A Triple Structure of Rough Sets Based on Selection Function

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Abstract Different from rough sets in Pawlak’s sense, which is a binary approximation operations based structure, in this paper, we propose a new rough equivalence relation based on triple approximation operations induced by selection function. The same as traditional rough sets research, we consider the algebra issue of new rough sets system and construct lattice structure in an algebraic fashion. An isomorphic relationship is studied between proposed rough sets algebra structure and that in Pawlak’s sense. In addition, some examples are shown and studied in order to indicate the effectiveness of new equivalence relation in distinguishing and describing subsets of universe. Besides, we also study restriction of selection congruence on subalgebra built by covering of universe. Finally, conclusion on the axioms of middle approximation operation are shown to clarify properties of new structure.

Keywords Rough sets · Selection function · Congruence · Lattice

1 Introduction

Based on two approximation operations, i.e. upper and lower approximations, rough sets offer an efficient tool to deal with incomplete and insufficient knowledge in classification and concept formation [13]. When confronting uncertainty in knowledge, rough sets could be regarded as an important complement for randomness and fuzzy-

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ness, which had been introduced into information granularity [22]. As a promising characterization of uncertainty, rough sets have been developed and applied to many real-life problems [9, 18].

Besides, lattice theory is a fundamental subfield in Algebra. The hierarchial structure of lattice permeates in the study of group, ring and field. From normal subgroup to quotient group, from ideal to quotient ring, from general integral domain to its quotient field, and also from polynomial ring on a field to its extension, lattice based on partial ordering relation is ubiquitous.

In this paper, we mainly concern algebra structure of rough sets based on lattice theory. In particular, we introduce an extended definition of rough sets based on rough equivalence relation induced by selection function. Depending on various relations in lattice theory, the hierarchial structure between rough sets in Pawlak's sense and that in our new definition will be studied. Before that, we firstly summarize some existing related researches related to our topic.

1.1 The Representation of Rough Sets

The concept of rough sets derives from knowledge representation. Based on approximation space (A, R) , rough sets offer a representation for every single element in $P(U)$, which is the power set of U . Additionally, different indiscernibility R relations lead to changeable granularity in quotient space U/R , which illuminates variable precision rough sets model [24]. Regarding the elements in U/R as fundamental sets in $P(U)$, Pawlak et al. and Yao et al. [14, 19] defined measure on σ -algebra $\sigma(U/R)$. Also, membership function could be equipped to set up a connection between rough sets and fuzzy sets [20]. Particularly, Iwinski [10] showed that rough sets system was equivalent to an interval algebra system with crisp lower and upper bound sets.

1.2 Classic Rough Sets Algebra

How to build the algebra structure of rough sets system is an fundamental issue. By defining minimal upper sample and selection function respectively, Bonikowski and Zhang et al. [3, 25] successfully defined intersection and union operations on rough sets. However, proper complement cannot be defined for every rough set. In fact, except the fundamental sets, only pseudo complement can be defined for each rough set. In this way, rough sets formed a complete atomic Stone algebra [3]. In addition, taking advantage of relative pseudo complement operation between two rough sets, Pagliani [12] proved that rough sets system constituted a Heyting algebra under disjoint representation and was isomorphic to a semi-simple Nelson algebra. Besides, rough sets system was also a regular double Stone algebra by introducing dual pseudo complement operation.

1.3 Covering Generalized Rough Sets

In Euclidean space, the class of neighborhoods of all the points in a set consists a covering of the corresponding set naturally. Inspired by covering concept, Zakowski

[23] introduced covering generalized rough sets by replacing equivalence relation with similar relation [15] and tolerance relation [11]. At the same time, Yao [21] used element, granule and subsystem to define upper and lower approximation operations respectively. In addition, Zhu and Wang [26–28] studied the attribute reduction for covering generalized rough sets and also established axiom system for covering based approximation operations. Slowinski and Vanderpooten [15] offered a new definition for rough sets to eliminate the ambiguous objects located in the boundary of a specific set. These researches constructed a necessary foundation for granularity computing.

On the other hand, in the study of algebra properties, Bonikowski et al. [5] considered similar relation based covering generalized rough sets and defined extension and intension of rough sets. They proved that when covering satisfied double representative or quasi double representative conditions, partial order based rough sets system could build a lattice.

1.4 Lattice Algebra

As an essential topic studied in Universal Algebra [6], lattice [2] (especially Boolean algebra [16]) was widely studied and applied to information science. For example, Formal Concept Analysis [8] offered a hierarchical relationship among different concepts.

In this part, we only introduce results in lattice algebra which are directly related to our topic. Firstly, Birkhoff and Frink [1] studied the relationship between set of subuniverses of algebra and algebraic lattice, which was complete and compactly generated by algebraic closure operator. Then, based on subuniverse generated operator, Tarski [17] proved irredundant basis theorem for a set, which offered the exact characterization of the number of elements in the basis. As a particular equivalence relation, congruence relation guaranteed the compatibility between algebra operations and equivalence relation on a nonempty set. As Cohn [7] pointed out, all the congruence relations on a specific nonempty set constructed an algebraic lattice. Furthermore, quotient algebra generated by congruence relation can give rise to generalizations of three homomorphism and isomorphism theorems in abstract algebra. These three theorems describes existence, hierarchical structure and roughness of information granulation, in perspectives of homomorphic kernel, relationship among quotient algebras and upper approximation of a set respectively. The main idea in establishing the algebraic structure of our new definition is extremely based on these three homomorphism and isomorphism theorems.

2 Rough Sets Algebra in Pawlak's Sense

In this section, we will review rough sets in Pawlak's sense. Particularly, as the mainly topic in this paper, algebra structure of rough sets will be considered. First of all, we start with introduction on the rough sets algebra.

2.1 Definition of Rough Sets in Pawlak's Sense

Let U be an universe. The equivalence relation R is equipped on U . An approximation space is denoted by $\mathcal{A} = (U, R)$. Every element in U/R is called elementary set and

the union of arbitrary finite elementary sets is called composed set, including the empty set. The family of composed sets is denoted by $ComR$. When cardinality of U is finite, we have a simple result: suppose that there are n equivalence classes in R , then the total number of composed sets is 2^n . Furthermore, a classic property can be described as follows.

Proposition 1 ([3]) *The algebra $\mathfrak{B} = (ComR, \cap, \cup, -, \emptyset, U)$ is the atomic, complete Boolean algebra. The inclusion relation $\subseteq$ is its natural order.*

Next, we can define approximation operations for an arbitrary certain subset $X \subseteq U$ (or $X \in P(U)$).

Definition 1 The lower approximation of X in the sense of extension is

$$\underline{P}X = \bigcup \{Y \mid Y \subseteq X \wedge Y \in U/R\}.$$

Similarly, the upper approximation of X in the sense of extension is

$$\overline{P}X = \bigcup \{Y \mid Y \cap X \neq \emptyset \wedge Y \in U/R\}.$$

Hence, the boundary of X can be defined in the following way:

$$Bn(X) = \overline{P}X \setminus \underline{P}X.$$

Remark 1 In this paper, approximations with respect to extension of any subset X of universe U is employed, which is based on granule-oriented point of view. While we can also define approximations with respect to extension based on element-oriented point of view [4].

In this way, we can directly describe approximations as elements in $P(U)$. Particularly, for the element in $ComR$, we have the following result.

Proposition 2 ([3]) *Suppose that $X \in P(U)$, the following conditions are equivalent:*

- (a) $X = \underline{P}X$;
- (b) $X = \overline{P}X$;
- (c) $X \in ComR$.

In addition, we would like to introduce the fundamental axiomatic properties of lower and upper approximation operations.

Theorem 1 ([3]) *Let $X, Y \in P(U)$, the following properties can be obtained.*

- (1) $\underline{P}X \subseteq \overline{P}X$; (2) $\underline{P}U = \overline{P}U = U$; (3) $\underline{P}\emptyset = \overline{P}\emptyset = \emptyset$;
- (4) $\underline{P}(\underline{P}X) = \overline{P}(\overline{P}X) = \underline{P}X$; (5) $\overline{P}(\underline{P}X) = \underline{P}(\overline{P}X) = \overline{P}X$;
- (6) $\underline{P}(X \cup Y) = \underline{P}X \cup \underline{P}Y$; (7) $\underline{P}(X \cap Y) = \underline{P}X \cap \underline{P}Y$;
- (8) $\overline{P}(X \cap Y) \subseteq \overline{P}X \cap \overline{P}Y$; (9) $\underline{P}X \cup \underline{P}Y \subseteq \underline{P}(X \cup Y)$;
- (10) $\overline{P}X = -(\underline{P}(-X))$; (11) $\underline{P}X = -(\overline{P}(-X))$; (12) $\overline{P}X \setminus \overline{P}Y = \overline{P}(X \setminus Y)$;
- (13) $X \subseteq Y \Rightarrow \underline{P}X \subseteq \underline{P}Y$; (14) $X \subseteq Y \Rightarrow \underline{P}X \subseteq \underline{P}Y$.

Based on the definition and discussion above, rough equivalence relation on $P(U)$ can be obtained.

Definition 2 For arbitrary $X, Y \in P(U)$, denote an rough equivalence relation $\tilde{R}$ on $P(U)$: $X \tilde{R} Y \Leftrightarrow \underline{P}X = \underline{P}Y \wedge \overline{P}X = \overline{P}Y$.

Then, denote the equivalence class as $[X]_{\approx}$. At the same time, partial order can be equipped on the equivalence classes $P(U)/\tilde{R}$ as follows:

$$[X]_{\approx} \leq [Y]_{\approx} \Leftrightarrow \underline{P}X \subseteq \underline{P}Y \wedge \overline{P}X \subseteq \overline{P}Y.$$

2.2 Algebra Structure of Rough Sets in Pawlak's Sense

Now, we show how the rough sets can construct different algebra systems, which leads to various algebra structures of rough sets in Pawlak's sense. To this end, we introduce two different definitions of the union and intersection of two arbitrary rough sets.

Definition 3 Let $X, Y \in P(U)$ and $Y \subseteq X$.

- (a) Y is called the lower sample of X iff $\overline{P}Y = \underline{P}X$;
- (b) Y is called the upper sample of X iff $\overline{P}Y = \overline{P}X$.

Definition 4 We call Y is the minimal lower sample of X ($Y = mls(X)$) iff Y is the lower sample of X and there is no lower sample Z of X such that $|Z| < |Y|$.

Similarly, Y is called the minimal upper sample of X ($Y = mus(X)$) iff Y is the upper sample of X and there is no upper sample Z of X such that $|Z| < |Y|$.

Particularly, for any composed set, the definition above is trivial.

Proposition 3 Suppose that $X \in ComR$, then $Y = mls(X) \Leftrightarrow Y = mus(X)$.

Based on Definition 3, we introduce the first definition of rough sets lattice by defining union and intersection of any two rough equivalence classes.

Definition 5 For any $X, Y \in P(U)$:

- (a) Let P be the minimal upper sample of set $\overline{P}X \cup \overline{P}Y$. Define

$$[X]_{\approx} \sqcup [Y]_{\approx} = \left[\underline{P}X \cup \underline{P}Y \cup P \right]_{\approx}.$$

- (b) Let P be the minimal upper sample of set $\overline{P}X \cap \overline{P}Y$. Define

$$[X]_{\approx} \cap [Y]_{\approx} = \left[\underline{P}X \cap \underline{P}Y \cap P \right]_{\approx}.$$

Theorem 2 ([3]) The algebra system $P_1^{\approx}(U) = (P(U)/\tilde{R}, \sqcup, \cap)$ is a distributive lattice, where $[\emptyset]_{\approx}$ and $[U]_{\approx}$ are identity and unit elements respectively.

Now, let $f : U/R \longrightarrow U$ be a selection function. That is to say, for every $[x]_R \in U/R$, exists $y \in U$, satisfying yRx , such that $f([x]_R) = y$. According to axiom of choice, it is well-defined. In addition, let $\{x_i\}_{i \in I} \subseteq U$, we have $f(\bigcup_{i \in I} [x_i]_R) = \bigcup_{i \in I} f([x_i]_R)$. Then, for arbitrary $x \in U$, we have $f(-[x]_R) = f(U/R) - f([x]_R)$. In this way, we can naturally extend f from U/R to $\sigma(U/R)$, which can be completely determined by the value of f on U/R and $f(\sigma(U/R)) = f(U/R)$. These results lead to the second definition of algebra structure of rough sets.

Definition 6 For any $X, Y \in P(U)$, we define

- (a) $[X]_{\approx} \cup [Y]_{\approx} = [\underline{P}X \cup \underline{P}Y \cup f(\overline{P}X \cup \overline{P}Y)]_{\approx}$.
- (b) $[X]_{\approx} \cap [Y]_{\approx} = [\underline{P}X \cap \underline{P}Y \cap f(\overline{P}X \cap \overline{P}Y)]_{\approx}$.

Theorem 3 ([25]) *The algebra system $P_2^{\approx}(U) = (P(U)/\tilde{R}, \cup, \cap)$ is a distributive lattice, where $[\emptyset]_{\approx}$ and $[U]_{\approx}$ are identity and unit elements respectively.*

Remark 2 In fact, lattices constituted based on two definitions are equivalent, if we notice that the value of selection function on any composed set constructs the minimal upper sample of the corresponding set. Thus, we argue that the second definition can be viewed as a specification of the first definition. We clearly state this conclusion in the following proposition.

Proposition 4 ([3]) *Let X be any subset of U . Every nonempty minimal lower (upper) sample of set X has exactly one common with every nonempty elementary set of relation R from $\underline{P}X(\overline{P}X)$.*

3 Rough Sets Based on Selection Function

Let R be the equivalence relation on nonempty set U . As we have introduced above, an equivalence relation on $P(U)$ can be induced by R , denoted by $\tilde{R}$, which is called rough equivalence relation. While we can also define another equivalence relation on $P(U)$, based on congruence relation. To this end, we introduce congruence relation on any nonempty algebra.

3.1 Congruence on a Nonempty Algebra

Firstly, we can define language on any nonempty set A .

Definition 7 A language (type) $\mathcal{F}$ is a collection of function symbols such that a nonnegative integer n is assigned to each member $f \in \mathcal{F}$. This integer is called the arity (or rank) of f , and f is said to be an n -ary function symbol. The subset of n -ary function symbols in $\mathcal{F}$ is denoted by $\mathcal{F}_n$.

A nonempty set A with certain language is called algebra, which is denoted by $(A, \mathcal{F})$. Sometimes, we denote arbitrary certain element in $\mathcal{F}$ as f^A .

Then, denote the collection of all the equivalence relations on nonempty set A as $Eq(A)$.

Fig. 1 Congruence relation

Definition 8 (*Congruence*) Let $\mathbf{A}$ be an algebra of type $\mathcal{F}$ and $\theta \in Eq(\mathbf{A})$. Then θ is a congruence on $\mathbf{A}$ if θ satisfies the following compatibility property. Compatibility Property (CP): for each n -ary function symbol $f^{\mathbf{A}} \in \mathcal{F}$ and $a_i, b_i \in A$, if $a_i \theta b_i$, $1 \leq i \leq n$, then, $f^{\mathbf{A}}(a_1, a_2, \dots, a_n) \theta f^{\mathbf{A}}(b_1, b_2, \dots, b_n)$ holds.

Remark 3 The compatibility property is a necessary condition for introducing an algebra structure on the set of equivalence classes A/θ , a quotient structure which is inherited from nonempty set A . In Fig. 1, we show a sketch for congruence relation. The dotted lines subdivide A into the equivalence classes of θ . We select a_1, b_1 and a_2, b_2 in the same equivalence classes respectively. For binary operation $f^{\mathbf{A}}$, compatibility property guarantees $f^{\mathbf{A}}(a_1, a_2)$ and $f^{\mathbf{A}}(b_1, b_2)$ to be in the same equivalence class.

3.2 Selection Congruence Relation on $P(U)$

Let $f(U/R)$ be the range of selection function f . Construct an equivalence relation on $P(U)$ as $\bar{R} : \forall X, Y \in P(U)$, $X \bar{R} Y \Leftrightarrow X \cap f(U/R) = Y \cap f(U/R)$.

Consider the operations on Boolean algebra $(P(U), \cap, \cup, -, \emptyset, U)$. It can be proven that for arbitrary $X_1 \bar{R} X_2, Y_1 \bar{R} Y_2$, we have

$$\begin{aligned} (X_1 \cap Y_1) \cap f(U/R) &= (X_2 \cap Y_2) \cap f(U/R), (X_1 \cup Y_1) \cap f(U/R) \\ &= (X_2 \cup Y_2) \cap f(U/R). \end{aligned}$$

In other words, $(X_1 \cap Y_1) \bar{R} (X_2 \cap Y_2)$, $(X_1 \cup Y_1) \bar{R} (X_2 \cup Y_2)$.

Besides, it is obviously that $(-X_1)\bar{R}(-X_2)$. Thus, $\bar{R}$ is a congruence relation on Boolean algebra $(P(U), \bigcap, \bigcup, -, \emptyset, U)$. We call $\bar{R}$ a selection congruence relation on $P(U)$.

Let $\sigma(U/R)$ be σ -algebra constructed by U/R . Consider the mapping

$$g_f : P(U)/\bar{R} \longrightarrow \sigma(U/R),$$

satisfying $g_f([X]_{\bar{R}}) = \bigcup \{[x]_R | f([x]_R) \in X\}$. Obviously, it is well-defined. Here, we interpret our definition of g_f by a simple example.

Example 1 Let $U = \{a, b, c, d, e, h\}$ be the universe and $R = \{(a, b, c), (d, e), (h)\}$ be an equivalence relation on U , in the view of partition. In addition, let f be a selection function on partition R as follows:

$$f((a, b, c)) = a, f((d, e)) = e, f((h)) = h.$$

For two subsets of U , $X = \{a, d, h\}$ and $Y = \{b, e, h\}$, we have $g_f([X]_{\bar{R}}) = \{a, b, c, h\}$ and $g_f([Y]_{\bar{R}}) = \{d, e, h\}$.

On one hand, based on the definition of g_f , we can define partially ordered relation on $P(U)/\bar{R}$ as follows: $[X]_{\bar{R}} \preceq [Y]_{\bar{R}} \Leftrightarrow g_f([X]_{\bar{R}}) \subseteq g_f([Y]_{\bar{R}})$.

On the other hand, consider the following operation property: for arbitrary family $\{X_i\}_{i \in I} \subseteq P(U)$, $X \subseteq U$, the union and complementation operations are well-defined as follows: $\bigcup_{i \in I} [X_i]_{\bar{R}} = [\bigcup_{i \in I} X_i]_{\bar{R}}$, $-[X]_{\bar{R}} = [-X]_{\bar{R}}$.

In this way, we extend g_f onto $\sigma(P(U)/\bar{R})$, which can be defined as:

$$g_f\left(\bigcup_{i \in I} [X_i]_{\bar{R}}\right) = \bigcup_{i \in I} g_f([X_i]_{\bar{R}}).$$

Besides, for arbitrary $X \subseteq U$, let $g_f(-[X]_{\bar{R}}) = g_f([U]_{\bar{R}}) - g_f([X]_{\bar{R}}) = U - g_f([X]_{\bar{R}})$. Consequently, we have the following result.

Proposition 5 $g_f : \sigma(P(U)/\bar{R}) \longrightarrow \sigma(U/R)$ is an isomorphism.

Remark 4 We call g_f the nature isomorphism based on selection function f .

3.3 Rough Equivalence Relation Based on Selection Function

Denote rough sets in Pawlak's sense as $[X]_{\approx} = \{Y \in P(U) | (Y, \bar{Y}) = (X, \bar{X})\}$, corresponding to equivalence relation $\tilde{R}$ on $P(U)$. Meanwhile, denote rough sets determined by selection function f as

$$[X]_f = \{Y \in P(U) | (Y, g_f([Y]_{\bar{R}}), \bar{Y}) = (X, g_f([X]_{\bar{R}}), \bar{X})\},$$

corresponding to equivalence relation R_f on $P(U)$. Here, we name $g_f([X]_{\bar{R}})$ middle approximation operation of X .

Example 2 Let U, R, f, X, Y be the same ones as in Example 1, we can obtain the rough description based on the approximation operations defined above.

Particularly, $\bar{X} = \bar{Y} = \{a, b, c, d, e, h\} = U$, $\underline{X} = \{h\} = \underline{Y}$, which means $[X]_{\approx} = [Y]_{\approx}$. However, when considering rough sets based on selection function f , we can easily get the corresponding sets based on middle approximation operation according to Example 1, that is, $g_f([X]_{\bar{R}}) = \{a, b, c, h\}$ and $g_f([Y]_{\bar{R}}) = \{d, e, h\}$. As a result, $[X]_f \neq [Y]_f$.

We can easily check the correctness of the following conclusion.

Proposition 6 Without considering the presence of singleton type equivalence classes, suppose that there are n equivalence classes in R , then the total number of rough equivalence classes is 3^n and the total number of rough equivalence classes based on selection function is 4^n .

Apparently, for arbitrary $X, Y \subseteq U$, $[X]_f = [Y]_f \Rightarrow [X]_{\approx} = [Y]_{\approx}$ holds. Defining union and intersection on $(P(U), \bar{R})$ and $(P(U), R_f)$ respectively, we can make the corresponding poset be a lattice.

Generally, let S be an arbitrary given selection function on U/R . According to Definition 6, we have

$$[X]_{\approx} \cup [Y]_{\approx} = [\underline{X} \cup \underline{Y} \cup S(\bar{X} \cup \bar{Y})]_{\approx}, [X]_{\approx} \cap [Y]_{\approx} = [\underline{X} \cap \underline{Y} \cup S(\bar{X} \cap \bar{Y})]_{\approx}.$$

Similarly, we define union and intersection on $P(U)/R_f$ as follows.

Definition 9 Assume that there are at least three elements in every equivalence class in U/R . Let f and f_c be two selection functions which value differently from each other on every equivalence class in U/R . Consequently, we can define

$$[X]_f \cup [Y]_f = [\underline{X} \cup \underline{Y} \cup f_{|X \cup Y}(\sigma(U/R)) \cup f_c(\bar{X} \cup \bar{Y})]_f, \\ [X]_f \cap [Y]_f = [\underline{X} \cap \underline{Y} \cup f_{|X \cap Y}(\sigma(U/R)) \cup f_c(\bar{X} \cap \bar{Y})]_f.$$

Here, $f_{|A}(B) = f(B) \cap A, \forall B \in \sigma(U/R), A \in P(U)$.

Example 3 Let $U = \{a, b, c, d, e, h, i, j, k, l, m, n, p\}$ be the universe, $X = \{b, e, h, i, j, n\}$, $Y = \{c, d, i, k, m, n, p\}$ be the subsets of U and $R = \{(a, b, c, d), (e, h, i), (j, k, l), (m, n, p)\}$ be the equivalence relation on U . Define two selection functions on R as follows:

$$f(a, b, c, d) = a, f(e, h, i) = e, f(j, k, l) = k, f(m, n, p) = n; \\ f_c(a, b, c, d) = c, f_c(e, h, i) = h, f_c(j, k, l) = j, f_c(m, n, p) = m.$$

As a result, we can obtain

$$[X]_f = (\{e, h, i\}, \{e, h, i, m, n, p\}, U), [Y]_f = (\{m, n, p\}, \{j, k, l, m, n, p\}, U); \\ [X]_{f_c} = (\{e, h, i\}, \{e, h, i, j, k, l\}, U), [Y]_{f_c} = (\{m, n, p\}, \{a, b, c, d, m, n, p\}, U).$$

Hence, we can calculate the union and intersection of $[X]_f$ and $[Y]_f$ as follows:

$$\begin{aligned}[X]_f \cup [Y]_f &= [\{e, h, i, m, n, p\} \cup \{e, k, n\} \cup \{c, h, j, m\}]_f \\ &= [c, e, h, i, j, k, m, n, p]_f; \\ [X]_f \cap [Y]_f &= [\{n\} \cup \{c, h, j, m\}]_f = [c, h, j, m, n]_f.\end{aligned}$$

It can be easily verified the rationality of defining $f(U) = \bigcup_{S \in R} f(S)$. Differently, we can also obtain the union and intersection with respect to rough sets in Pawlak's sense as follows:

$$\begin{aligned}[X]_{\approx} \cup [Y]_{\approx} &= [\{e, h, i, m, n, p\} \cup f(U)]_{\approx} = [a, e, h, i, k, m, n, p]_{\approx}; \\ [X]_{\approx} \cap [Y]_{\approx} &= [f(U)]_{\approx} = [f_c(U)]_{\approx} = [a, e, k, n]_{\approx} = [c, h, j, m]_{\approx}.\end{aligned}$$

We declare that the upper and lower approximations of rough equivalence classes keep the same, under union and intersection with respect to two different senses of rough sets respectively. Here, we conclude this property in the following Proposition 7 and the proof is straightforward.

Proposition 7 Assume that $X, Y \subseteq U$.

Let A_1, A_2 be two arbitrary elements in $[X]_f \cup [Y]_f$ and $[X]_{\approx} \cup [Y]_{\approx}$ respectively, then $\overline{A_1} = \overline{A_2}$, $\underline{A_1} = \underline{A_2}$.

Similarly, let B_1, B_2 be two arbitrary elements in $[X]_f \cap [Y]_f$ and $[X]_{\approx} \cap [Y]_{\approx}$ respectively, then $\underline{B_1} = \underline{B_2}$, $\overline{B_1} = \overline{B_2}$.

Definition 10 Define the partial ordering relation on $P(U)/R_f$: $[X]_f \preceq [Y]_f \Leftrightarrow \underline{X} \subseteq \underline{Y} \wedge \overline{X} \subseteq \overline{Y} \wedge g_f([X]_{\overline{R}}) \subseteq g_f([Y]_{\overline{R}})$.

Lemma 1 $(P(U)/R_f, \preceq)$ is a lattice.

Proof Firstly, we show the rationality of intersection and union operations defined in Definition 9. To this end, let $[X_1]_f = [Y_1]_f$, $[X_2]_f = [Y_2]_f$, consider the equivalence between $[X_1]_f \cup [X_2]_f$ and $[Y_1]_f \cup [Y_2]_f$. According to the definition, $\overline{X_1} = \overline{Y_1}$, $\underline{X_1} = \underline{Y_1}$, $\overline{X_2} = \overline{Y_2}$, $\underline{X_2} = \underline{Y_2}$. Thus, we just need to check the equivalence between the middle approximation operation on $[X_1]_f \cup [X_2]_f$ and $[Y_1]_f \cup [Y_2]_f$.

Noting g_f is an isomorphism from $\sigma(P(U)/\overline{R})$ onto $\sigma(U/R)$, as well as:

$$g_f([X_1]_{\overline{R}}) = g_f([Y_1]_{\overline{R}}), g_f([X_2]_{\overline{R}}) = g_f([Y_2]_{\overline{R}}).$$

In addition, $\overline{R}$ is a congruence on $P(U)$. Then,

$$\begin{aligned}g_f([X_1]_{\overline{R}}) \cup g_f([X_2]_{\overline{R}}) &= g_f([X_1]_{\overline{R}} \cup [X_2]_{\overline{R}}) = g_f([X_1 \cup X_2]_{\overline{R}}), \\ g_f([Y_1]_{\overline{R}}) \cup g_f([Y_2]_{\overline{R}}) &= g_f([Y_1]_{\overline{R}} \cup [Y_2]_{\overline{R}}) = g_f([Y_1 \cup Y_2]_{\overline{R}}).\end{aligned}$$

In short,

$$\begin{aligned} g_f \left([X_1] \cup [X_2]_{\bar{R}} \right) &= g_f \left([Y_1] \cup [Y_2]_{\bar{R}} \right). \\ f_{\upharpoonright_{X_1 \cup X_2}}(\sigma(U/R)) &= f_{\upharpoonright_{Y_1 \cup Y_2}}(\sigma(U/R)). \end{aligned}$$

Therefore, $[X_1]_f \cup [X_2]_f = [Y_1]_f \cup [Y_2]_f$. Similarly, we can prove $[X_1]_f \cap [X_2]_f = [Y_1]_f \cap [Y_2]_f$.

Assume that $[X_1]_f \preceq [Z]_f$, $[X_2]_f \preceq [Z]_f$. Notice that the value of middle approximation operation on $[X_1]_f \cup [X_2]_f$ is $g_f([X_1] \cup [X_2]_{\bar{R}})$. Because g_f is an isomorphism, we have $g_f([X_1] \cup [X_2]_{\bar{R}}) \subseteq g_f([Z]_{\bar{R}})$. This means $[X_1]_f \cup [X_2]_f$ is the supremum of $[X_1]_f$ and $[X_2]_f$.

Similarly, for arbitrary $[W]_f \preceq [X_1]_f$, $[W]_f \preceq [X_2]_f$, we have $g_f([W]_{\bar{R}}) \subseteq g_f([X_1] \cap [X_2]_{\bar{R}})$. This means $[X_1]_f \cap [X_2]_f$ is the infimum of $[X_1]_f$ and $[X_2]_f$. $\square$

Notice that $R_f \subseteq \tilde{R}$, we can build a binary relation on the direct product space of quotient set as follows:

$$\tilde{R}/R_f = \left\{ ([X]_f, [Y]_f) \in (P(U)/R_f)^2 \mid [X]_{\approx} = [Y]_{\approx} \right\}.$$

Lemma 2 $\tilde{R}/R_f$ is a congruence relation on lattice $P(U)/R_f$.

Proof Because $\tilde{R}$ is an equivalence relation on $P(U)$, it is easy to see $\tilde{R}/R_f$ is an equivalence relation on $P(U)/R_f$.

Let $([X_1]_f, [Y_1]_f), ([X_2]_f, [Y_2]_f) \in \tilde{R}/R_f$. According to the definition, for arbitrary $X \in [X_1]_f \cup [X_2]_f$ and $Y \in [Y_1]_f \cup [Y_2]_f$, $[X]_{\approx} = [Y]_{\approx}$ holds. In other words, $([X_1]_f \cup [X_2]_f, [Y_1]_f \cup [Y_2]_f) \in \tilde{R}/R_f$.

Similarly, we can prove $([X_1]_f \cap [X_2]_f, [Y_1]_f \cap [Y_2]_f) \in \tilde{R}/R_f$. Thus, equivalence relation $\tilde{R}/R_s$ preserves operations.

Because $\tilde{R}/R_f$ is a congruence relation on lattice $P(U)/R_s$, we can define the operations on quotient algebra $(P(U)/R_s)/(\tilde{R}/R_s)$ as follows:

$$\begin{aligned} [X]_f/(\tilde{R}/R_f) \cap [Y]_f/(\tilde{R}/R_f) &= ([X]_f \cap [Y]_f)/(\tilde{R}/R_f), \\ [X]_f/(\tilde{R}/R_f) \cup [Y]_f/(\tilde{R}/R_s) &= ([X]_f \cup [Y]_f)/(\tilde{R}/R_f). \end{aligned}$$

Inspired by Second Isomorphism Theorem, we have the following result.

Theorem 4 $(P(U)/R_f)/(\tilde{R}/R_f) \cong P(U)/\tilde{R}$.

Proof Define $\alpha: (P(U)/R_f)/(\tilde{R}/R_f) \rightarrow P(U)/\tilde{R}$, $\alpha([X]_f/(\tilde{R}/R_f)) = [X]_{\approx}$.

Firstly, $[X]_{\approx} = [Y]_{\approx} \Leftrightarrow [X]_f/(\tilde{R}/R_f) = [Y]_f/(\tilde{R}/R_f)$.

Secondly, $\alpha([X]_f/(\tilde{R}/R_f) \cap [Y]_f/(\tilde{R}/R_f)) = \alpha([X]_f \cap [Y]_f/(\tilde{R}/R_f))$.

On one hand,

$$[(\underline{X} \cap \underline{Y} \cup f_{\upharpoonright_{X \cap Y}}(\sigma(U/R)) \cup f_c(\bar{X} \cap \bar{Y}))]_{\approx} = [(\underline{X} \cap \underline{Y} \cup f_c(\bar{X} \cap \bar{Y}))]_{\approx}.$$

On the other hand,

$$\begin{aligned} \left[\left(\underline{X} \cap \underline{Y} \cup f_c(\overline{X} \cap \overline{Y}) \right) \right]_{\approx} &= [X]_{\approx} \cap [Y]_{\approx} \\ &= \alpha([X]_f / (\tilde{R}/R_f)) \cap \alpha([Y]_f / (\tilde{R}/R_f)). \end{aligned}$$

Then, $\alpha([X]_f / (\tilde{R}/R_f)) \cap ([Y]_f / (\tilde{R}/R_f)) = \alpha([X]_f / (\tilde{R}/R_f)) \cap \alpha([Y]_f / (\tilde{R}/R_f))$. In a similar way, we can prove:

$$\alpha([X]_f / (\tilde{R}/R_f)) \cup ([Y]_f / (\tilde{R}/R_f)) = \alpha([X]_f / (\tilde{R}/R_f)) \cup \alpha([Y]_f / (\tilde{R}/R_f)).$$

Consequently, α is an isomorphism between the lattices. $\square$

Remark 5 In the theorem above, we show a hierarchical structure between lattices $(P(U)/R_f)$ and $P(U)/\tilde{R}$. In other words, rough equivalence relation R_f based on selection function f indeed achieves a variable precision description of data space, which results in multiple-granularity.

4 Discussion on Selection Congruence and Axioms of Middle Approximation Operation

In this section, we would like to discuss certain properties of selection congruence. Particularly, restriction of selection congruence on subalgebra of Boolean algebra $(P(U), \cap, \cup, -, \emptyset, U)$ will be studied. In addition, as it is shown in Pawlak's rough sets, we also build axioms of middle approximation operation.

4.1 Restriction of Selection Congruence on Subalgebra

Let U be a nonempty universe, and C is an arbitrary covering of U , i.e. $C \subseteq P(U)$ and $U \subseteq \bigcup C$. Also, denote the selection congruence relation on Boolean algebra $(P(U), \cap, \cup, -, \emptyset, U)$ as $\overline{R}$.

In fact, $[X]_{\overline{R}}$ can be regarded as a subset of $P(U)$ in the sense of partition. Denote $C^- = \{X \in P(U) \mid C \cap [X]_{\overline{R}} \neq \emptyset\}$ as the rough upper approximation of covering C in $P(U)$. For every element X in C^- , there is at least one element C_1 in C such that $X \overline{R} C_1$.

Meanwhile, let $\overline{R}|_C = \overline{R} \cap C$ be the restriction of $\overline{R}$ on covering C . For every element $[X]_{\overline{R}|_C}$ in $C/\overline{R}|_C$, there is one element (equivalence class) $[X]_{\overline{R}}$ in $P(U)/\overline{R}$, such that $[X]_{\overline{R}|_C} = [X]_{\overline{R}} \cap C$.

Lemma 3 Consider $\sigma(C)$, which is generated by covering C , as the subalgebra of $P(U)$. For the selection congruence relation $\overline{R}$ on $(P(U), \cap, \cup, -, \emptyset, U)$, we have the following results.

- $\sigma(C)^-$ is closed under intersection, union and complementation operations.
- $\overline{R}|_{\sigma(C)}$ is a congruence relation on $\sigma(C)$.

Here, $\sigma(C)^- = \{X \in P(U^\infty) | \sigma(C) \cap [X]_{\bar{R}} \neq \emptyset\}$, $\bar{R}_{|\sigma(C)} = \bar{R} \cap \sigma(C)$.

The proof of Lemma 3 is straightforward. Meanwhile, based on Third Isomorphism Theorem [6], we have the following isomorphic relationship between two quotient algebras.

Theorem 5 $\sigma(C)/\bar{R}_{|\sigma(C)} \cong \sigma(C)^-/\bar{R}_{|\sigma(C)^-}$.

Proof Define $\alpha: \sigma(C)/\bar{R}_{|\sigma(C)} \rightarrow \sigma(C)^-/\bar{R}_{|\sigma(C)^-}$, satisfying:

$$\alpha(c/\bar{R}_{|\sigma(C)}) = c/\bar{R}_{|\sigma(C)^-}, \quad \forall c \in \sigma(C).$$

Firstly, α is well-defined. To confirm this, notice that for arbitrary $c_1, c_2 \in \sigma(C)$, if $c_1 \bar{R} c_2$, then $c_1/\bar{R}_{|\sigma(C)^-} = c_2/\bar{R}_{|\sigma(C)^-}$.

Secondly, for arbitrary $X \in \sigma(C)^-$, exists $c \in \sigma(C)$ and $c \in [X]_{\bar{R}}$, i.e. $c/\bar{R}_{|\sigma(C)^-} = X/\bar{R}_{|\sigma(C)^-}$.

In addition, if $c_1, c_2 \in \sigma(C)$ and $c_1/\bar{R}_{|\sigma(C)} \neq c_2/\bar{R}_{|\sigma(C)}$, then, $c_1/\bar{R}_{|\sigma(C)^-} \neq c_2/\bar{R}_{|\sigma(C)^-}$.

Next, for arbitrary $c_1, c_2 \in \sigma(C)$, we have:

$$\alpha\left(c_1/\bar{R}_{|\sigma(C)} \cup c_2/\bar{R}_{|\sigma(C)}\right) = \alpha((c_1 \cup c_2)/\bar{R}_{|\sigma(C)}) = (c_1 \cup c_2)/\bar{R}_{|\sigma(C)^-}.$$

Consequently, $\alpha\left(c_1/\bar{R}_{|\sigma(C)} \cup c_2/\bar{R}_{|\sigma(C)}\right) = \alpha(c_1/\bar{R}_{|\sigma(C)}) \cup \alpha(c_2/\bar{R}_{|\sigma(C)})$. Similarly, we can prove that:

$$\alpha\left(c_1/\bar{R}_{|\sigma(C)} \cap c_2/\bar{R}_{|\sigma(C)}\right) = \alpha(c_1/\bar{R}_{|\sigma(C)}) \cap \alpha(c_2/\bar{R}_{|\sigma(C)}).$$

Besides, for arbitrary $c \in \sigma(C)$, $\alpha(-c/\bar{R}_{|\sigma(C)}) = -\alpha(c/\bar{R}_{|\sigma(C)})$ holds. $\square$

4.2 Axioms of Middle Approximation Operation

Similar to the axioms of upper and lower approximation operations in Pawlak's sense, in this part, we show the axioms of middle approximation operation with respect to rough equivalence relation based on selection function.

Definition 11 Let U be a nonempty set. Denote the middle approximation operation as follows: $M: P(U) \rightarrow P(U)$, $M(X) = g([X]_{\bar{R}})$.

Theorem 6 Middle approximation operation defined above satisfies the following axiomatic properties:

- (1) $M(U) = U$; (Co-normality)
- (2) $M(\emptyset) = \emptyset$; (Normality)
- (3) $\underline{X} \subseteq M(X) \subseteq \bar{X}$;
- (4) $X \subseteq Y \Rightarrow M(X) \subseteq M(Y)$; (Monotone)

- (5) $M(X \cup Y) = M(X) \cup M(Y)$; (*Addition*)
- (6) $M(X \cap Y) = M(X) \cap M(Y)$; (*Multiplication*)
- (7) $M(M(X)) = M(X)$; (*Idempotency*)
- (8) $M(-X) = -M(X)$; (*Duality*)
- (9) $M(-M(X)) = -M(X)$; (*Middle-complement relation*)
- (10) $\forall K \in U/R, M(K) = K$; (*Granularity*)

Proof Firstly, it is apparent to see the correction of (1), (2), (3), (7), (10).

Consider (4), let $X \subseteq Y$, then $\{[x]_R | f([x]_R) \in X\} \subseteq \{[x]_R | f([x]_R) \in Y\}$. This leads to $g([X]_{\bar{R}}) \subseteq g([Y]_{\bar{R}})$. Consequently, $M(X) \subseteq M(Y)$.

Consider (5), for arbitrary $x \in M(X \cup Y)$, exists $[y]_R \in g([X \cup Y]_{\bar{R}})$, satisfying $x \in [y]_R, f([y]_R) \in X \cup Y$. Without loss of generality, suppose that $f([y]_R) \in X$, then $x \in M(X)$. On the other hand, based on (4), we have $M(X) \subseteq M(X \cup Y), M(Y) \subseteq M(X \cup Y)$. Consequently, $M(X \cup Y) = M(X) \cup M(Y)$ holds.

For (6), if $X \cap Y = \emptyset$, assume that there is an element $x \in M(X) \cap M(Y)$. As a result, there is an element $y \in U$, satisfying $f([y]_R) \in X \cap Y = \emptyset$, which leads to a contradiction. Combining with (2), we can conclude that $M(X \cap Y) = M(X) \cap M(Y)$. If $X \cap Y \neq \emptyset$, according to (4), we just need to show: $M(X \cap Y) \supseteq M(X) \cap M(Y)$. To this end, assume that $x \in M(X) \cap M(Y)$, exists $y \in U$, satisfying $x \in [y]_R, f([y]_R) \in X \cap Y$. Hence, $M(X \cap Y) = M(X) \cap M(Y)$.

Considering (8), for arbitrary $x \in M(-X)$, exists $y \in U$, satisfying $x \in [y]_R, f([y]_R) \in (U - X)$. Thus, $f([y]_R) \notin X$, i.e. $[y]_R \in U - M(X), x \in -M(X)$, and vice versa. In addition, (9) can be inferred directly from (7) and (8).

Remark 6 The names of axioms are similar to Zhu and Wang's discussion [26]. Besides, (3) is essential property of rough sets based on selection function.

5 Conclusion

This paper proposes a triple structure of rough sets based on selection function. A middle approximation operation is employed to fulfill the construction of new rough sets algebra structure, as a supplementation to upper and lower approximation operations constructed in rough sets in Pawlak's sense. This kind of rough sets algebra, arisen from selection congruence relation on the power set of universe, builds a more meticulous granularity structure comparing with that in traditional rough sets algebra. First, lattice structure of the proposed rough sets is discussed. Second, based on homomorphism and isomorphism theorems in Universal Algebra, isomorphic relationships between different rough sets algebra structures are also studied and axioms of middle approximation operation are proven in this paper.

In the future work, we will consider how to extend our new rough sets structure to covering generalized rough sets, which is more useful in many real-life problems. Particularly, we will focus on how to apply our structure to irredundant covering based generalized rough sets system. Also, as same as the discussion in rough sets in Pawlak's sense, more algebra systems based on the proposed rough sets will be studied, such as Heyting and Nelson algebras.

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