A General Statistic for Testing the Validity of a Model's Forecasts

Hadi Alizadeh Noughabi¹

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Abstract The paper introduces a new approach for testing the validity of a model's forecasts and it is established that the proposed test is exact for location–scale family. The results are used to introduce goodness-of-fit tests for the normal, exponential, uniform and Laplace distributions. Through Monte Carlo simulation, the power values of the proposed test under various alternatives are compared with the other tests. The proposed test has higher power than the competing tests against some alternatives. The use of the proposed test is shown in real examples.

Keywords Goodness of fit tests · Model validity · Location–scale family · Exact test · Power comparison · Nonparametric kernel density estimation

1 Introduction

In reliability studies, engineering and management sciences, it is important to test whether the underlying distribution has a particular form. Most statistical methods assume an underlying distribution in the derivation of their results and inferences. Therefore approaches for determining the underlying distribution, i.e. goodness of fit test, is necessary.

Some general goodness of fit tests can be found in the literature, namely, Kolmogorov–Smirnov, Cramér–von Mises, Kuiper and Anderson–Darling. These tests are common and famous tests which are widely used in practice. For further study about these tests, see [17].

✉ Hadi Alizadeh Noughabi
alizadehhadi@birjand.ac.ir

¹ Department of Statistics, University of Birjand, Birjand, Iran

Different methods for goodness of fit tests are introduced by researchers such as goodness of fit tests based on the empirical distribution function, empirical characteristic function, entropy and Kullback–Leibler information, maximum correlations, and divergences.

Comparison of goodness of fit tests has received attention in the literature. Stephens [35] by Monte Carlo simulation presented comparisons for some tests for normality. Moreover, some authors compared the power of goodness of fit tests for other distributions.

The goodness of fit tests has been discussed by many authors including, [1–9, 13–18, 20, 22, 23, 26–28, 31–33, 37]. Moreover, some tests for censored data are proposed by authors; see, for example, [10, 11, 21, 25, 29, 30].

In Sect. 2, we introduce an exact goodness of fit test by using the kernel density estimation. Also, properties of proposed test are discussed. Section 3 discusses the special case of location–scale family and we used the results for introducing tests for normal, exponential, uniform and Laplace distributions. In Sect. 4, by simulation, the power values of the proposed test are compared with the power values of the competitor tests. Section 5 contains the use of the proposed test in real examples.

2 The Test Statistic

Let $X_1, \dots, X_n$ be a random sample from an unknown distribution F with a probability density function $f(x)$. Let $F_0(x; \theta)$ be a parametric family of distributions with probability density function $f_0(x; \theta)$. The hypothesis of interest is

$$H_0 : f(x) = f_0(x; \theta), \quad \text{for some } \theta \in \Theta,$$

and the alternative to H_0 is

$$H_1 : f(x) \neq f_0(x; \theta), \quad \text{for any } \theta \in \Theta.$$

To discriminate between the two hypotheses H_0 and H_1 , we apply the following method.

We generate the following data ($Y_{(i)}$'s) under the null distribution F_0 as

$$Y_{(i)} = F_0^{-1} \left(\frac{i}{n+1}, \theta \right), \quad i = 1, 2, \dots, n.$$

If θ be unknown, we will estimate it by a reasonable estimator.

In order hand, $X_{(i)} = F^{-1} \left(\frac{i}{n+1} \right)$, where $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ are the order statistics of sample.

Let β be a constant. We have under H_0 ,

$$\begin{aligned} \forall i \ X_{(i)} = Y_{(i)} &\Leftrightarrow \forall i \ F(X_{(i)}) = F_0(Y_{(i)}) \Leftrightarrow \forall i \ f(X_{(i)}) = f_0(Y_{(i)}) \\ &\Leftrightarrow \forall i \ \left(\frac{f(X_{(i)})}{f_0(Y_{(i)})} \right)^\beta - 1 = 0 \\ &\Leftrightarrow T_\beta(f, f_0) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{f(X_{(i)})}{f_0(Y_{(i)})} \right)^\beta - 1 \right]^2 = 0. \end{aligned}$$

And under H_1 ,

$$\begin{aligned} \exists i \ X_{(i)} \neq Y_{(i)} &\Leftrightarrow \exists i \ F(X_{(i)}) \neq F_0(Y_{(i)}) \Leftrightarrow \exists i \ f(X_{(i)}) \neq f_0(Y_{(i)}) \\ &\Leftrightarrow \exists i \ \left(\frac{f(X_{(i)})}{f_0(Y_{(i)})} \right)^\beta - 1 \neq 0 \\ &\Leftrightarrow T_\beta(f, f_0) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{f(X_{(i)})}{f_0(Y_{(i)})} \right)^\beta - 1 \right]^2 > 0. \end{aligned}$$

It is obvious that $T_\beta(f, f_0) \geq 0$ and the equality holds if and only if $f(x) = f_0(x)$ almost everywhere. Therefore, $T_\beta(f, f_0)$ measures the discrepancy between F and F_0 . Under H_0 , $T_\beta(f, f_0) = 0$, and $T_\beta(f, f_0) > 0$ if H_0 be false.

We consider the kernel density estimator for estimating $f(x_i)$ and $f_0(y_i)$. The kernel density function estimator is defined by

$$\hat{f}(X_i) = \frac{1}{nh} \sum_{j=1}^n k\left(\frac{X_i - X_j}{h}\right),$$

where h is a bandwidth and k is a kernel function which satisfies

$$\int_{-\infty}^{\infty} k(x)dx = 1.$$

Usually, k will be a symmetric probability density function.

Therefore, we propose the following test statistic for goodness of fit test.

$$T_\beta(f, f_0) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{\hat{f}(X_{(i)})}{\hat{f}_0(Y_{(i)}; \hat{\theta})} \right)^\beta - 1 \right]^2,$$

where

$$\hat{f}(x_{(i)}) = \frac{1}{nh_x} \sum_{j=1}^n k\left(\frac{x_{(i)} - x_j}{h_x}\right), \quad \hat{f}_0(y_{(i)}) = \frac{1}{nh_y} \sum_{j=1}^n k\left(\frac{y_{(i)} - y_j}{h_y}\right),$$

and the kernel function is chosen to be the standard normal density function and the bandwidth h is chosen to be the normal optimal smoothing formula, $h = 1.06sn^{-\frac{1}{5}}$, where s is the sample standard deviation.

Since large values of $T_\beta(f, f_0)$ favor the alternative hypothesis to H_0 , we reject H_0 if $T_\beta(f, f_0)$ is large, i.e., reject H_0 if $T_\beta(f, f_0) \geq C_\alpha$ for some critical value C_α .

Since

$$\hat{f}(x) \xrightarrow{P} f(x) \quad \text{as } n \rightarrow \infty,$$

[34] and, by of LLN,

$$\frac{1}{n} \sum_{i=1}^n \left[\left(\frac{\hat{f}(x_{(i)})}{\hat{f}_0(y_{(i)}; \hat{\theta})} \right)^\beta - 1 \right]^2 \rightarrow E_f \left\{ \left[\left(\frac{f(x)}{f_0(y; \theta)} \right)^\beta - 1 \right]^2 \right\},$$

the test based on $T_\beta(f, f_0)$ is consistent.

Let G be the group of transformations, we have

$$\forall g \in G, \quad T_\beta(g(x)) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{\hat{f}(g(x_{(i)}))}{\hat{f}_0(g(y_{(i)}); \hat{\theta})} \right)^\beta - 1 \right]^2,$$

where

$$\begin{aligned} \hat{f}(g(x_{(i)})) &= \frac{1}{nh_{g(x)}} \sum_{j=1}^n k \left(\frac{g(x_{(i)}) - g(x_{(j)})}{h_{g(x)}} \right), & h_{g(x)} &= 1.06S_{g(x)}n^{-\frac{1}{5}}, \\ S_{g(x)}^2 &= \frac{1}{n-1} \sum_{i=1}^n \left(g(x_{(i)}) - \overline{g(x)} \right)^2, & \overline{g(x)} &= \frac{1}{n} \sum_{i=1}^n g(x_{(i)}), \\ \hat{f}(g(y_{(i)})) &= \frac{1}{nh_{g(y)}} \sum_{j=1}^n k \left(\frac{g(y_{(i)}) - g(y_{(j)})}{h_{g(y)}} \right), & h_{g(y)} &= 1.06S_{g(y)}n^{-\frac{1}{5}}, \\ S_{g(y)}^2 &= \frac{1}{n-1} \sum_{i=1}^n \left(g(y_{(i)}) - \overline{g(y)} \right)^2, & \overline{g(y)} &= \frac{1}{n} \sum_{i=1}^n g(y_{(i)}). \end{aligned}$$

Generally, the proposed test is not exact, because

$$\frac{\hat{f}(g(x_{(i)}))}{\hat{f}_0(g(y_{(i)}); \hat{\theta})} \neq \frac{\hat{f}(x_{(i)})}{\hat{f}_0(y_{(i)}; \hat{\theta})}.$$

However, if $g(x) = x + d$, $g(x) = cx$ or $g(x) = cx + d$, which is the case for location, scale and location–scale families respectively, the test is exact. Because we have

$$\begin{aligned}
h_{g(x)} &= ch_x, \quad \hat{f}(g(x_{(i)})) = \frac{1}{nch_x} \sum_{j=1}^n k \left(\frac{(cx_{(i)} + d) - (cx_j + d)}{ch_x} \right) = \frac{1}{c} \hat{f}(x_{(i)}), \\
h_{g(y)} &= ch_y, \quad \hat{f}(g(y_{(i)})) = \frac{1}{nch_y} \sum_{j=1}^n k \left(\frac{(cy_{(i)} + d) - (cy_j + d)}{ch_y} \right) = \frac{1}{c} \hat{f}(y_{(i)}), \\
\frac{\hat{f}(g(x_{(i)}))}{\hat{f}_0(g(y_{(i)}); \hat{\theta})} &= \frac{\hat{f}(x_{(i)})}{\hat{f}_0(y_{(i)}; \hat{\theta})} \Rightarrow T_{\beta}(g(x)) = T_{\beta}(x).
\end{aligned}$$

Several examples of scale and location–scale families are considered in Sect. 3. Section 4 contains a power study for goodness of fit tests introduced in the preceding section. Applications of the proposed test in real examples are given in Sect. 5.

3 Goodness of Fit Tests for Location–Scale Family

Suppose $X_1, \dots, X_n$ are a random sample from a continuous probability distribution F with a density $f(x)$, the hypothesis of interest is

$$H_0 : f(x) = f_0(x; \alpha, \beta) = \frac{1}{\beta} h \left(\frac{x - \alpha}{\beta} \right), \quad (\alpha, \beta) \in \Theta = \mathbb{R} \times \mathbb{R}^+,$$

where α and β are unspecified. The alternative to H_0 is

$$H_1 : f(x) \neq f_0(x; \alpha, \beta), \quad \text{for any } (\alpha, \beta) \in \Theta.$$

Note that the location and scale families are two special cases of the location–scale family.

Example 1 Let

$$f_0(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right\}, \quad -\infty < x < \infty, \mu \in \mathbb{R}, \sigma > 0.$$

We have

$$T_{\beta}(f, f_0) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{\hat{f}(X_{(i)})}{\hat{f}_0(Y_{(i)}; \hat{\theta})} \right)^{\beta} - 1 \right]^2,$$

where $y_{(i)} = F_0^{-1}(\frac{i}{n+1})$, $i = 1, 2, \dots, n$ and

$$\hat{\theta} = \left(\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, S^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right).$$

Example 2 Let

$$f_0(x; \theta) = \frac{1}{\theta} \exp \left\{ -\frac{x}{\theta} \right\}, \quad x > 0, \theta > 0.$$

We have

$$T_{\beta}(f, f_0) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{\hat{f}(X_{(i)})}{\hat{f}_0(Y_{(i)}; \hat{\theta})} \right)^{\beta} - 1 \right]^2,$$

where $y_{(i)} = F_0^{-1} \left(\frac{i}{n+1} \right) = -\hat{\theta} \log \left(1 - \frac{i}{n+1} \right)$, $i = 1, 2, \dots, n$ and $\hat{\theta} = \bar{x}$.

Example 3 If

$$f_0(x) = 1, \quad 0 < x < 1,$$

then

$$T_{\beta}(f, f_0) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{\hat{f}(X_{(i)})}{\hat{f}_0(Y_{(i)})} \right)^{\beta} - 1 \right]^2,$$

where $y_{(i)} = F_0^{-1} \left(\frac{i}{n+1} \right) = \frac{i}{n+1}$, $i = 1, 2, \dots, n$.

Example 4 This example introduces goodness of fit test for Laplace distribution. The hypothesis of interest is

$$H_0 : f(x) = f_0(x; \alpha, \beta) = \frac{1}{2\beta^2} \exp \left\{ -\frac{|x - \alpha|}{\beta} \right\}, \\ -\infty < x < \infty, \text{ for some } (\alpha, \beta) \in \Theta = \mathbb{R} \times \mathbb{R}^+,$$

where α and β are unknown. The alternative to H_0 is

$$H_1 : f(x) \neq f_0(x; \alpha, \beta), \quad \text{for any } (\alpha, \beta) \in \Theta.$$

The test statistics is

$$T_{\beta}(f, f_0) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{\hat{f}(X_{(i)})}{\hat{f}_0(Y_{(i)})} \right)^{\beta} - 1 \right]^2,$$

where $y_{(i)} = F_0^{-1} \left(\frac{i}{n+1} \right)$, $i = 1, 2, \dots, n$ and $\hat{\alpha}$ and $\hat{\beta}$ are the maximum likelihood estimates of α and β ,

$$\hat{\alpha} = \text{median}(x_1, x_2, \dots, x_n), \quad \hat{\beta} = \frac{1}{n} \sum_{i=1}^n |x_i - \hat{\alpha}|.$$

Table 1 Critical values of the proposed statistic for location–scale group family and $\beta = 1$

<i>n</i>	Normality test		Exponentiality test		Uniformity test		Testing Laplace	
	Significance level		Significance level		Significance level		Significance level	
	0.05	0.10	0.05	0.10	0.05	0.10	0.05	0.10
5	0.0904	0.0811	0.5393	0.2853	1.2040	0.5522	0.3525	0.3376
10	0.1095	0.0869	0.3314	0.1934	0.3439	0.2010	0.2942	0.2698
15	0.1153	0.0889	0.2947	0.1747	0.1991	0.1345	0.3423	0.2799
20	0.1134	0.0866	0.2614	0.1584	0.1531	0.1051	0.3779	0.2984
25	0.1067	0.0845	0.2369	0.1519	0.1227	0.0873	0.3778	0.3057
30	0.1034	0.0806	0.2159	0.1357	0.1072	0.0767	0.3831	0.2996
40	0.0981	0.0727	0.1814	0.1240	0.0824	0.0603	0.3776	0.2992
50	0.0879	0.0673	0.1610	0.1030	0.0673	0.0502	0.3608	0.2906

4 Simulation Study

Monte Carlo methods were used to obtain the critical values of our procedure. Table 1 gives the critical values of the proposed statistic for testing normal, exponential, uniform, and Laplace distributions, respectively.

The competitor tests are choose from the class of tests discussed in D'Agostino and Stephens [17]. The test statistics of competitor tests are as follows.

The Kolmogorov–Smirnov, Cramer–von Mises, Kuiper and Anderson–Darling test statistics are respectively:

$$\begin{aligned}
 KS &= \max \left\{ \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - Z_{(i)} \right\}, \max_{1 \leq i \leq n} \left\{ Z_{(i)} - \frac{i-1}{n} \right\} \right\}, \\
 CH &= \frac{1}{12n} + \sum_{i=1}^n \left(\frac{2i-1}{2n} - Z_{(i)} \right)^2, \\
 V &= \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - Z_{(i)} \right\} + \max_{1 \leq i \leq n} \left\{ Z_{(i)} - \frac{i-1}{n} \right\}, \\
 AD &= \left[-n - \frac{\sum_{i=1}^n (2i-1) \{ \ln(Z_{(i)}) + \ln(1 - Z_{n-i+1}) \}}{n} \right],
 \end{aligned}$$

where $Z_i = F_0(x_{(i)}; \hat{\theta})$, $i = 1, \dots, n$ and F_0 is the cdf of the null distribution.

The power values of the tests based on $T(f, f_0)$, KS , V , CH and AD statistics for samples of size $n = 10, 20$ are computed. Tables 2, 3, 4 and 5 show the power values of goodness of fit tests for normal, exponential, uniform and Laplace distributions at the significance level $\alpha = 0.05$, respectively. For each alternative, the bold type in these tables indicates the test achieving the maximal power.

We observe that the proposed tests perform very well compared with the other tests for some alternatives. No single test can be said to perform the best against all alternatives. Generally, the proposed test has the most power against some alternatives. For

Table 2 Power comparisons of normal goodness of fit tests with size 0.05

Alternatives	$n = 10$							$n = 20$						
	KS	CH	V	AD	$T_{0.5}$	T_1	T_2	KS	CH	V	AD	$T_{0.5}$	T_1	T_2
Normal	0.051	0.051	0.054	0.047	0.050	0.050	0.050	0.049	0.051	0.048	0.051	0.050	0.050	0.050
Gamma(2)	0.175	0.210	0.180	0.225	0.247	0.264	0.255	0.326	0.425	0.353	0.467	0.570	0.576	0.567
Gamma(3)	0.129	0.151	0.130	0.165	0.180	0.185	0.177	0.239	0.297	0.234	0.330	0.381	0.388	0.373
Weibull(2)	0.068	0.072	0.070	0.081	0.087	0.087	0.082	0.116	0.118	0.146	0.120	0.154	0.163	0.164
Weibull(3)	0.048	0.046	0.047	0.046	0.047	0.046	0.043	0.051	0.050	0.049	0.049	0.042	0.049	0.053
Exponential	0.301	0.390	0.360	0.416	0.475	0.498	0.501	0.586	0.724	0.696	0.773	0.869	0.897	0.902
Uniform	0.066	0.074	0.081	0.080	0.048	0.067	0.073	0.100	0.144	0.150	0.171	0.155	0.229	0.267
Beta(2,1)	0.100	0.115	0.109	0.126	0.123	0.141	0.153	0.174	0.232	0.202	0.261	0.351	0.422	0.441
Beta(2,2)	0.046	0.044	0.048	0.046	0.032	0.038	0.042	0.053	0.058	0.064	0.058	0.041	0.067	0.076
Logistic	0.073	0.080	0.071	0.083	0.078	0.076	0.072	0.087	0.106	0.090	0.113	0.102	0.080	0.059
$t_{(1)}$	0.580	0.618	0.594	0.618	0.590	0.551	0.499	0.847	0.880	0.865	0.882	0.865	0.818	0.763
$t_{(3)}$	0.164	0.182	0.163	0.190	0.188	0.170	0.145	0.260	0.309	0.277	0.327	0.307	0.248	0.183
$\chi^2_{(1)}$	0.537	0.663	0.668	0.702	0.745	0.785	0.795	0.889	0.820	0.953	0.971	0.991	0.994	0.995
$\chi^2_{(2)}$	0.300	0.387	0.361	0.417	0.465	0.504	0.503	0.595	0.734	0.707	0.787	0.875	0.894	0.906

Table 3 Power comparisons of exponential goodness of fit tests with size 0.05

Alternatives	$n = 10$							$n = 20$						
	KS	CH	V	AD	$T_{0.5}$	T_1	T_2	KS	CH	V	AD	$T_{0.5}$	T_1	T_2
Exponential	0.051	0.051	0.054	0.047	0.050	0.050	0.050	0.051	0.048	0.049	0.052	0.050	0.050	0.050
Gamma(2)	0.211	0.240	0.200	0.176	0.222	0.230	0.216	0.406	0.486	0.372	0.441	0.347	0.324	0.281
Gamma(3)	0.457	0.536	0.445	0.448	0.463	0.462	0.433	0.811	0.891	0.780	0.876	0.720	0.673	0.576
Weibull(2)	0.500	0.609	0.515	0.518	0.585	0.579	0.555	0.848	0.930	0.850	0.915	0.864	0.846	0.773
Weibull(3)	0.899	0.968	0.931	0.947	0.962	0.962	0.943	0.999	1.000	1.000	1.000	1.000	1.000	0.997
Normal(5,1)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Uniform	0.280	0.362	0.356	0.289	0.574	0.598	0.601	0.528	0.673	0.668	0.619	0.931	0.929	0.929
Beta(2,1)	0.837	0.947	0.934	0.921	0.985	0.986	0.988	0.995	0.999	0.999	0.999	1.000	1.000	1.000
Beta(2,2)	0.622	0.764	0.699	0.691	0.824	0.830	0.818	0.933	0.987	0.971	0.983	0.993	0.992	0.987
Log-normal	0.097	0.101	0.088	0.079	0.070	0.051	0.053	0.138	0.152	0.143	0.134	0.063	0.036	0.034
$\chi^2_{(1)}$	0.260	0.290	0.217	0.475	0.141	0.010	0.007	0.471	0.524	0.391	0.701	0.122	0.005	0.006
$\chi^2_{(2)}$	0.050	0.050	0.051	0.048	0.053	0.047	0.052	0.052	0.048	0.051	0.048	0.052	0.049	0.050

Table 4 Power comparisons of uniform goodness of fit tests with size 0.05

Alternatives	$n = 10$						$n = 20$							
	KS	CH	V	AD	$T_{0.5}$	T_1	T_2	KS	CH	V	AD	$T_{0.5}$	T_1	T_2
Uniform	0.051	0.051	0.052	0.048	0.050	0.050	0.050	0.048	0.049	0.049	0.053	0.050	0.050	0.050
Beta(2,1)	0.390	0.440	0.245	0.421	0.299	0.293	0.297	0.686	0.759	0.474	0.752	0.573	0.572	0.587
Beta(3,1)	0.806	0.866	0.569	0.857	0.688	0.679	0.675	0.988	0.996	0.934	0.996	0.958	0.959	0.969
Beta(3,2)	0.204	0.180	0.349	0.122	0.487	0.493	0.488	0.499	0.486	0.682	0.443	0.797	0.802	0.808
Beta(3,.5)	0.990	0.997	0.917	0.998	0.931	0.936	0.937	1.000	0.999	1.000	1.000	0.999	0.999	1.000
Beta(2,.5)	0.894	0.938	0.666	0.960	0.675	0.675	0.689	0.995	0.999	0.959	0.999	0.966	0.968	0.977
Beta(2,2)	0.041	0.026	0.182	0.014	0.284	0.277	0.277	0.065	0.047	0.358	0.039	0.469	0.486	0.492
Beta(3,3)	0.046	0.026	0.410	0.012	0.571	0.576	0.565	0.148	0.136	0.769	0.151	0.868	0.877	0.880

Table 5 Power comparisons of Laplace goodness of fit tests with size 0.05

Alternatives	$n = 10$							$n = 20$						
	KS	CH	V	AD	$T_{0.5}$	T_1	T_2	KS	CH	V	AD	$T_{0.5}$	T_1	T_2
Normal	0.051	0.051	0.054	0.047	0.011	0.026	0.051	0.089	0.084	0.103	0.073	0.032	0.069	0.078
Gamma(2)	0.104	0.121	0.093	0.131	0.265	0.342	0.243	0.183	0.212	0.193	0.267	0.773	0.726	0.591
Gamma(3)	0.076	0.089	0.076	0.094	0.201	0.254	0.163	0.133	0.153	0.142	0.185	0.610	0.534	0.389
Weibull(2)	0.063	0.069	0.068	0.067	0.010	0.027	0.083	0.116	0.118	0.146	0.120	0.033	0.159	0.225
Weibull(3)	0.051	0.057	0.059	0.052	0.007	0.030	0.081	0.102	0.090	0.125	0.081	0.028	0.126	0.165
Exponential	0.241	0.245	0.202	0.269	0.062	0.107	0.337	0.473	0.437	0.461	0.535	0.217	0.834	0.908
Uniform	0.099	0.116	0.138	0.106	0.000	0.014	0.141	0.244	0.256	0.364	0.253	0.023	0.332	0.489
Beta(2,1)	0.105	0.131	0.114	0.128	0.000	0.100	0.242	0.216	0.242	0.290	0.267	0.058	0.418	0.576
Beta(2,2)	0.065	0.071	0.082	0.064	0.000	0.000	0.060	0.154	0.137	0.200	0.129	0.001	0.089	0.178
Logistic	0.046	0.050	0.046	0.046	0.019	0.030	0.045	0.064	0.056	0.066	0.052	0.035	0.055	0.065
Lognormal(0, .5)	0.092	0.106	0.091	0.117	0.134	0.179	0.173	0.163	0.188	0.158	0.239	0.466	0.530	0.474
$t_{(1)}$	0.325	0.336	0.371	0.357	0.253	0.210	0.084	0.516	0.544	0.607	0.565	0.259	0.183	0.141
$t_{(3)}$	0.055	0.053	0.064	0.056	0.051	0.052	0.053	0.071	0.068	0.079	0.075	0.062	0.064	0.071
$\chi^2_{(1)}$	0.565	0.521	0.516	0.553	0.075	0.221	0.642	0.900	0.820	0.907	0.886	0.503	0.993	0.996
$\chi^2_{(2)}$	0.233	0.236	0.196	0.259	0.360	0.476	0.452	0.491	0.448	0.469	0.542	0.903	0.957	0.925

example, in testing normality, the proposed test is powerful against Gamma, Weibull and Beta families. This result also is held for test of fit for Laplace distribution.

5 Real Examples

Here, we present two real examples to show the behavior of the proposed test in real cases.

Example 1 The following data represent active repair times (in hours) for an airborne communication transceiver.

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

First time the data were analyzed by von Alven [36] who fitted successfully the Lognormal distribution.

The implementation of the proposed test can be used to investigate the above conclusion. We obtain the value of the test statistic $T_1 = 0.0283$ and also the critical value at the 5 % is 0.3934. The value of the proposed test statistic is smaller than the critical value so that the fit of the lognormal distribution is accepted at the 5 % level.

Example 2 We consider one real-life data analysis from Lawless [24]. The following data are failure times for 36 appliances subjected to an automatic life test.

11, 35, 49, 170, 329, 381, 708, 958, 1062, 1167, 1594, 1925, 1990, 2223, 2327, 2400, 2451, 2471, 2551, 2565, 2568, 2694, 2702, 2761, 2831, 3034, 3059, 3112, 3214, 3478, 3504, 4329, 6367, 6976, 7846, 13403.

Ebrahimi et al. [19] concluded that the fit of the exponential distribution to these data is satisfactory. Recently, the same conclusion has been drawn by Baratpour and Habibi Rad [12].

For this example, we have observed that the value of the proposed test statistic is smaller than the critical value, at 5 % level; The value of the test statistic is $T_1 = 0.1766$ and the critical value is equal to 0.1945. Then the null distribution is accepted and the results obtained by previous authors are confirmed.

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