of the accumulation concept is challenging for students (Hall Jr 2010). In natural language, the term “accumulation” is usually interpreted as acquiring more of a given quantity (Thompson and Silverman 2008). The mathematical meaning of the accumulation function considers the summation of both positive and negative values. In addition to this tension, students must maintain at least three levels of mathematical ability to understand the accumulation function. Thompson and Silverman (2008) summarized these abilities: (a) the process conception of a formula, which means that students must make an effort to calculate any particular value of the formula; (b) a covariational understanding of the relationship between x and $f(x)$, which means that as the x value varies, the function varies accordingly; and (c) the ability to coordinate the variation of the accumulated quantity with the variation of x and $f(x)$, that is, to coordinate between the three values in the formula mentioned above.

Study Design

Participants

The present study explored approximately 12 h of learning of 13 pairs of 17-year-old students from two public schools in Israel; a rural one and an inner-city one. All participants were high achievers in mathematics and studied mathematics at the highest level in their schools. All participants returned signed consent forms and completed all the learning tasks assigned to them. At the time the meetings took place, the students had already learned the concepts of function, derivative, and indefinite integral. The students were also familiar with the conventional function graph artefact, which was

part of their previous study of functions within the framework of the formal school curriculum. They were familiar with using the derivative symbolically and were able to analyze functions by finding extreme points. The students studied the function and the derivative from a textbook that is available for high school students in Israel. Generally, the textbooks contain a brief theoretical discussion of the concept being considered and provide formulas to apply the concept and strategies to solve exercises. The students were familiar with the indefinite integral concept graphically, as the inverse of derivative. They had learned it graphically with an iconic, dynamic, and interactive artefact, as part of a research project (for details about the ways in which the students learned the indefinite integral as the inverse of derivative, see (Swidan and Yerushalmy 2014).

Procedures

The participants volunteered to participate in four after-school meetings. The learning took place in the computer lab at the schools. Each pair of students shared one computer. The first author introduced them briefly to the interface and illustrated how to use it. He explained, for example, how to input the symbolic expression, how to change the parameters in the value-control tool, and how to change the scale of the Cartesian systems. In particular, the students were told about the technical functionality of the artefact. The first author was present as an observer and provided technical and miscellaneous clarifications. To study the objectification of accumulation function by varying the upper limit, we asked the students to explain and explore the possible connection between two given function graphs and the table of values. They were given the following instructions:

Because the product sign (positive or negative) depends on both the sign of Δx and of the y -ordinate, we designed the task in such a way that students could attend to the four possible signs resulting from the product of two numbers (Tall 2009). For this reason, the recommended “ X ” value was (-3) . For example, in the neighborhood of $x = -3$ the function $f(x) = x^2$ presents two possibilities. On the right side of -3 , the Δx and the ordinate y are positive, whereas on the left side the y -ordinate is positive and Δx is negative. The function $f(x) = x^2 - 9$ presents also two possibilities in the neighborhood of -3 , which are different from those of the function $f(x) = x^2$. For example, on the right side, the Δx is positive and the y ordinate is negative, whereas on the left side the Δx is negative and the y ordinate is positive.

Data Collection and Analysis

To collect the data, we video-recorded all the pairs of students in each session as they engaged in solving the task assigned to them. We also captured the corresponding computer screens. In total, we video-recorded 13 films, which document the entire learning process. The films range from 42 to 67 min and are 55 min long on average.

We conducted the data analysis at three levels. The macro level identifies the mathematical elements involved in objectifying the accumulation function. Mathematical elements were defined as segments of discourse in which the students sought to discover the mathematical relationship inherent the accumulation function, for example, Δx , product, sum of products, and the positions of the accumulation function graph. The meso level presents learning trajectories and the complexity of each mathematical element. At

the micro level, the analysis detailed the ways in which the artefact features supported the students' work in objectifying the accumulation function.

We performed the triple analysis because although the macro analysis enables us to identify the mathematical elements involved in the learning process, it is limited and does not permit the identification of the complexity of the mathematical elements, and does not shed light on the evolution of personal meanings into mathematical ones. The meso analysis, however, enabled us to learn about the complexity of the mathematical elements and about the evolution of personal meanings into mathematical ones, but it did not shed light on the ways in which the students used the interactive and multi-representational artefact in the objectification of the accumulation function. To better understand this process, we looked closer at the ways in which the students used the dynamic manipulation features, the transition between representations, and specific features of the artefact.

At the macro level, we watched the videos repeatedly to detect the mathematical elements that students apply in order to explain possible connections between the representations. The unit of analysis is the collated transcript of all 13 pairs of students in order to identify the mathematical elements used by them. We divided the recording into episodes, which we coded based on the mathematical element under consideration. An episode begins when the students detect a specific mathematical element and ends when we identify a transition in the element. For example, episodes that contain statements like " Δx splits the x -axis into segments of half" were coded as " Δx ;" episodes that contain statements like "at negative three, which is the lower value, it is zero, which means that the lower is a limit" were coded as "lower limit;" and episodes that contain statements like "it is increasing because it is the sum of both" were coded as "accumulation as sum of product."

At the meso level, we analyzed the data in two stages. At the first stage, we distinguished personal meanings that are accepted mathematically from those that are not. Students' statements (usages) that in the context of the accumulation function are not congruent with the usage of the mathematical community were defined as personal meanings that are not accepted mathematically. By contrast, usages that are congruent with those of the mathematical community were defined as personal meanings that are accepted mathematically. For example, statements like "the y -value of the initial point in the accumulation function is equal to the height of the rectangle," which refers to the objectification of the initial value point (category B2) were defined as personal meanings that are not accepted mathematically. For each value of Δx other than 1, the initial point in the accumulation function is equal to the product of the rectangle dimensions, but not equal to the height of the rectangle. For this reason, we defined this statement as a personal meaning that is not accepted mathematically. Statements like "the y -value of the initial point in the accumulation function is the same as the first rectangle area" were defined as personal meanings that are accepted mathematically. Statements like "the lower limit is like zero. After zero the accumulation function is positive, before zero it is negative," which refer to the objectification of the accumulation function, as the upper limit is smaller than the lower limit (category C2), were defined as personal meanings that are not accepted mathematically. The lower limit splits the x -axis into two sections, left and right. The sign of Δx (positive or negative) on the right is determined as positive, and on the left as negative. But the function sign (positive or negative) and the sign of Δx determine the sign of the accumulation function (positive or negative). We defined these statements as personal

meanings that are not accepted mathematically because the students determined the sign of the accumulation function based on the sign of Δx alone. By contrast, we defined statements like “[points at the section where Δx and the function are negative]: Negative times negative is positive. Here [the section where Δx is positive and the function is negative] it is positive times negative is negative. And positive times positive it is also positive” as personal meanings that are accepted mathematically, because the students determined the sign of the accumulation function based on the sign of Δx and of the function.

Distinguishing personal meanings that are accepted mathematically from those that are not helped us identify the complexity of mathematical elements and to learn about the evolutionary path of the learning process. To better understand the complexity of each element and to analyze the evolution of personal meanings into mathematical ones over time, we applied the second stage of the meso analysis, using the strategy of evolution over time mentioned in Yerushalmy (2006) to each pair of students.

To examine the ways in which the students used the interactive and multi-representational artefact in the objectification of the accumulation function, we micro-analyzed the data. At this level, the unit of analysis is an individual episode. We transcribed all the excerpts in Arabic and translated them into English. We used the semiotic mediation model (Bartolini Bussi and Mariotti 2008) to analyze the learning processes in each excerpt. We used the basic components of the semiotic tool, which are the students’ progressive attention and awareness of the mathematical object, as suggested by Radford (2003), to analyze the dynamically evolving relationship between personal and mathematical meanings. We focused especially on the interplay between gestures, artefactual actions, and the use of words to analyze the evolutionary meaning of the accumulation function.

To establish reliability in identifying the mathematical elements involved in objectifying the accumulation function, two coders coded the data independently. Each coder identified 10 elements. After discussing the elements that had been identified, we agreed on the nine mathematical elements that are presented in this study.

To establish reliability in identifying statements as personal meanings that are accepted mathematically or as personal meanings that are not accepted mathematically, a second coder recoded a subset of statements for each mathematical element. This subset included some statements that were coded by the first author as being accepted mathematically and other statements that were coded as not being accepted mathematically. The second coder recoded a total of 75 statements from among all the mathematical elements. Forty-eight statements were coded by the first coder as accepted and 27 as not accepted mathematically. In identifying statements that are accepted mathematically, agreement was about 92 % ($N=48$). In identifying statements that are not accepted mathematically, agreement was 96 % ($N=27$).

Findings

Mathematical Elements Involved in Objectifying the Accumulation Function Graph

Through the macro level, we identify the mathematical elements involved in learning the accumulation function. The mathematical elements were grouped into four

categories: (a) dividing the x -axis into equal-length segments, (b) identifying the product of the dimensions of the rectangles as an area, (c) identifying the accumulation as a sum of products, and (d) addressing the properties of the accumulation function graph. Categories (b), (c), and (d) were divided into subcategories based on the mathematical element under consideration, as shown in Fig. 6 (Fig. 7).

In the graph below, we summarized the frequencies of pairs of students who used each element at least once in the learning process.

We found that all the students tried to identify the product of the dimensions of the rectangles which are represented by the initial value point in the accumulation function graph. Furthermore, all of them tried to identify the accumulation function in a domain, in which the upper limit values are bigger than the lower limit values (B2, C1). The majority of students (11 pairs) considered the reflection of the lower limit value in the accumulation function graph (B1). Eleven pairs of students also tried to identify the accumulation function in a domain, in which the upper limit values are smaller than the lower limit values (C2). Nine pairs tried implicitly to identify Δx as dividing the x -axis into segments of equal length (A1). Nine pairs also considered the position of the accumulation function graph relative to the x -axis (D1). Six pairs tried to identify the concavity of the accumulation function graph (D3). Five pairs considered the accumulated zero (C3), and three pairs considered the tendency of the accumulation function graph (D2).

Paths Followed to Learn the Accumulation Function

In Table 2, we present the distinction between personal meanings that are accepted mathematically and those that are not for each pair of students. The number of episodes associated with each category is also shown in Table 2. The shaded gray columns represent personal meanings that are not accepted mathematically; the non-shaded columns represent personal meanings that are accepted mathematically. For example, the number 3 in the shaded gray cell of column B2, row “Narmin & Maram,” indicates that three episodes belonging to the pair of students Narmin and Maram are associated

Fig. 6 Mathematical elements identified in the first round

Fig. 7 Frequencies of pairs of students who used each mathematical element at least once

with the mathematical element “Initial value point in the accumulation function” and are not accepted mathematically. In the “Total” row, we summarize all the episodes that are associated with each category, and distinguish between personal meanings that are and are not accepted mathematically. For example, in column B2, row Total, the number 41 indicates that 41 of the episodes are associated with the mathematical element “Initial value point in the accumulation function;” 18 of these were considered to be accepted mathematically and 23 were considered not accepted. In the last row, we present the percentage of cases in each category. In the Total column we list the overall number of episodes that belong to each pair of students. Empty cells indicate that the students did not address the corresponding mathematical elements in their exploration (Table 1).

About 5 % (10/211) of the episodes were devoted to identifying Δx as dividing the x -axis into intervals of equal length. About 13 % (27/211) were devoted to identifying the reflection of the lower limit value in the accumulation function graph. All of these but one were coded as meanings accepted mathematically. Because the majority of the students addressed these elements, the findings may indicate that the identification of these two mathematical elements was not challenging for them. It is likely that the design of the artefact helped students identify these mathematical elements. About 20 % of the episodes (41/211) were devoted to the identification of the initial value point. Twenty-three of these were coded as personal meanings that are not accepted mathematically. This finding may indicate that identification of the initial value point was challenging for the students.

Twenty seven percent of the episodes (57/211) were devoted to identification of the accumulated values when the upper limit value is bigger than the lower one. Of these, forty-four episodes were coded as personal meanings that are accepted mathematically.

Table 1 The task given to the students versus the instructions given to the students verbally by the first author

The task given to the students	Instructions given to the students
Explore and explain the connections between graphs in the upper and the lower Cartesian systems and the linked table of values (Fig. 1). You should be able to input function expressions [from the functions listed below] and obtain function graphs such as $f(x)=x^2$, x^2-9 , x^3 , $(x+3)(x-1)(x+4)$. While working on the task, you may use the CUL software to generate function graphs by means of symbolic expression, and change the value of “X” by pressing the $\triangleright \triangleleft$ arrows in the upper part of the value box [tool for value-control], as needed. As default values, it is recommended to start with 0.5 in Δx box, -3 in the lower box, and -3 in the “X” box.	<p>Before the students began their exploration activity, the first author provided the following instructions to them:</p> <ul style="list-style-type: none"> • Start the exploration by plotting the function $f(x)=x^2$ and conjecture about the mathematical relationship between the graphs and the table of values in this case. • Next, plot other functions from the given list to look for additional relationships and to verify or refute the previous conjectures.

In contrast, among the 31 episodes that were devoted to identification of the accumulated values when the upper limit value is smaller than the lower one, only four were coded as personal meanings that are accepted mathematically. This finding indicates that the students were better able to identify the accumulation function when the upper limit is bigger than the lower limit than the other way around.

About 17 % of the episodes (35/211) were devoted to identification of the accumulation function properties. Only three episodes of these were coded as personal meanings that are not accepted mathematically. Twenty two episodes were devoted to identifying the position of the accumulation function graph. Two of these were coded as personal meanings that are not accepted mathematically. Four episodes were devoted to the tendency of the accumulation function, and all of them were coded as accepted mathematically. Nine episodes were devoted to the concavity of the accumulation function graph and coded as accepted mathematically. This finding indicates that the students who noticed the last two elements also became aware of their mathematical meaning.

Distinguishing the personal meanings that are accepted mathematically from those that are not enabled us to learn about the complexity of the mathematical elements, but it did not shed light on the evolution of personal meanings into mathematical ones. To learn about the latter, we analyzed the evolutionary processes of the personal meanings for each pair of students. Thus, the analysis unit of this level was that of student pairs.

Below we present an evolutionary path in the learning process, which we identified through the data analysis. We consider this path to be representative because we found a tendency similar to the one illustrated in Fig. 8 in the learning paths of ten pairs of students. The chart in Fig. 8 shows the order in which students were engaged with various mathematical elements, and the time elapsed. The data are color-coded. The mathematical elements appear in the left column. The numbers at the bottom represent the time in minutes. We divided each minute into four squares, each one representing 15 s. The colors distinguish between personal and mathematical meanings. The gray

Analysis of the charts suggests a possible path for learning the accumulation function graphically. The chart illustrates the suggested path, which consists of four phases: (A) identifying Δx , (B) identifying the product of rectangles, (C) identifying the accumulation as the sum of products, and (D) addressing the accumulation function properties. Despite the differences between the charts, the similarity between them is apparent in the trend line.

The identification of the sum of the products usually occurred after the students have identified the product $f(x)\Delta x$. Students who tried to identify the sum of the products before becoming aware of the product $f(x)\Delta x$ were not able to do so (gray segments in row C1 in Fig. 8). Analysis of the charts reveals that only three episodes out of 13 that were coded as not accepted mathematically in category C1 (accumulated value in which the upper limit is bigger than the lower, non-shaded C1 column in Table 2)

Table 2 Frequency of mathematical elements distinguishing personal meanings that are and are not accepted mathematically

Students	Categories												lr					
	A1	B1	B2	C1	C2	C4	D1	D2	D3									
Lin & Bian	1	0	2	0	3	2	2	0	0	4	1	0	1	0	16			
Saja & Salma					1	1	3	2	1	1	3	0			13			
Hamza & Mahmoud	1	0	2	0	1	2	5	0	0	2	1	0	1	0	14			
Ans & Misba	1	0	3	0	1	2	4	2			2	0	2	0	17			
Narmin & Maram	1	0	3	1	0	3	5	1	0	1	2	0	3	0	21			
Aroob & Namat	1	0	2	0	1	1	4	1	0	2	5	0	2	0	20			
Mohamed & Ahmed					1	0	3	1	0	2					7			
Nagam & Nihaia	1	0	2	0	3	1	6	0	1	0	2	2			18			
Adi & Adham		1	0	0	2	0	5	0	0	2				1	0	11		
Mariam & Zoubi		2	0	0	3	1	1	0			1	0		3	0	10		
Mohand & Moatasim	2	0	3	0	2	1	3	2	2	5	2	0	1	0		23		
Mkazha & Slemam	1	0	3	0	2	3	1	2	0	2				1	1	16		
Remy & Shahed	1	0	3	0	1	4	2	2	0	6	1	0	4	0	1	0	25	
Total	10	0	26	1	18	23	44	13	4	27	10	0	20	2	4	0	8	1
	10		27		41		57		31		10		22		4		9	
% (N=211)	5 %	0 %	12 %	0.5 %	8 %	11 %	21 %	6 %	2 %	13 %	5 %	0 %	9 %	1 %	2 %	0 %	4 %	0.5 %
	5 %		12.5 %		19 %		27 %		15 %		5 %		10 %		2 %		4.5 %	

occurred after the students objectified the product. This finding indicates that identifying the product $f(x)\Delta x$ made possible the identification of the sum of the products.

In general, identifying the accumulation function properties occurred at advanced stages in the learning intervention, after the students objectified the sum of the products and plotted a function graph with positive and negatives values. This finding highlights the role that function properties played in the students becoming aware of the accumulation function.

The Ways in which Students Used the Artefact to Objectify the Accumulation Function

In this section we illustrate the microanalysis of the ways in which the students used the artefact to objectify the accumulation function. This type of analysis sheds light on the role played by the artefact in the learning process of the accumulation function. We paid a special attention to the role played by the upper limit parameter in creating learning situations that allow students to become aware of the mathematical elements involved in the accumulation function. We also focused on the transitions between the multiple representations and the artefactual signs.

Objectifying the Accumulation Function as a Sum of Products

This subsection contains two excerpts. The first one illustrates the assignment of meaning to the value of the points of the accumulation function graph in which the upper limit values are bigger than the lower limit values. The excerpt illustrates the role played by the upper limit and by the multiple linked representations in making students aware of the accumulation as a sum of products. In the first excerpt the students vary the upper limit by clicking the accumulated values in the table of values, not by repeatedly clicking the ► button. The second excerpt illustrates the assignment of meaning to the value of the points of the accumulation function graph in which the upper limit values are smaller than the lower limit values. The excerpt shows how varying the upper limit makes it possible to ascribe a meaning of relative zero to the lower limit value. We also show how changing the function graph makes possible the evolution of personal meanings that are not accepted mathematically into personal meanings that are accepted mathematically by objectifying the accumulation function whenever the upper limit is smaller than the lower limit.

Accumulated Values in which the Upper Limit is Bigger than the Lower Limit The students noticed the similarities between the structure of the accumulated numbers in the table of values and the accumulation function graph. The relation between the first point in the accumulation function graph, the area of the first rectangle, and the accumulated number in the table of values are artefact signs that promote the development of the mathematical meaning of accumulation as a sum of products.

Remy: We have concluded that this number (*the number 3.125 in the table of values*) is the y-value of this function (*points to the first point in the accumulation function graph* (Fig. 9a) (and is the area of the rectangle (*points to the first*

Fig. 9 **a** Remy points at the first point in the accumulation function graph; **b** points at the first rectangle in the upper Cartesian system; **c** points at the value 5.125 in the table of values; **d** points at the second point in the accumulation function graph; **e** points at the first point in the accumulation function graph

rectangle, (Fig. 9b)). This number (*the number 5.125 in the table of values* (Fig. 9c)) is bigger than this number (*the number 3.125*), and this point (*the second point in the accumulation function graph* (Fig. 9d)) is above this point (*the first point in the accumulation function graph* (Fig. 9e)). We may add the area of the rectangles.

The students related the value 3.125 in the table of values to the y-value of the first point in the accumulation function graph. They also compared two successive accumulated values (5.125 and 3.125) and two successive points in the accumulation function graph. This comparison appears to have helped them to hypothesize that the number 5.125 may correspond to the y-value of the second point in the accumulation function graph. Furthermore, Remy's utterance ("This number is bigger than this number, and this point is above this point") suggests that the structure of the accumulated values in the table of values and the shape of the accumulation function graph helped the students formulate their hypothesis. To test their hypothesis, they computed the sum of the areas of the two rectangles. After they applied the formula for the area of the rectangle to the second rectangle and added the area of the first rectangle to it, they found a number equal to the Δ one that appears in the table of values.

Accumulation Values in which the Upper Limit is Less than the Lower Limit The function $f(x)=x^2$ appears on the upper Cartesian system. As instructed, the students set the lower limit value to -3 and the upper limit value to -2 . The students clicked the button repeatedly to obtain -3.5 in the upper limit. They noticed that the rectangles changed color when the upper limit value became smaller than the lower limit value. The students appear to have objectified the lower limit value as dividing the x-axis into two sections, distinguishing between two situations designated by dark and bright colors.

Saja: Why did the rectangle change its color? Salma: Because its value (*the upper limit value*) passes the lower value. -3 (*the lower limit value* (Fig. 10a)) is like zero. After zero (*points to the right of -3 on the x-axis* (Fig. 10b)) it is positive, before zero (*points to the left of -3 on the x-axis* (Fig. 10c)) it is negative.

Salma's utterance (" -3 is like zero. After zero it is positive, before zero it is negative") suggests that she divided the x-axis into two sections, to the right and to the left of the lower limit value. It appears that the students signified the rectangle areas that are to the right of the lower limit value to be positive, and those to the left of the lower limit value to be negative. Salma's last utterance and gesture suggest that she determined the sign of the product (positive or negative) based on its location relative to

Fig. 10 **a** Salma points at the lower limit value (-3) on the upper Cartesian system; **b** points to the right of -3 on the x -axis; **c** points to the left of -3 on the x -axis

the lower limit value. But Salma's determination is not accepted mathematically, because what determines the product sign are the signs of Δx and of the function.

In the course of the learning intervention, the student plotted the function $y=x^3$ and set as a default the lower limit value at $x=-3$. They repeatedly clicked the $\blacktriangleleft$ button to reach from 4 to -4 . As the upper limit value became smaller than the lower limit value, dark rectangles appeared because the function and Δx value were negative (Fig. 11a and b).

Salma: You see it is less in the lower one. But it is dark. Here (points at the section where Δx and the function it is negative (Fig. 11c). Negative times negative is positive. Here (points at the section where Δx is positive and the function is negative (Fig. 11d) it is positive times negative is negative. And positive times positive it is also positive (Fig. 11e).

The students have identified the artefact sign (dark rectangles) as a positive product. Salma noticed that the rectangles located in the $x < -3$ domain have a dark color. This observation enabled her to refute their previous hypothesis. The cubic function appears to have enabled the students to revise their hypothesis. Salma's gestures and utterances suggest that they tried to explain the positive sign of the product, represented by the dark color, by the fact that "negative times negative is positive." Thus, they used the positive and negative mathematical signs in three ways: (a) signing Δx , (b) signing the function, and (c) signing the $f(x)\Delta x$. To objectify the signing of the product, they signed Δx relatively to the lower limit value (to the left of the lower limit it is positive, otherwise it is negative), and signing the function graph relatively to the x -axis

Fig. 11 **a** Dark rectangle left of the lower limit value; **b** positive value point left of the lower limit value in the accumulation function graph; **c** Salma points at the section where Δx and the function are negative; **d** points at the section where Δx is positive and the function is negative; **e** points at the section where Δx and the function are positive

(the upper x -axis is positive, otherwise it is negative). Objectifying the artefact signs and endowing them with mathematical meaning appears to have allowed the students to revise their hypothesis and to confirm it. They applied the same process of objectifying the artefact sign to the other domains of the function graph.

Objectifying the Accumulation Function Graph Properties

This subsection contains two excerpts. The first one illustrates the role played by the upper limit in objectifying the positional element of the accumulation function graph relative to the x -axis. The second excerpt illustrates the role played by the upper limit in the students' becoming aware of the concavity of the accumulation function and connecting it to the change in the rectangles.

Ascribing Meaning to the Position of the Accumulation Function Graph The students plotted the function $f(x)=x^2-9$ and set -3 as the default lower and upper limit. They clicked the button $\triangleright$ repeatedly to obtain 6 in the upper limit.

Nihaia: When this area (points at the area with bright color (Fig. 12a)) is equal to this area (points at the dark colored area (Fig. 12b)), the y -value is zero here (points at the accumulated zero in the accumulation function graph (Fig. 12c)).

Nagam: Would you repeat that?

Nihaia: Here (points at the bright area) we add, add, and add negative areas (Nagam uses her index finger to count (Fig. 12d)). Here (points at the dark colored rectangle) we start to add positive numbers. It is still negative, but the y -value is getting bigger. When the positive area becomes equal to the negative area here we get zero. That is, the sums of both areas are zero. After that, the sum of the areas becomes positive.

Nihaia's first utterance ("when this area is equal to this area") suggests that she distinguished between two artefact signs (bright and dark colors) that signify negative and positive areas. Her use of the word "equal" suggests that she is comparing the negative and positive products. Following Nagam's request for additional information about her finding, Nihaia explains again, in greater detail, the process of obtaining the accumulated zero. Nihaia's utterance and Nagam's gesture "here we add, add, and add areas" suggest that they accumulated products step by step. Nihaia concludes her process of accumulation by stating the result of the process as "the result of the addition must be negative." Nihaia's utterance ("we start to add positive numbers") suggests that she considered three aspects at the same time: her use of the verb "add" suggests

Fig. 12 a Nihaia points at the bright-colored area; b Nihaia points at the dark-colored area; c Nihaia points at the accumulated zero in the accumulation function graph; d Nagam counts the rectangles using her index finger

that she was accumulating values, her use of the noun “positive” suggests that she distinguished between the signs of the products based on rectangle color, and her use of the noun “number” suggests that she signified the filled rectangles as the area value. Her utterance “it is still negative but the y-value is getting bigger” suggests that she was considering the process of accumulation as the rectangles move from the left to the right side, and it also suggests that the students were aware that the dark rectangle changes the tendency of the accumulation function graph but not its position relatively to the x -axis. The change in position of the accumulation function graph is articulated explicitly by Nihai “When the positive area becomes equal to the negative area here we get zero. That is, the sums of both areas are zero. After that, the sum of the areas becomes positive.”

Ascribing Meaning to the Concavity of the Accumulation Graph The students clicked the accumulated values in the table of values one at a time. With each click on the upper Cartesian system a new rectangle appeared. The graph of the function $f(x)=x^2$ appeared in the upper Cartesian system. The default of the lower limit was set to -3 and of the upper limit to 2 .

Shahed: Here zero (clicks the accumulated value, 0 , in the table of values), nothing here (points at the upper Cartesian system then clicks the accumulated value, -3.125 , and one rectangle appears on the upper Cartesian system (Fig. 13a and b)). Here it is big (clicks the accumulated value, 5.125 , in the table of values (Fig. 13c and d)). It gets smaller. How much did it add? It added the difference. The difference From 3 to 5 is 2 ; (she points at 3.125 , then at 5.125 , then clicks the accumulated value, 6.25 , in the value table (Fig. 13e) here (points at the third rectangle in the upper Cartesian system (Fig. 13f)) it is adding less because the rectangle area is becoming smaller, so the number we add is less. (Clicks the next accumulated value, 6.875 (Fig. 13g).) You see, because it is a small area we add less. Here, at zero (around the origin) there are no rectangles (points at two equal accumulated numbers in the table of values (Fig. 13g and h)), so the values do not change. Here (the first rectangle after the parabola vertex) the rectangle appears again, but it is small, so we add a few here (points at the number

Fig. 13 a Shahed clicks on 3.125; b one rectangle appears on the upper Cartesian system; c Shahed clicks on 5.125; d two rectangles appear on the upper Cartesian system; e Shahed clicks on 6.25; f three rectangles appear on the upper Cartesian system; g Shahed clicks on the accumulated value 6.875; h no rectangles appear around the origin; i Shahed points at the number 8.625 in the table of values; j rectangles appear as the student clicks on 8.625

in the accumulated values (Fig. 13k)). But here the rectangle gets bigger (*the next rectangle*), the number we add here is greater. Look at the graph (*points at the accumulation function graph*); it is the same principle. The differences between the points are the same as in the rectangle area.

When the students clicked on the accumulated value 0 in the table of values, no rectangle appeared in the upper Cartesian system. When they clicked the following accumulated number, 3.125, one rectangle appeared in the upper Cartesian system. The students signified the filled color rectangle as an amount. They compared the amounts using the words “big” and “less.” Regarding the first rectangle they said “here it is big,” and regarding the second rectangle they said “It gets smaller.” The students focused on two consecutive accumulated values and evaluated quantitatively the difference between them: “The difference From 3 to 5 is 2”. They continued clicking the ► button, adding a rectangle with a smaller area. Based on qualitative consideration, the students hypothesized about the difference between the third and the last two accumulated values (the areas of the first and second rectangles): “here it is adding less because the rectangle area is becoming smaller, so the number we add is less.” To confirm their hypothesis, they paid attention to the neighborhood of the origin in the function graph and made the connection between the disappearance of the rectangles and the stability of the rate of change: “Here, at zero there are no rectangles, so the values do not change.” The students focused on the domain where x was bigger than zero in the upper Cartesian system: “But here the rectangle gets bigger, the number we add here is greater.” Their last utterance (“look at the graph; it is the same principle”) suggests that they shifted their attention to the accumulation function graph in an attempt to apply their hypothesis to it. Their utterance (“The differences between the points are the same as in the rectangle area”) suggests that they considered qualitatively the rate of change between two consecutive accumulated values. Their focus on the rectangles appears to have helped them formulate the explanation of the rate of change of the accumulation function graph.

Discussion

In this study we introduced the accumulation function as a cognitive base for learning the concept of integral. We consider the accumulation function to be essential for learning the integral because it focuses on the multiplicatively accruing bits. For example, understanding the accumulation function may help students better understand the connection between accumulation and rate of change. Seeing the interrelation between the concepts of accumulation and rate of change may, in turn, help them understand the fundamental theorem of calculus. Learning the accumulation function, however, is challenging for many students, who find it difficult to understand if they are not aware of the bits that are being accrued. Furthermore, the accumulation function, which is represented as the sum of products, contains many interrelated mathematical elements, of which the students must become aware. For example, to objectify the product, students need to objectify the mathematical elements of which it is composed. To objectify the sum of the products they must first objectify the products themselves.

Therefore, objectifying delta x is essential for objectifying the product, and objectifying the product $f(x)\Delta x$ is essential for objectifying the sum of products.

In the present study we identified the mathematical elements involved in learning the accumulation function and have suggested a learning path for teaching it. This path describes a developmental progression toward understanding the accumulation function and sheds light on the interrelations between the mathematical elements involved in it. The discussion that follows summarizes these mathematical elements and the role played by the artefact in the objectification of the accumulation function.

Objectifying the Product of $f(x)\Delta x$

Thompson and Silverman (2008) considered objectifying delta x as a length to be an essential part in understanding the accumulation function. The majority of the students who participated in this study have objectified delta x as a segment. Our decision to determine that delta x has a fix-equal value was inspired by pedagogical rather than mathematical considerations. This decision made the students aware that delta x is dividing the x -axis into intervals of equal length. In this case, describing the manner in which the numbers in the value-control tool varied by equal differences, and linking these varying numbers with the dynamic rectangles of equal width appears to be an important aspect in objectifying delta x as dividing the x -axis into segments of equal length.

Despite the common opinion that the lower limit plays a minor role in objectifying the concept of integration (e.g., Sealey 2014), our study shows that the lower limit value plays a central role in objectifying the accumulation function. The lower limit value is considered a *relative zero* that divides the x -axis into two domains. Delta x has a positive sign for all the numbers that are bigger than the lower limit value. Otherwise, the signing of delta x is negative. The accumulated values consist of a sum of products, and each product consists of delta x multiplied by the function value. Therefore, signing both the function value and delta x is an important step in objectifying the accumulation as the sum of products. Although signing the function value is not a complex task, signing delta x was found to be a complex one because the students were familiar with the fact that the origin divides the x -axis into two domains.

The emergence of the area concept appears to have guided students to consider the area of the rectangle that reappeared on the screen. But ascribing the meaning of a rectangle area to the initial value point, following the assignment of meaning to the reflection of the lower limit value, did not occur early in the learning process; it happened after several unsuccessful attempts to endow the initial value point with mathematical meaning. This finding is consistent with those of Sealey's (2014) study, claiming that difficulties in understanding the product are not necessarily related to performing calculations but rather to how the product is formed. In agreement with Sealey, our findings indicate that the complexity of becoming aware of the product was not related to performing the calculation but to the semiotic structure of the multi-representations. Nevertheless, the students were able to correlate the accumulated values in the table of values with the y -value of the accumulated points in the accumulation function graph, but they were not able to connect the accumulated values with the area of the rectangle.

Objectifying the Accumulation Function

Thompson and Silverman (2008) argued that to conceive of an accumulation function defined as x is to objectify the accumulated quantity for each value of x . In our case, which considers the learning of the approximation of the accumulation function, it means that the students must coordinate three values simultaneously $(x, f(x), \sum_{i=1}^n f(a + i\Delta x) \Delta x)$. Objectifying the accumulation function means becoming aware of the accumulated value, whether it is represented as a number in the table of values or as a point in the accumulation graph. Our analysis shows that all the students in our study have become aware of the accumulated value as representing the sum of the areas of the rectangles. Indeed, objectifying the accumulation idea was not particularly challenging for the students (Table 2). Although the study conducted by Sealey (2014) focuses on the context of applying integration to solving word problems, our findings are consistent with Sealey's findings that "none of the students spent much time explicitly discussing the concepts represented in the summation layer" (p. 240).

The main complexity in objectifying the accumulation as the sum of products was the students' focusing on a single rectangle (usually the second one). In this case, the decreasing domain of the function graph produced a cognitive conflict having to do with the contrast between the decreasing rectangle area and the increase of the accumulation function graph. The accumulation graph played a central role in resolving this conflict. Ainsworth (2006) argued that the inherent properties of representation may constrain the interpretation of other unfamiliar representations. The inherent properties of the increasing accumulation function graph appear to have constrained the interpretation of the accumulation as a sum of the products. In this context, the structure of the table of values as a representation of the sequence of accumulated numbers, and the properties inherent in the accumulation function graph, which was considered to be a complementary representation of the same phenomena, played a central role in promoting the evolution of the mathematical meanings of the accumulation as a sum of the products. In addition, the interactivity of the table of values and its relation to other representations seems to have promoted the objectification of the accumulation as a sum of rectangle areas. The appearance of the appropriate rectangles after the students clicked each number in the table of values was effective in making students aware of the accumulation as the sum of products.

Objectifying the accumulation function required students to become aware of accumulation values for each value of the variable x . This overview of the accumulation function should allow students to understand the concept of integral when the upper limit value is smaller than the lower limit value. Despite our attempt to provide students with a wide variety of graphs to help them objectify the meaning of the accumulation function for each value of x , our data analysis reveals that they were not able to objectify the accumulation graph when the upper limit value is smaller than the lower limit value. The majority of students noticed the change that occurs in the signing of the product, represented by the artefactual sign (bright vs. dark-colored rectangles). Only two pairs of students became aware of the accumulated values when the upper limit becomes less than the lower limit. These pairs had objectified the reflection of the lower limit value in the accumulation function *as a relative zero*, whereas the others had not. For those

students who were not able to objectify the lower limit value as a relative zero, their personal meanings did not evolve into mathematical meanings. Therefore, based on our data analysis and on another study (Yerushalmy and Swidan 2012), it was reasonable to conclude that objectifying the reflection of the lower limit as a *relative zero* is an essential element in objectifying the overview of the accumulation function.

The conflict between the natural and the scientific language is well documented in the literature (Hall Jr 2010). In the context of natural language, the accumulation concept is considered to be an increasing quantity. As Thompson and Silverman (2008) noted, “you accumulate a quantity by getting more of it. We accumulate injuries as we exercise. We accumulate junk as we grow older. We accumulate wealth by gaining more of it” (2008, p. 3). Because the scientific meaning of accumulation considers the accumulation of both positive and negative products, objectifying the accumulation mathematically requires students to become aware of the positive and negative values of the accumulation function. Our data analysis reveals that nine pairs of students objectified the positiveness and the negativeness of the accumulation function graph. The artefactual signs that distinguish between the positive and the negative products by the color of the rectangles were found to have a strong influence in drawing the students’ attention to the positiveness and negativeness of the accumulation function. Moreover, the accumulated zero in the accumulation function graph as an intersection point on the x -axis played a central role in objectifying the positiveness and negativeness of the accumulation function graph. Explaining the meaning of the accumulated zero required students to use what we called in our previous study the “balance metaphor.” In the current study, as in the previous one, the students considered the balance metaphor as they distinguished between two equal quantities with different signs. Using this metaphor appears to have helped students objectify the artefact signs and endow them with the mathematical meaning of positiveness and the negativeness of the accumulation graph.

Learning the accumulation function required students to become aware of inverse processes (addition and subtraction) inherent in the accumulation concept. The inverse process is inherent in the accumulation function graph where the graph changes its tendency (e.g., from increasing to decreasing). For this reason, objectifying the tendency of the accumulation function graph is considered essential for learning the accumulation concept. Our data analysis shows that three pairs of students objectified the tendency of the accumulation function graph. For this element as well, the artefactual signs played an important role in the objectification of the accumulation function. The dynamically-linked representation, the value-control tool, and the colored rectangles combined to explain the tendency of the accumulation function graph.

Pedagogical Implications

Our study suggested a learning path for the accumulation function in specific cases in which the two components of the product are lengths and the product is the area of a rectangle. This path may be smoothly followed with other applications of integrals, as for example, when students learn about the position function as derived from the velocity function. Indeed, the fundamental idea is the same, and what changes is the quantity that is created from the bits, which themselves consist of measures of two quantities.

In the accumulation function approach, the definite integral is expressed differently than in the traditional approach of Riemann sums. In the traditional approach, we ask whether there is a number that is the limit of the Riemann sums as Δx approaches zero. Conversely, in the accumulation function approach we ask whether there is a function that is the limit of the family of the approximate accumulation function, as Δx approaches zero. This change in the convergence process is more than a cosmetic one. It purports to enable students to maintain the notion of multiplication as a transition from a finite to an infinite sum of products. We believe that this issue remains a challenge for calculus students. The main question in this context is still whether and how using linked multiple representations helps maintain the concept of multiplication. We answered this question partially in Swidan and Yerushalmy (2013) and presented evidence of maintaining the concept of multiplication. But further studies are needed to fully answer this question.

A well-developed meaning of the accumulation function can be part of a coherent calculus that focuses on students recognizing the connections between rates of change and the accumulation of quantities (Thompson et al. 2013). In this context Park et al. (2013) noted that the fundamental theorem of calculus is a generalization of the relationship between “the rate of change” and “the change.” Nevertheless, the students’ awareness of the relationships between the change, the rate of change, and the area between the graph of the rate of change and the horizontal axis was not well established, and their perception of integration of the rate of change remained a simple calculation of the area between the graph of the rate of change and the horizontal axis. Indeed, the initial purpose of the task given to the students was to explain the accumulation function graph from the exact rate of change of the function (the function graph). Our data analysis, however, showed that six pairs of students objectified the rate of change of the accumulation function graph, which means that they described how the accumulation function changes. This connection between the accumulation function and the original (rate of change) function is fundamental in understanding the FTC. We do not claim that the students became aware of the FTC, but we argue that the design of the artefact allowed them to notice the relationship between the change in the rectangles and the rate of change of the accumulation function. Because helping undergraduate students appreciate the FTC is an essential challenge, noticing the relationship seems to be an important step. Therefore, this finding has direct implications for designing learning units that use the accumulation function approach to help students appreciate the connections between the rates of change function and the change function.

The results of the present study should help educators become aware of the semiotic potential of interactive, multi-representational artefacts in learning advanced mathematical concepts. This study also generates ideas for curriculum designers to develop curricula for the learning of advanced mathematical concepts using interactive and multi-representational artefacts and make the mathematics of change and variation available to all students.

The present study did not require students to use the symbolic notation of the accumulation function to represent it. This, however, does not diminish the quality of the learning. On the contrary, the absence of symbolic requirements may be a reason for participants to express verbally their valuable understanding of the accumulation function and of its mathematical elements. In any case, we do not claim that there is no need for a symbolic notation of the accumulation function. But we believe that

perceiving the inherent ideas of mathematical concepts, even those that are learned as part of undergraduate studies, through other actions and representations, can improve the meaning of the notations of these concepts, and it should precede the symbolic procedures. Naturally, to shed light on the transition to symbolizing the notation of the accumulation function, future studies must explore how students come to understand the notation involved in it after they have learned it with interactive and multi-representational artefacts, and how students link the multi-representations with a symbolic notation that makes sense to them.

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