Factors Influencing Students' Propensity for Semantic and Syntactic Reasoning in Proof Writing: a Case Study

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Abstract In this paper we present a case study of an individual student who consistently used semantic reasoning to construct proofs in calculus but infrequently used semantic reasoning to produce proofs in linear algebra. We hypothesize that the differences in these reasoning styles can be partially attributed to this student's familiarity with the content, the teaching styles of the professors who taught him, and the amount of time he was given to complete the tasks. We suggest that there are factors that have been ignored in previous research, including domain, instruction, and methodological constraints, that researchers should consider when ascribing to students a proving style.

Keywords Proof · Proving styles · Semantic proof productions · Syntactic proof productions

Introduction

Overview

In recent years, several mathematics educators have noted that one can productively engage in advanced mathematics in two different ways. An individual can focus on the formal aspects of mathematics by understanding statements in terms of their logical structure and the formal definitions of their terms, and by using calculation and formal rules of inference to produce new mathematical ideas. Alternatively, an individual can understand and reason about mathematical concepts using informal representations of

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these concepts, such as thinking about these concepts in terms of graphs, diagrams, gestures, or examples (Pinto and Tall 1999, 2001; Raman 2003; Vinner 1991; Weber and Alcock 2004, 2009). Recently a number of research reports have identified individual students or groups of students who predominantly engage in one of these two types of reasoning while rarely engaging in the other (e.g., Alcock and Inglis 2008; Alcock and Simpson 2004, 2005; Alcock and Weber 2010; Duffin and Simpson 2006; Moutsios-Rentzos 2009; Pinto and Tall 1999, 2002; Weber 2009). However, in these studies, students' reasoning styles were often identified by their performance on a small number of tasks and these tasks were nearly always situated in a single mathematical domain.

In this paper, we present a case study of an individual who approaches proof production tasks in linear algebra and calculus with very different reasoning styles. We use this case study to illustrate that an individual's reasoning in advanced mathematics (and proof production in particular) is not necessarily consistent across domains, and we suggest factors that should be taken into account when considering a student's reasoning style.

Different Modes of Reasoning in Advanced Mathematics

The products of advanced mathematical reasoning—definitions, conjectures, theorems, and proofs—are typically expressed in a unique representation system. This representation system uses a combination of specialized words and logical syntax, where the nouns and adjectives in this system have precise definitions that are agreed upon by the wider mathematical community, and the rules of inference are based on logical deduction (Weber and Alcock 2009). This is also the representation system in which statements to be proven are typically presented to university students in advanced mathematics courses.

One way that mathematics can be understood is working within this system. For instance, a concept can be understood by studying its definition, making deductions from this definition to see what properties are true about that concept, reformulating the definition into equivalent statements, and comparing the logical structure of the definition to other concepts that one is aware of (e.g., Pinto and Tall 1999; Weber 2009). When constructing proofs, one can start with definitions and permissible assumptions and use logical inference (including applying known theorems) and calculation to deduce the desired conclusion, without considering informal representations of that concept such as graphs, diagrams, or specific examples (e.g., Vinner 1991; Weber and Alcock 2004). Weber and Alcock (2009) refer to this type of reasoning as *syntactic reasoning*.

Alternatively, a concept can be interpreted in other representation systems, and understood in terms of gestures, graphs, diagrams, or prototypical examples. An individual may try to understand a concept by constructing links between the definition of the concept and these informal representations, perhaps refining their informal understanding if necessary (e.g., Pinto and Tall 1999). When constructing a proof, one can interpret the statement to be proven in one of these informal representation systems, justify it through informal arguments that use these representations, and use these arguments as a basis for constructing a formal proof (e.g., Raman 2003; Weber and Alcock 2004). Weber and Alcock (2009) refer to this type of reasoning as *semantic reasoning*.

For instance, consider the following proof construction task that was used in the study reported in this paper:

Suppose T is a non-invertible matrix. Prove that there exists a nonzero matrix S such that TS is the zero matrix.

Based on actual students' work we now illustrate the employment of syntactic and semantic reasoning to address this task. One way in which students approach this task is by representing the matrix T symbolically as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, recalling the fact that a matrix is invertible if and only if its determinant is not zero, and representing this statement by writing $\det(T) = ad - bc = 0$. Some students then represent the matrix S symbolically as $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and calculate the product TS , writing $TS = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$. At this point they set this product equal to the zero matrix and come up with the following four equations: $ae + bg = 0$, $af + bh = 0$, $ce + dg = 0$, and $cf + dh = 0$. Using the fact that $\det(T) = ad - bc = 0$ some solve for the entries of the matrix S by writing $e = f = d$ and $g = h = -c$. Finally, they conclude that for any 2×2 non-invertible matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the matrix $S = \begin{pmatrix} d & d \\ -c & -c \end{pmatrix}$ meets the required conditions.¹ Notice that in this approach students engage productively in the task employing syntactic reasoning: they use symbolic representations of the matrices, their product, and the determinant, and work from a recalled theorem regarding non-invertible matrices to deduce what the entries of the required matrix need to be.

In another approach, students start working with a particular example of a non-invertible matrix, such as $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. At that point they begin constructing a matrix S that would meet the required conditions, eventually coming up with a matrix such as $\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$. Searching for an explanation for why this particular construction works, some observe that the columns of non-invertible matrices do not form a linearly independent set, saying that one column is a multiple of the other one. The matrix T is then symbolized as $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & ca_{11} \\ a_{21} & ca_{21} \end{pmatrix}$ and, following the construction in the case of the specific example, construct the matrix $S = \begin{pmatrix} c & c \\ -1 & -1 \end{pmatrix}$.² In contrast to the previous approach, these students do not reason primarily in a syntactic manner (i.e., within the representation system of mathematical proof). Instead, they decide to represent a non-invertible matrix in terms of a specific example and proceed to solve

¹ We note here that this answer is technically incorrect, as the matrix S could be the zero matrix (in the case $c = d = 0$).

² Like in the previous case, this is not a completely correct answer: technically, cases in which $a_{11} = a_{21} = 0$, and either $a_{12} \neq 0$ or $a_{22} \neq 0$ have not been considered.

the problem for that particular matrix, in that representation system. Crucially, this specific construction provides some of them with the main idea of how to approach the more general one, back in the representation system of mathematical proofs.

It is important to note that, as illustrated above, that semantic and syntactic reasoning can complement each other in students' proof productions. In particular, the employment of semantic reasoning in a proof construction task in which the to-be-proven statement is presented in the representation system of mathematical proofs (and the expected product is a proof in that same representation system) induces a translation back and forth between informal and formal arguments. Furthermore, the interaction between the two types of reasoning may not be as linear as in the example above: a student may begin employing one type of reasoning, get stuck, start over using the other type of reasoning, get stuck again, decide to continue where she left off with her first approach, and so on. Similarly, both types of reasoning may lead to important contributions in the proof construction attempt of a student.

Cognitive Styles in Advanced Mathematics

To define *cognitive styles*, we adopt the perspective of Riding and Cheema (1991), as used in mathematics education by Duffin and Simpson (2006). A *cognitive strategy* is a general cognitive approach that an individual can use to accomplish a class of mathematical tasks. An individual is said to have a *cognitive style* if he or she consistently invokes the same cognitive strategy when working on a class of mathematical tasks. Semantic and syntactic reasoning constitute different cognitive strategies that an individual may use to engage in advanced mathematical thinking. Recently, several researchers have suggested that students and mathematicians may have corresponding cognitive styles in advanced mathematics—i.e., that students and mathematicians may show a strong propensity to engage in either syntactic or semantic reasoning when approaching tasks in advanced mathematics.

Pinto and Tall (1999) hypothesized that this may be the case with mathematicians, citing Poincare (1913), who referred to Reimann as an intuitive thinker “who calls geometry to his aid” and Hermite as a logical thinker who “never invoked a sensuous image” when discussing mathematics (Pinto and Tall 1999, p. 281). Similar contrasts can be found when mathematicians reflect on their own practice. For instance, Weyl (1940) claimed that, when doing mathematics, a mathematician “forgets what the symbols stand for... there are many operations that he can carry out with these symbols, without ever having to look at the things they stand for”. In contrast, William Thurston (1994) stated that when doing or reading about mathematics, he is continually trying to interpret ideas in terms of his mental models that are not represented by formal mathematical syntax. In a large-scale interview study, Burton (2004) identified some mathematicians whose reasoning was predominantly symbolic and others whose reasoning was mostly visual.

In mathematics education, researchers have also hypothesized that, like mathematicians, students may have cognitive styles in advanced mathematics (e.g., Alcock and Inglis 2008; Alcock and Simpson 2004, 2005; Alcock and Weber 2010; Duffin and Simpson 2006; Moutsios-Rentzos 2009; Pinto and Tall 2002; Raman 2003; Weber 2009), and have begun to explore what makes students with either cognitive style successful or unsuccessful.

Difficulties with Documenting Cognitive Styles

In a comprehensive synthesis of the research on general cognitive styles, Coffield et al. (2004) concluded that both researchers and teachers frequently label individuals as having a particular cognitive style based on insufficient evidence. They warn that one should not infer that students have a cognitive style due to their behavior completing a small number of tasks, or tasks within a narrow setting. The cognitive strategy that one invokes may be dependent upon the demands of the task, as well as other cultural or environmental factors (Entwistle 1998).

When assigning cognitive styles to students in advanced mathematics, mathematics educators have generally made this judgment based on students' behavior on tasks within a single mathematical domain. Table 1 briefly summarizes the research articles that we are aware that assign a cognitive style to students in the context of undergraduate or graduate proof-oriented mathematics courses. What is notable in Table 1 is that only Moutsios-Rentzos (2009) analyzed students' cognitive strategies in two different mathematical domains (three tasks in abstract algebra and three tasks in analysis) before assigning to them a cognitive style in advanced mathematics. Further, while Alcock and Simpson (2004, 2005) are careful to indicate that their findings are specific to real analysis, others are less specific about the limitations of their findings and often imply they are discussing students' use of cognitive strategies in advanced mathematics in general.

Purposes of this Paper

In the present study, we investigate the extent to which the choice of mathematical domain influences the cognitive strategies that students employ in proof construction tasks. As pointed out above, we have very little evidence that students identified as having a particular cognitive style in mathematics invoke the same cognitive strategy when working on mathematical tasks in different domains. On the one hand, it is possible that these students' tendency to work in a given representation system does not depend on the mathematical subject of the given tasks. However, it is also possible that this consistency with which they employ a particular type of reasoning strategy would disappear when working on similar tasks in another domain.

Thus, this investigation will help us start delineating the boundaries and stability of cognitive styles in advanced mathematics, which in turn will allow us to better interpret the results from previous research and inform the design of future studies on the subject.

Methods

Materials

The case study reported in this paper is part of a larger study on students' proving processes. In this study, 12 students were asked to write seven proofs in linear algebra and seven proofs in calculus. In each mathematical domain, we included tasks of different difficulty level that could reasonably be approached by our participants by employing semantic or syntactic reasoning. Both judgments (the level of difficulty and the existence of different, but accessible approaches) were based on interviews with

Table 1 Research articles assigning a cognitive style to university students in the context of proof-oriented mathematics

Paper	General claim regarding cognitive styles	Context and evidence
Pinto and Tall (1999, 2001)	Based on the ways in which they learn new concepts in advanced mathematics courses, some undergraduate students can be classified as formal thinkers, and others as natural thinkers.	Eleven students were interviewed seven times during their first-year real analysis course. Three students learned about convergence, continuity and differentiation by making deductions from the given definitions (formal), while three others did so by linking the definitions to their concept images (natural).
Weber (2001)	Undergraduate students rely on syntactic proving strategies in abstract algebra.	Four undergraduates were asked to think-aloud as they solved seven tasks in group theory. These students' proof attempts relied entirely on their syntactic knowledge of the relevant concepts in abstract algebra.
Pinto and Tall (2002)	Some undergraduate students successfully learn multi-quantified definitions in analysis by consistently linking these definitions to their concept images.	One student (Chris) was interviewed seven times during his first-year real analysis course. Chris successfully learned about convergence, continuity and differentiation by giving meaning to definitions through graphs and diagrams.
Raman (2002)	For many undergraduate students, proof is not about "key ideas"—i.e., many students do not believe proof construction is related to informal investigations of graphs and examples.	Eleven first-year undergraduate students were asked to think-aloud (and were later interviewed) as they solved one proving task in calculus. These students produced either informal or formal arguments and did not believe the two could be linked.
Alcock and Simpson (2004)	Some students learn real analysis by consistently employing visual imagery in their reasoning.	Eighteen real analysis students were interviewed five times in one term. Ten of these students regularly employed visual reasoning when approaching problems about the convergence of sequences and series.
Alcock and Simpson (2005)	Some students have a tendency to learn real analysis almost exclusively by means of verbal and algebraic reasoning, with almost no reference to visual imagery.	Eighteen real analysis students were interviewed five times in one term. Seven of these students rarely employed visual reasoning when approaching problems about the convergence of sequences and series.
Duffin and Simpson (2006)	Some graduate mathematics students can be classified as natural or alien learners.	Thirteen PhD mathematics students were interviewed on how they learn new mathematics. Some talked about trying to connect new knowledge to their existing understanding (natural), while others talked about accepting new ideas without being too concerned about their meaning (alien).
Alcock and Inglis (2008)	Successful doctoral students show different propensities to use semantic and syntactic reasoning.	Two doctoral students were asked to think aloud as they estimated the truth of five number theory conjectures. One of them displayed a considerably higher rate of example usage than the other.

Table 1 (continued)

Paper	General claim regarding cognitive styles	Context and evidence
Moutsios-Rentzos (2009)	Undergraduates can be classified as different types of syntactic or semantic reasoners.	Fifteen undergraduates were asked to think-aloud as they approached six proving tasks in abstract algebra and analysis. Students in different groups were found to employ either a semantic or a syntactic strategy as their initial strategy choice.
Weber (2009)	Some undergraduate students successfully learn new mathematical concepts by relying primarily on syntactic reasoning.	One student (Issac) was interviewed as he learned a new definition in analysis and completed eight proof tasks related to this concept. Isaac successfully learned the new concept by developing formal representations of mathematical ideas and making logical deductions.
Alcock and Weber (2010)	Some undergraduate students consistently and successfully approach proof-related tasks in a semantic manner, and others do so in a syntactic manner.	Two students were asked to think-aloud (and later interviewed) as they solved four proof tasks on equivalence relations and functions. One of them (Brad) consistently used examples, while the other (Carla) consistently worked from the syntax and form of the given definitions and questions.

mathematicians, the authors' own experience doing and teaching mathematics, and feedback from an advisory board providing guidance on this project. Regarding the level of difficulty of these tasks, we verified our judgment by giving these task items to two classes of undergraduate math majors (all of whom had successfully completed a transition to proof course and at least one calculus and one linear algebra course) and examining the proportion of students who were able to complete or make significant progress on each task. The 14 tasks that we used are given in the Appendix.

Procedure

Twelve advanced undergraduate mathematics majors from a research-intensive university agreed to participate in this study and were paid a moderate fee for their participation. Each participant met individually with an interviewer for two 90-min sessions. Participants were told that they would be asked to write proofs and to “think aloud” as they constructed the proofs. They were informed they would be given 10 min to complete each proof and that they should write up their final proofs as if they were going to be graded in a mathematics exam.

In the first interview session, the participant began by completing a practice problem to become accustomed to the interview format. The participant was then given one of the study tasks. The participant was permitted to work on a proof until he or she wrote a proof that he or she was satisfied with, the participant felt that he or she could not make any more progress, or 10 min elapsed (whichever event occurred first). The interviewer then asked the participant questions about their proving process, including a summary of what the participant did, what the main ideas of the proof were, and how the main ideas of the proof were generated. This process was repeated six more times for other tasks.

In the second interview session, the participant attempted the remaining seven tasks using the same protocol as above. The participant was then asked general questions about his or her proving process, knowledge of linear algebra and calculus, and experience as a student in courses in these areas. All interviews were videotaped.

At any point in the study, two resources were available to the participants. First, if participants could not recall the definition of a relevant concept, they could ask the interviewer for the definition. At that point, the interviewer would hand them a sheet of paper with the definition of the concept and an example of the concept. Second, participants had access to a computer with a graphing calculator application that enabled participants to make basic calculations and view the graph of any function that they wished.

The Case Study of Caleb

In this paper, we report on the performance of one student, Caleb.³ Caleb was a mathematics major who was studying to be a secondary mathematics teacher. At the time of the study, he had just completed an undergraduate degree in mathematics and was beginning the fifth year of a 5-year mathematics education program that leads to a master's degree in mathematics education and state certification to teach secondary mathematics. In the first interview session, Caleb completed the seven linear algebra tasks. In the second interview session, he completed the seven calculus tasks.⁴

As is common in case study research (Bromley, 1986; Yin 2003), this single-case study is not presented here with the intent of generalizing Caleb's behavior to a wider population, but to inform theory development (in this case the construct of cognitive styles in advanced mathematics) by comparing the empirical results of the case study to the broader theory. Thus, Caleb was chosen primarily because his approach to different tasks presented a theoretically interesting contrast: his proving strategies for the linear algebra tasks and the calculus tasks were highly consistent within each domain, but were also quite different between domains. Furthermore, when answering questions about why this might have occurred, Caleb was articulate in describing interesting reasons for these differences.

Analysis

The first two authors independently coded Caleb's entire interview and met afterwards to discuss and reach an agreement on the few differences in their coding. Each coder conducted the analysis in four stages. First, Alcock and Inglis (2009) argued that when evaluating whether syntactic or semantic reasoning was used in a proof construction, what is most important is to determine if the prover represented a concept in a different representation system than the formal mathematical system used in mathematical proof. For this analysis, we flagged every instance in which Caleb drew a diagram or graph, or constructed a specific example of a general concept.

Second, Weber and Mejia-Ramos (2009) argued that it is not only important to know *if* a diagram, graph, or example was considered by a student when attempting to construct a proof. Analyzing *how* these informal representations of concepts were used, including the participants' intended purpose for considering these representations and

³ A pseudonym.

⁴ The order of domains (and the order of tasks in each domain) was randomized in the larger study.

how insights gained from these representations related to the final proof that was produced can also provide useful information. In this second level of analysis, for each informal representation of a concept that Caleb considered, we describe why he used this representation and how (if at all) it aided his proof construction. In particular, we coded for each instance Caleb attempted to interpret the statement to be proven in terms of informal representations, and whether or not he managed to construct an informal explanation of the statement to be proven by drawing inferences from previously constructed informal representations.

Third, Weber and Mejia-Ramos (2009) further noted that the absence of semantic reasoning in a student's mathematical work does not necessarily imply the employment of syntactic reasoning, as the student may not have engaged in much reasoning of any type. Thus, it is also important to check the extent to which syntactic reasoning plays a role in a given proof construction attempt. In this third level of analysis, each coder focused on Caleb's general approach to each task, flagging every instance in which Caleb began working with definitions of concepts in the statement to be proven, and used current assumptions and logical inference (including applying known theorems and calculation) to deduce new information.

Finally, as Table 1 illustrates, when researchers claim that students have a semantic or syntactic reasoning style in advanced mathematics, they often base these findings as much on interview data as on performance on mathematical tasks (e.g., Alcock and Weber 2010; Duffin and Simpson 2006; Pinto and Tall 2002; Raman 2003; Weber 2009). For Caleb, we document patterns of reasoning he used in his calculus and linear algebra tasks and report on interview data when he was asked how he commonly approached these proofs.

Results

Like most other participants in the study, Caleb performed better in the Calculus tasks than in the Linear Algebra ones (see Table 2).

As Table 2 indicates, Caleb correctly interpreted and meaningfully engaged in all but one task (L3). Also, even though he only produced one fully valid proof (L4), Caleb constructed four mostly correct proofs (L1, C1, C3, C4), and made substantial progress on three other tasks (L6, C5, C6). We now focus on Caleb's reasoning strategies when approaching these tasks.

Table 2 Caleb's performance on all tasks

	Linear algebra tasks	Calculus tasks
Incorrect interpretation of the statement or no valid inference.	L3	
No substantial progress, but correct interpretation of the statement and at least one valid inference.	L2, L5, L7	C2, C7
Incorrect or incomplete proof with substantial progress.	L6	C5, C6
Mostly correct proof, with only minor errors.	L1	C1, C3, C4
Completely correct proof.	L4	

Linear Algebra Tasks

Use of Graphs, Diagrams, and Examples

As Table 3 illustrates, Caleb did not draw or discuss a graph or a diagram in any of his seven proof attempts. He considered examples in two of the seven proof attempts: on problems L2 and L6. We now focus on the role played by these examples in these two proof attempts.

Problem L2 asked Caleb to prove that if a square matrix A had the property that $A^3 = 0$, then A had no non-zero eigenvalues. Caleb conjectured that a square matrix whose cube is the zero matrix might have a special property that was useful to this proof. In an attempt to figure out what that special property was, Caleb produced an arbitrary 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and began using matrix multiplication to compute A^3 , but quickly abandoned this attempt before completing the calculation. He did not verbalize any inferences from his work. We coded this as using an example since he reduced the abstraction of the problem by looking at a 2×2 matrix, where the A in the problem statement could have been any square matrix.

Problem L6 asked Caleb to prove that if a 2×2 matrix T was not invertible, then there existed a 2×2 matrix S such that $TS = 0$. When Caleb began his work, he asked for the definition of invertible. The sheet containing the definition contained the theorem giving several equivalent conditions to a matrix being invertible. Caleb attempted to use the fact that since T was not invertible, $\det(T) = 0$. He then recalled the theorem that $\det(TS) = \det(T)\det(S)$, but was not sure if he recalled the theorem correctly. To verify the theorem, he chose two 2×2 matrices and multiplied them together, writing: $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 16 & 17 \\ 6 & 7 \end{pmatrix}$. He then verified that this computation satisfied $\det(TS) = \det(T)\det(S)$, providing him with confidence that this theorem was true. However, he did not use this theorem in his proof attempt.

Table 3 Caleb's use of informal representation in Linear Algebra tasks and its contribution to his proof attempts

Use of informal	Contribution to representation proof attempt
<i>L2</i>	
Caleb attempted to find a general expression for A^3 when A was a 2×2 matrix.	Caleb announced he was hoping to find a special property A would have if $A^3 = 0$, but never found a general form for A^3 so his investigation yielded no contribution.
<i>L6</i>	
Caleb multiplied two 2×2 matrices to check that $\det(ST) = \det(S)\det(T)$ was correct.	Caleb successfully verified that a theorem that he recalled, $\det(ST) = \det(S)\det(T)$, was correct in this instance. This increased Caleb's confidence in the theorem, although the theorem was not useful for proving the task.

Caleb's General Approach in the Linear Algebra Tasks

As seen above, Caleb rarely engaged in semantic reasoning when attempting to construct proofs in the linear algebra tasks. However, this was not due to lack of work of any kind. Caleb worked hard on each of these seven tasks, spending the full 10 min per task on five of them and 6 min per task on the remaining two. Furthermore, as Table 2 indicates, his work on these tasks was at least partially successful, as Caleb correctly interpreted and meaningfully engaged in all but one of these tasks (L3).

Instead of working from diagrams or specific examples, Caleb consistently employed syntactic reasoning when working on these linear algebra tasks. One of Caleb's most extreme use of syntactic reasoning was displayed on L1, a task in which Caleb was asked

to prove that $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R}, xyz \geq 0 \right\}$ was not a subspace of $\mathbb{R}^3$. Although a

single counterexample would have sufficed, Caleb instead used deduction to produce what he believed was a class of counterexamples. When Caleb first read the problem statement, he wrote down the definition of subspace, saying that he “starts proof by writing down what [he] knows”. He wrote “W is a subspace of $\mathbb{R}^3$ iff for any two vectors $u_1, u_2 \in \mathbb{R}^3$, $(\forall c \in \mathbb{R})(\forall d \in \mathbb{R})(cu_1 + du_2 \in \mathbb{R}^3)$ ”.⁵ He then announced: “to prove it's not a subspace, I need to find one instance where this wouldn't work out [...] what I can do is pick negative

scalars and just show that this is not going to be in it”. He then deduced that if $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and

$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ are in W, then $-\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -x_1 - x_2 \\ -y_1 - y_2 \\ -z_1 - z_2 \end{pmatrix}$ is not in W. His final

proof attempt consisted of verifying that the product of the terms in the final vector was a negative number.⁶ Again, even though he was aware that he only needed to find a specific counterexample, Caleb used syntactic reasoning throughout this proof construction: he applied the definition of subspace to see what W being a subspace of $\mathbb{R}^3$ meant in set theoretic language, he negated the statement to see how this would be false, and he used algebra to verify that he had found a class of counterexamples.

In general, Caleb's initial approach to each of the seven linear algebra tasks (see Table 5) was to write down the definitions of the concepts involved in the problem, either by recalling these definitions or asking the interviewer for the sheet containing the definitions. After this, Caleb tried to work forward from these definitions, and backward from what he intended to prove, in an attempt to find a bridge between the two. This general approach is clearly illustrated in his work on L5, a task that asked Caleb to prove that a particular subspace U of $\mathbb{R}^4$ had dimension 2. He began his proof attempt by asking for the definitions of subspace and dimension, and then inferring that

⁵ Although this is incorrect (the vectors and their linear combination should be in W, not $\mathbb{R}^3$), Caleb wrote it correctly in his final proof.

⁶ We note here there is a flaw in Caleb's argument, as x_1, y_1 , and z_1 , as well as x_2, y_2 , and z_2 , need not all be positive, so $-x_1 - x_2$, $-y_1 - y_2$, and $-z_1 - z_2$, need not all be negative. If Caleb had specified his argument only pertained to elements in W with three positive terms, his argument would have been valid and produced a general class of counterexamples.

any linear combination of vectors in U was still contained in U . He struggled with how he should proceed, saying, “I want to go very mechanical with the definitions, but there’s so much floating around in my head that I can’t get a focal point”. He then focused on what he needed to prove, writing “Prove that $\dim(U)=2$ ” and saying “this is where I get very mechanical”. He reinterpreted the claim to be proven as “ $\dim(U)=2$ means that there are two vectors in any basis of U ”. He was unsure of how to proceed, and tried representing his previous work as “ $B=\{v_1, v_2\}$ is a basis for U ”. He was unable to make further progress on his proof attempt. When asked to describe his thought processes, he said, “I would almost write that U is a subspace down here [referring to the bottom of the page where he was writing his proof] and try to make them meet”. And then, describing why he focused on the statement to be proven, Caleb explained:

“[...] sometimes, if I do that, and I have this part [referring to the conclusion], that’s kind of like the end of the proof on one side, and the beginning of the proof, and I kind of want to make them meet somewhere in the middle”.

We also thought it was interesting that at two points during the interview, when Caleb was struggling writing proofs, he complained that what he lacked was experience syntactically manipulating the definitions. At one point, Caleb commented, “I’m not so familiar with the definitions. One of the things that makes proof writing easier is manipulating the definitions again and again”. At another point, he lamented: “the notions are familiar. It’s the routine manipulations that I would have picked up if I had worked with them [the concepts] more”.

Summary of Caleb’s Reasoning Strategies for Linear Algebra Tasks

Based on his behavior on the seven proving tasks, we would classify Caleb as a student with a propensity for syntactic reasoning who would also rarely engage in semantic reasoning. Even though Caleb made substantial progress in only three of the Linear Algebra tasks (and employed one informal representation in one of those tasks), he meaningfully engaged in three of the other tasks by reasoning in a predominantly syntactic manner, and the only two informal representations he constructed while working on the linear algebra tasks played a very limited role in his reasoning. In general, Caleb’s proof attempts consisted mostly of syntactic reasoning: most of his proof attempts began with him writing the definitions of the concepts involved in the statements to be proven and drawing basic deductions from these statements, at one point noting that “once again I go mechanical to a certain extent”.

Calculus Tasks

Use of Graphs, Diagrams, and Examples

As Table 4 illustrates, Caleb considered at least one informal representation on each of the seven calculus tasks.

Further, Caleb’s graphs and examples played a significant role in his proof attempts. In four tasks (C1, C2, C3, and C5), Caleb graphed functions satisfying the hypotheses of the statement that he was trying to prove, used insights from these graphs to

Table 4 Caleb's use of informal representations in Calculus tasks and its contribution to his proof attempts

Use of informal representation	Contribution to proof attempt
<i>C1</i>	
Caleb graphed a linear function satisfying $f(0) = 1$ and $f(1) = 2$ (i.e., $f(x) = x + 1$)	Caleb observed that a linear function would satisfy $f(2) = 3$ and deduced that $f(2) > 3$ for a concave up function. He used this as the basis for the proof.
<i>C2</i>	
Caleb graphed $f(x) = x^3 - 3x^2 + 5x$.	Caleb observed that the only solution for $f(x) = 0$ is $x = 0$.
Caleb added the graph of $h(x) = \sin x$ onto the same set of axes.	Caleb noted that $f(x) = h(x) = 0$ and $f'(x) > h'(x)$ and used this as the basis for his proof attempt.
<i>C3</i>	
Caleb drew a prototypical graph of an even function symmetric about the y -axis.	Caleb formulated an argument for why the slopes at x and $-x$ are negatives of one another and used the graph to guide the algebraic work that constituted his proof.
<i>C4</i>	
Caleb plugged in numbers to explore the claim that $a^2 + b^2 > ab$.	By plugging in numbers, Caleb observed that ab would only be negative if a and b had opposite signs, so he only needed to consider cases where a and b had the same sign.
<i>C5</i>	
Caleb graphed $f(x)$ and $g(x)$ fitting the hypotheses of the task.	Caleb used the graph to see why the statement was true. He later expressed a desire to use the Mean Value Theorem as a tool in his proof and used the graph to see how MVT would be used, but could not finish his proof because he ran out of time.
<i>C6</i>	
Caleb graphed $\sin x$.	Caleb realized $\sin x$ is an odd function and hypothesized that $\sin^3 x$ is odd because the product of odd functions is odd.
Caleb graphed $\sin^2 x$.	Caleb realized the above hypothesis is false, but then set as a subgoal to prove that the product of two odd functions is an even function and the product of an odd and even function is odd.
<i>C7</i>	
Caleb tried to graph a generic $f(x)$ and $g(x) = f(x)/x$, but he could not do it satisfying all the hypotheses.	As Caleb could not graph functions satisfying all the given hypotheses, this representation was not helpful in his proof attempt.

intuitively explain why the statement was true, and then used (or attempted to use) this informal explanation as a basis for his proof. For instance, immediately after reading the problem statement for C1, which asserts that a function with $f(0) = f'(0) = 1$ and $f''(x) > 0$ for all positive x would have, Caleb began drawing a graph, saying:

“ $f(0) = 1$ [plots the point $(0, 1)$ on a coordinate system], $f'(0)$ is 1 so the slope right here is going to be a 1 [draws a short line segment around $(0, 1)$ with a slope of 1] and $f''(x) > 0$ for all positive x and that means that from here [referring to $(0, 1)$], the slope is only going to get steeper, so from the y -axis onwards, it's only going to be concave down [sic], and they're saying prove that $f(2)$ is going to be

greater than 2 ... if this was a straight line [draws a straight line with a slope of 1 from (0, 1)], then we're going to get $f(1)$ is 2 and $f(2)$ is 3 ... if this was just a straight line, which it can't be, then we would have (1, 2) and (2, 3) [plots (1, 2) and (2, 3)]. So, okay, to go by proof, hopefully I can make this rigorous enough".

In our interpretation of this excerpt, Caleb argued that if $f(x)$ was a line, then $f(2) = 3$, but since f is concave up, then $f(2)$ has to be greater than 3. Shortly after giving this explanation, Caleb explained, "I can prove it graphically, but I just want to be sure that this would be acceptable to a professor in real analysis". He proceeded to write a proof by contradiction using the Mean Value Theorem. When asked to give a general description of what he was thinking when working on this proof, Caleb replied, "I was thinking how was I going to translate my graphical understanding into a rigorous symbolic proof that would be accepted by a professor".

Similarly, on C2, Caleb was asked to show that the equation $x^3 + 5x = 3x^2 + \sin x$ has no non-zero solutions. Caleb recognized this was equivalent to showing $G(x) = x^3 - 3x^2 + 5x - \sin x$ only had a root at zero. Caleb first graphed $f(x) = x^3 - 3x^2 + 5x$ using the graphing application in the provided computer, saw that $f(x)$ had a root at zero, and then verified this result algebraically. However, he remarked "that sine could do something and give this [the general equation] a second solution theoretically". Caleb then graphed $f(x)$ along with $h(x) = \sin x$ using the graphing application, and noted that " $f'(x)$ is greater than $h'(x)$ because I see it on the graph". He later gave the following intuitive explanation for why the problem statement was true:

"I graphed these two functions and my only issue was that whatever values the sine function can take on, as long as it can't compensate for how fast the $f(x)$ was growing, then it wasn't going to produce any more solutions. So in my head, I had these two [$f(x)$ and $h(x)$] and that was giving me the shape of the third graph [$G(x)$]".

Caleb ran out of time and wrote a proof attempt that claimed $f'(x) > h'(x)$, but did not justify this claim.⁷ Describing his proof-writing processes, Caleb said, "once again it was a process of me turning my explanatory kind of proof into a symbolic kind of proof."

For two other proofs, C4 and C6, Caleb's exploration of specific example objects provided him with insights that helped him make substantial progress on his proof attempts. For instance, for C6 Caleb had conjectured that he could deduce that $\sin^3 x$ was an odd function from $\sin x$ being an odd function using the warrant that the product of odd functions was odd. However, upon inspecting the graph of $\sin^2 x$ and realizing that this was an even function, Caleb changed his conjecture to the product of two odd functions is even and the product of an odd function and an even function is odd.

Caleb's General Approach in the Calculus Tasks

For five of the seven tasks that Caleb attempted (see Table 5), shortly after reading a description of the problem statement, Caleb attempted to interpret the statement

⁷ One possible justification was to verify that $f'(x) = 3x^2 - 6x + 5 > 1$ for all real-valued x and hence was always greater than $h'(x) = \cos x$.

Table 5 Differences in Caleb’s behavior in Linear Algebra and Calculus tasks (number of tasks for which a behavior occurred)

Behavior	Linear algebra Tasks	Calculus Tasks
Tasks in which an informal representation of an object was constructed.	2	7
Tasks begun by writing definitions of concepts in the statement to be proven.	7	1*
Tasks begun by attempting to informally interpret the statement to be proven.	0	5
Tasks in which an informal explanation of the statement was provided.	1	4
Description of his proving process.	“When there’s a lot of stuff like this, I go mechanical to a certain extent”	“Once again it was a process of turning my explanatory kind of proof into a formal kind proof”

*- Caleb began his work on C3 by writing the definition of an even function

graphically. If he was successful in doing so, he would try to intuitively explain why the statement to be proven was true and then express this informal explanation using the language and deductive structure required of a mathematical proof. When asked to summarize his proof construction attempts, for five tasks Caleb declared that his proof writing consisted of trying to formalize what he saw in pictures. For instance, he said, “once again I was just thinking how I can bridge the gap between what I clearly saw on the graph into symbols” and “the entire time I saw it on the graph, plain as day, but putting it into symbols is a translation process”. Two similar quotations are given in the preceding sub-section. Caleb’s proving processes, as well as his descriptions of them, indicate that Caleb attempts to write proofs in calculus using what Raman (2003) referred to as a “key idea”, where the activity of proving can be viewed as building informal justifications of mathematical claims and then translating these into the language and structure of formal proof.

Summary of Caleb’s Reasoning on Calculus Tasks

We contend that the evidence above shows Caleb as having a strong semantic reasoning style for writing proofs in calculus. For all seven tasks, Caleb considered informal representations of the mathematical concepts and, in each case, these allowed Caleb to draw useful inferences that shaped his proof attempt. Further, Caleb’s actions and his reflections upon them indicate he views the task of proving as generating intuitive graphically-based arguments and translating them into the language of proof.

Comparing Caleb’s Behavior on the Linear Algebra and Calculus Tasks

We believe Caleb’s propensity to engage in semantic reasoning differed depending on whether he was working on linear algebra or calculus tasks. We contend that Caleb had

a semantic proving style for the calculus tasks but engaged in semantic reasoning only occasionally for the linear algebra tasks. The differences in Caleb's behavior are summarized in Table 5. This result is theoretically interesting in itself in that it informs mathematics educators that proving style might not solely be a function of the student, but may also depend on the mathematical domain being studied.

Other Factors that Could have Influenced Caleb's Reasoning Strategies

In this section, we use Caleb's comments to hypothesize on *why* his reasoning strategies were so different in the calculus and linear algebra tasks. As we are basing this on a single data source, Caleb's response to our interview questions about his general proving processes, our analysis is necessarily speculative. We advance these hypotheses to inform future research.

One possibility is simply that calculus is more amenable to semantic reasoning. Just as we would expect almost any students to use more diagrams when writing proofs in Euclidean geometry than in, say, number theory, perhaps most students use more semantic reasoning in calculus than in linear algebra. Although we cannot rule out this possibility, we use Caleb's responses to our interview questions to suggest other factors, including his depth of knowledge of calculus and linear algebra, how he was taught these two domains, and the time pressure he felt while completing these tasks.

Caleb's Proving Strategies in Other Domains

After Caleb completed the calculus tasks, the interviewer noted that Caleb often used graphs to complete the calculus tasks and asked if this was something that Caleb did in general. Caleb responded "yes, I think very graphically" adding that in calculus, he used graphs both to interpret statements and as a basis for his proofs. He was later asked if his more syntactic approaches to writing proofs in linear algebra were because of the differences in the domains of linear algebra and calculus or his lack of familiarity with the linear algebra material (discussed more below). Caleb replied:

"I really do think that it's because of the lack of familiarity and not the difference in subject area cause if I'm trying to imagine writing a proof about abstract algebra, which I did take the class and I enjoyed it so I got into it a little, I more often than not for an abstract algebra proof am not willing to just put the definitions down but rather to kind of like sink my teeth into it and understand it in my own constructive way and then go about trying to make the proof."

According to Caleb, his consistent use of semantic reasoning in calculus extended to other domains. Indeed, Caleb suggested that he employs similar strategies in abstract algebra, a domain that is less obviously visual than calculus, and arguably less visual than linear algebra as well. As we did not observe Caleb writing proofs in other domains such as abstract algebra, we cannot be certain if this interpretation is correct and hence view this data as merely suggestive.

Familiarity

As seen above, when asked about his proof construction in linear algebra, Caleb claimed that his lack of semantic reasoning and his reliance on syntactic reasoning were a consequence of his lack of familiarity with the material.

Interviewer: For the linear algebra problems, I noticed that you rarely used specific examples, specific matrices... Can you tell me a little bit more about that?

Caleb: I think that it probably had a lot to do with my lack of familiarity with that material. But I think also, I don't know, calculus I like to go to the graphs. And because of my lack of familiarity [with linear algebra], one of my general proof writing strategies is, if I really have no idea, just, you know, write down all the definitions and see if you can, you know, make them meet in the middle. Or a lot of times just using the logic of the symbolism, you know, can get you... I've written proofs where I have no idea what's going on. Just manipulating the definitions enough and knowing that I'm being logically sound, I can make a proof that is acceptable whereas in these proofs, I understood each one so I wanted to understand them before I wrote them and I knew that wasn't going to happen on the linear algebra proofs so I wrote down the definitions and see where that would take me.

Caleb seemed to lack a deep understanding in linear algebra, which for him meant that he was unable to approach proofs in a meaningful way and had to resort to what sounded like the default strategy for him in these situations: writing definitions as starting points and ending points, and trying to meet in the middle. It is unclear whether or not Caleb's use of semantic reasoning in a particular domain is actually tied to his level of familiarity with that domain. On the one hand, Caleb demonstrated he could construct and operate with semantic representations of concepts in the linear algebra tasks; it is possible that more experience with these concepts was necessary for him to be able to use this type of representations more productively. On the other hand, not all students will abandon semantic reasoning on concepts that are new or unfamiliar to them, and previous research has shown that students can creatively generate examples and diagrams to understand definitions of concepts that are new to them in a short period of time (Dahlberg and Housman 1997; Weber et al. 2008). Future research could focus on the interaction between students' employment of a particular type of reasoning in proof construction tasks and their familiarity with the general mathematical domain and the specific concepts involved in these tasks.

The Influence of Teaching and Cognitive Resources

A natural question to ask is whether, and how, Caleb's proof-writing strategies in linear algebra and calculus were influenced by how Caleb was taught these subjects. Unfortunately, we do not have any data to corroborate Caleb's brief accounts of what happened in his advanced calculus and linear algebra courses.

When asked to describe his linear algebra course, Caleb noted that he had taken linear algebra several years ago, and while there were proofs both in the lectures and as homework problems, the course was less proof-oriented than his advanced calculus course. He described the course as “a pretty straightforward lecture class..., pretty rote, and not very engaging... like he [the professor] was reading out of the book”. Caleb’s description does not provide us with enough detail about his professor’s proof presentation style, but it may suggest that his class followed the formal definition-theorem-proof followed by most advanced mathematics textbooks (Raman 2004; Weber 2004).

When asked about his advanced calculus course, Caleb said that his professor encouraged students to follow a prototypical syntactic strategy when constructing proofs:

It’s funny because my [advanced] calculus⁸ class, I loved it, I thought the professor was amazing, but my professor said to me right in the beginning, every definition I put on the board, I want you to put on an index card and I want you to have all of them memorized, and at the beginning of every proof, I want you to literally just write down what you know, write down what you need to get to, and try to have them meet in the middle.

However, Caleb later indicated that his advanced calculus professor also gave him “access to the material” in terms of graphical representations.

Caleb: For my advanced calculus class, [my professor] was just like, you know, let’s draw a graph and let’s really look at it. He was just very animated and engaging and explained something three different ways.

Interviewer: How would he use diagrams and examples in that class?

Caleb: He would use them very often. He would give us the definition of something, say a neighborhood, and he would draw a graph. I remember there was something really illuminating that he did. I can’t remember but it was like Cauchy sequences and even if it was not a connected function and they were like getting closer like this and getting closer like this [sketches a sequence on a Cartesian plane that appear to consist of two subsequences approaching the same limit, one from above and one from below], he was just very demonstrative on the board, multiple examples. If you asked him a question, he would go to the board and draw a picture. Very graph oriented [...] I do think that his mental framework was that there was proof-writing and the material. They converged, but they were like two separate things.

Thus, it seems Caleb’s professor also helped him build informal representations of the concepts he was studying. In other words, his professor helped Caleb build the cognitive resources necessary to produce proofs semantically.

⁸ Follow-up questions from the interviewer confirmed that when Caleb referred to his calculus professor, he was referring to the professor of his proof-oriented junior-level advanced calculus course.

Even though it is unclear how and to what extent instruction had an effect on Caleb's proof construction, previous researchers have found that professors' teaching can influence their students' proof production (e.g., Weber 2004). In particular, while students may easily develop the skill of representing concepts and statements in certain subjects to certain informal representational systems (e.g., interpreting general statements in number theory in terms of specific numbers), this interpretation may be harder to do in other subjects and to other systems (e.g., visualizing general statements in linear algebra) without some guidance. Thus, by helping students build the cognitive resources necessary to produce proofs semantically, a professor may provide students with a reasoning strategy that they would not have access to otherwise.

Time as a Factor in Choosing a Proving Strategy

In this study, Caleb was given 10 min to work on each proving task. We chose this time allotment because we thought this would be analogous to students taking an exam in a proof-oriented class, in which students are typically asked to write four or five proofs in a 50-min period. In his interview, Caleb indicated that he preferred semantic proving strategies because they were based on his understanding, but at several points, he said he regretted using these strategies since they might have inhibited his performance in his mathematics courses. After articulating how graphs aided his proof writing in calculus, Caleb said:

Caleb: I don't know about if that is always conducive for me. Like you saw a couple of times, I'll kind of get caught up in, well I can explain it to you in 2 s if you just let me draw a graph and I think that sometimes actually hinders me rather than helps me when it comes to sitting down for an exam in math [...] The way I was doing with the linear algebra proofs yesterday, I'm going to go into mechanical mode and the definitions are the definitions, I don't think I could do this with calculus proofs. And maybe I should. Maybe it would help me. I don't know [...] A lot of times I have to fight it because it holds me back in a lot of cases.

Interviewer: You mean in terms of time constraints?

Caleb: Time constraints. Not in terms of understanding, but in terms of, here's an exam, you have this long to do it, and you have to maximize your score.

In contrast to his preferred method, when working on L2 Caleb was tempted to use a common student strategy that might obtain partial credit for a proof attempt, but was not conducive toward understanding. When struggling to prove that a square matrix A with the property that $A^3=0$ could not have a non-zero eigenvalue, Caleb said, "In a limited amount of time like this, my inclination would be to gloss over the fact that I'm not sure what's going on and I'm going to write a statement that says 'since $A^3=0$, then A is the zero matrix'. No, I can't really say that. That's such a broad leap."

We find it disappointing that Caleb feels the need to "fight" trying to write proofs meaningfully on his exams due to time constraints, and that he feels his failure to fight this impulse holds him back in his courses. However, since we did not observe Caleb

constructing proofs under different time constraints, we cannot be certain that time would actually affect his reasoning strategies.

Discussion

In this paper, we have described a student who consistently employed syntactic reasoning to construct proofs in linear algebra, but relied on semantic reasoning to approach similar tasks in calculus. This is an existence proof that for some students the types of reasoning they use in proof construction tasks change from one domain to another.

We argue that Caleb's case informs the development of the construct of cognitive style (particularly proving style) in advanced mathematics. With a few exceptions, studies on this construct do not specify the limitations of their findings and often imply they are describing a person's cognitive strategies in advanced mathematics in general. In a sense, most existing research presents cognitive styles as being solely a function of the student: the student is classified as a particular type of learner, thinker, or reasoner. Caleb's behavior demonstrates that researchers need to be aware of these limitations: indeed, we argue that by looking only at the linear algebra tasks Caleb would have been classified as a syntactic reasoner, whereas looking only at his work on calculus he would have been classified as a semantic reasoner. This does not mean that cognitive styles in advanced mathematics do not exist, but it does put into question the stability of classifications made by previous researchers on the basis of students' performance in a small number of tasks in a single mathematical domain. Based on this case study, we can also make some recommendations for future research.

Generally, we need to be more cautious when assigning cognitive styles in advanced mathematics, acknowledging that just like in the study of general cognitive styles (Entwistle 1998), the cognitive strategies that someone invokes to solve tasks in advanced mathematics may be dependent upon environmental factors. In particular, researchers studying students' cognitive strategies in one mathematical domain should avoid making generalizations to other domains, and specify that any cognitive styles identified are potentially restricted to that particular domain.

Finally, this case study suggests interesting avenues for further research. In our interviews, Caleb was articulate in describing interesting reasons for his different proving strategies in the linear algebra and calculus tasks. However, we do not have further data to corroborate that the factors he mentioned did indeed influence his behavior in these two domains. Future research could test the extent to which teaching style, level of familiarity, and the time given to participants influence their strategies on proof construction tasks.

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Appendix A. Tasks, in the Order Presented to Caleb.

Session 1: Linear Algebra

- L1: Prove that $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R}, xyz \geq 0 \right\}$ is not a subspace of $\mathbb{R}^3$.
- L2: Suppose that A is a square matrix for which $A^3 = 0$, and r is any non-zero real value. Prove that r is not an eigenvalue of A .
- L3: Suppose $\{v_1, v_2, \dots, v_n\}$ forms a basis for a vector space V . Prove that $\{v_1, v_1 + v_2, \dots, v_1 + v_2 + \dots + v_n\}$ also forms a basis for V .
- L4: Let $u = (u_1, u_3, u_3)$ and $v = (v_1, v_3, v_3)$ be unit vectors in $\mathbb{R}^3$ with $|u_3| > |v_3|$. Prove that the projection of u onto the x - y plane is shorter than the projection of v onto the x - y plane.
- L5: Given that $U = \left\{ \begin{pmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$, prove that $\dim(U)=2$.
- L6: Suppose T is a non-invertible 2×2 matrix. Prove that there exists a nonzero 2×2 matrix S such that TS is the zero matrix.
- L7: Suppose U and W are subspaces of $\mathbb{R}^4$, $\dim(U)=2$ and $\dim(W)=3$. Prove that the intersection of U and W must contain more than one vector.

Session 2: Calculus

- C1: Suppose $f(0) = f'(0) = 1$. Suppose $f''(x) > 0$ for all positive x . Prove that $f(2) > 2$.
- C2: Prove that the only real solution to the equation $x^3 + 5x = 3x^2 + \sin x$ is $x = 0$.
- C3: Suppose $f(x)$ is a differentiable even function. Prove that $f'(x)$ is an odd function.
- C4: Prove that $a^2 + ab + b^2 \geq 0$ for all real numbers a and b .
- C5: Suppose $f''(x) > 0$ for all real numbers x . Suppose a and b are real numbers with $a > b$. Define $g(x)$ as the line through the points $(a, f(a))$ and $(b, f(b))$. Prove that for all $x \in [a, b]$, $f(x) \leq g(x)$.
- C6: Prove that $\int_{-a}^a \sin^3(x) dx = 0$ for any real number a .
- C7: Let f be differentiable on $[0, 1]$, and suppose that $f(0) = 0$ and f' is increasing on $[0, 1]$. Prove that $g(x) = \frac{f(x)}{x}$ is increasing on $(0, 1)$.

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