

Estimation of Stress-Strength Reliability for the Generalized Pareto Distribution Based on Progressively Censored Samples

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Abstract This paper deals with the estimation of stress-strength reliability parameter, $R = P(Y < X)$, based on progressively type II censored samples when stress, strength are two independent generalized Pareto random variables. The maximum likelihood estimators, their asymptotic distributions, asymptotic confidence intervals, bootstrap based confidence intervals and Bayes estimators are derived for R . Using Monte Carlo simulations, the MSE, Bayes risk estimators, credible sets and coverage probabilities are computed and compared.

Keywords Generalized Pareto distribution · Maximum likelihood estimator · Progressively type II censored samples · Uniformly minimum variance unbiased estimator

1 Introduction

Estimation of $R = P(Y < X)$ is widely known as stress-strength modeling: if X denotes the stress that a system is subjected to and Y the strength of the system then R is the probability of the failure of the system. Many authors have discussed parametric and non-parametric inference on R when complete random samples are available on each X and Y . We mention: [16] for the case X, Y are independent gamma random variables; [4] for the case X, Y are independent exponential random variables with common scale; [17] for the case X, Y are independent logistic random variables; [18] for the case X, Y are independent Laplace random variables; [12]

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for the case X, Y are independent generalized exponential random variables; [19] for the case X, Y are independent beta random variables; [20] for the case X, Y are dependent gamma random variables; [15] for the case X, Y are independent Burr type III random variables; [13] for the case X, Y are independent Weibull random variables; [21] for the case X, Y are dependent exponential random variables; [11] for the case X, Y are independent exponential random variables; [24] for the case X, Y are independent Gompertz random variables; [22] for the case X, Y are independent generalized exponential random variables; [23] for the case X, Y are independent generalized Pareto random variables; [1] for the case X, Y are independent generalized logistic random variables. We refer the readers to [10] for a review of other papers.

There has been little work on parametric and non-parametric inference of $R = P(Y < X)$ when samples available on X and Y are not complete. In reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Some real life examples giving censored data are: (i) sometimes a failure is planned and expected but does not occur due to operator error, equipment malfunction, test anomaly, etc; (ii) sometimes engineers plan a test program so that, after a certain time limit or number of failures, all other tests will be terminated. Among various censoring schemes, the type II progressive censoring scheme has become very popular. It can be described as follows: Let N items be put in a life time study and $n (< N)$ items be completely observed; At the time of the first failure, r_1 surviving units are removed from the $N - 1$ remaining items; At the time of the next failure, r_2 items are randomly withdrawn from the $N - r_1 - 2$ remaining items; When the n th failure occurs all the remaining $N - n - r_1 - \dots - r_{n-1}$ items are removed. See [2] for more details.

The aim of this paper is to extend the results in [23] for progressively type II censored samples, i.e., we develop inference procedures for $R = P(Y < X)$ when X and Y are independent generalized Pareto random variables and samples available on them are progressively type II censored.

The contents of this paper are organized as follows: the generalized Pareto distribution is introduced in Sect. 2; estimation of R when X and Y are generalized Pareto random variables with common scale parameter is addressed in Sect. 3, giving the maximum likelihood estimator (MLE), its asymptotic distribution and bootstrap confidence intervals for R ; estimation of R when the common scale parameter is known is addressed in Sect. 4, giving maximum likelihood, uniformly minimum variance unbiased and Bayes estimators of R ; estimation of R in the general case is addressed in Sect. 5, giving maximum likelihood and Bayes estimators of R ; Monte Carlo simulations are used to compare these estimators in Sect. 6.

The original contributions of this paper are Sects. 3 to 6. These include the algorithm presented in Sect. 4.4.

2 Preliminaries

A random variable X is said to have the generalized Pareto (GP) distribution if its probability density function (pdf) is given by

$$f_X(x) = \frac{1}{\sigma} \left(1 + \xi \frac{x - \mu}{\sigma} \right)^{-\left(\frac{1}{\xi} + 1\right)} \tag{1}$$

for $\xi \in \Re, \mu \in \Re$ and $\sigma > 0$. Exponential, Pareto and logistic distributions are special cases of the GP distribution. The GP distribution has the decreasing failure rate (DFR) property.

A reparametrization of the GP distribution has the pdf and cumulative distribution function (cdf) specified by

$$f_X(x) = \alpha\lambda(1 + \lambda x)^{-(\alpha+1)} \tag{2}$$

and

$$F_X(x) = 1 - (1 + \lambda x)^{-\alpha}, \tag{3}$$

respectively, for $x > 0, \alpha > 0$ and $\lambda > 0$. The parameter α is a shape parameter and λ is a scale parameter.

We shall write $X \sim GP(\alpha, \lambda)$ if X has the pdf and cdf specified by (2) and (3), respectively. Let $X \sim GP(\alpha, \lambda_1)$ and $Y \sim GP(\beta, \lambda_2)$ be independent random variables. The aim of this paper is to estimate $R = P(Y < X)$ based on progressively type II censored samples. We consider three cases.

3 Estimation of R with Same Scale Parameter

Here, we investigate properties of R when $\lambda_1 = \lambda_2 = \lambda$. In this case,

$$R = P(Y < X) = \frac{\beta}{\alpha + \beta}. \tag{4}$$

3.1 Maximum Likelihood Estimator

Suppose $X_{1:N}, X_{2:N}, \dots, X_{n:N}$ is a progressively type II censored sample from $GP(\alpha, \lambda)$ and $Y_{1:M}, Y_{2:M}, \dots, Y_{m:M}$ a progressively type II censored sample from $GP(\beta, \lambda)$. The likelihood function is (see [3])

$$L(\alpha, \beta, \lambda) = K_1 K_2 \prod_{i=1}^n f_X(x_i) [1 - F_X(x_i)]^{r_i} \times \prod_{j=1}^m f_Y(y_j) [1 - F_Y(y_j)]^{r'_j}, \tag{5}$$

where

$$K_1 = (N - 1 - r_1)(N - 2 - r_1 - r_2) \cdots (N - n + 1 - r_1 - \cdots - r_{n-1}), \tag{6}$$

$$K_2 = M(M - 1 - r'_1)(M - 2 - r'_1 - r'_2) \cdots (M - m + 1 - r'_1 - \cdots - r'_{m-1}), \tag{7}$$

and r, r' denote the number of items randomly withdrawn from samples corresponding to X and Y , respectively.

By (2) and (3), (5) can be expressed as

$$L(\alpha, \beta, \lambda) = K_1 K_2 \prod_{i=1}^n \alpha \lambda (1 + \lambda x_i)^{-(\alpha+1)} (1 + \lambda x_i)^{-\alpha r_i} \quad (8)$$

$$\times \prod_{j=1}^m \beta \lambda (1 + \lambda y_j)^{-(\beta+1)} (1 + \lambda y_j)^{-\beta r'_j} \quad (9)$$

$$= K_1 K_2 \lambda^{n+m} \alpha^n \beta^m \prod_{i=1}^n (1 + \lambda x_i)^{-\alpha(1+r_i)+1} \quad (10)$$

$$\times \prod_{j=1}^m (1 + \lambda y_j)^{-\beta(1+r'_j)-1}. \quad (11)$$

The log likelihood function is

$$\begin{aligned} \ell(\alpha, \beta, \lambda) &= \log K_1 + \log K_2 + (n + m) \log \lambda + n \log \alpha + m \log \beta \\ &\quad - \sum_{i=1}^n [\alpha (1 + r_i) + 1] \log (1 + \lambda x_i) \\ &\quad - \sum_{j=1}^m [\beta (1 + r'_j) + 1] \log (1 + \lambda y_j). \end{aligned} \quad (12)$$

The MLEs of parameters can be derived as the simultaneous solutions of

$$\frac{\partial \ell(\alpha, \beta, \lambda)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n (1 + r_i) \log (1 + \lambda x_i), \quad (13)$$

$$\frac{\partial \ell(\alpha, \beta, \lambda)}{\partial \beta} = \frac{m}{\beta} - \sum_{j=1}^m (1 + r'_j) \log (1 + \lambda y_j) \quad (14)$$

and

$$\frac{\partial \ell(\alpha, \beta, \lambda)}{\partial \lambda} = \frac{m + n}{\lambda} - \sum_{i=1}^n \frac{\alpha [(1 + r_i) + 1] x_i}{1 + \lambda x_i} - \sum_{j=1}^m \frac{\beta [(1 + r'_j) + 1] y_j}{1 + \lambda y_j}. \quad (15)$$

Rearranging (13)–(15), we can obtain the MLEs of α and β as

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^n (1 + r_i) \log (1 + \lambda x_i)}, \quad (16)$$

$$\hat{\beta} = \frac{m}{\sum_{j=1}^m (1 + r'_j) \log (1 + \lambda y_j)}. \quad (17)$$

The MLE of λ can be obtained as the root of

$$(n + m) \left[\frac{n}{\sum_{i=1}^n (1 + r_i) \log(1 + \lambda x_i)} \sum_{i=1}^n \frac{(1 + r_i) x_i}{1 + \lambda x_i} - \sum_{i=1}^n \frac{x_i}{1 + \lambda x_i} \right] \tag{18}$$

$$- \frac{m}{\sum_{j=1}^m (1 + r'_j) \log(1 + \lambda y_j)} \sum_{j=1}^m \frac{(1 + r'_j) y_j}{1 + \lambda y_j} - \sum_{j=1}^m \frac{y_j}{1 + \lambda y_j} \Big]^{-1} = 0. \tag{19}$$

3.2 Asymptotic Distribution

Here, the asymptotic distribution of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ and the asymptotic distribution of \hat{R} are derived. The latter can be used to derive an asymptotic confidence interval of R and this in turn requires the Fisher information matrix of $\theta = (\alpha, \beta, \lambda)$.

If $X_{1:N}, X_{2:N}, \dots, X_{n:N}$ is a progressively type II censored sample from $GP(\alpha, \lambda)$ with censored scheme (r_1, r_2, \dots, r_n) , then $Z_{1:N}, Z_{2:N}, \dots, Z_{n:N}$ is a progressively type II censored sample from the unit exponential distribution with censored scheme (r_1, r_2, \dots, r_n) , where $Z_{i:N} = \alpha \log(1 + \lambda X_{i:N})$. By [3],

$$E[\log(1 + \lambda X_{i:N})] = \frac{1}{\alpha} E(Z_{i:N}) = \frac{1}{\alpha} \sum_{k=1}^i \frac{1}{N - \left(\sum_{s=0}^{k-1} r_s\right) - k + 1} \tag{20}$$

for $i = 1, 2, \dots, n$. By [9], the pdf of $X_{i:N}$ is

$$f_{X_{i:N}}(x) = c_{i-1} \sum_{k=1}^i a_{k,i} \alpha \lambda (1 + \lambda x_i)^{-(\alpha \gamma_k + 1)}, \tag{21}$$

where

$$\gamma_k = N - k + 1 + \sum_{s=k}^n r_s, \tag{22}$$

$$a_{k,i} = \prod_{\substack{s=1 \\ s \neq k}}^i \frac{1}{\gamma_s - \gamma_i}. \tag{23}$$

The expected Fisher information matrix is

$$I(\theta) = - \begin{bmatrix} E \left(\frac{\partial^2 \ell}{\partial \alpha^2} \right) & E \left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta} \right) & E \left(\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \right) \\ E \left(\frac{\partial^2 \ell}{\partial \beta \partial \alpha} \right) & E \left(\frac{\partial^2 \ell}{\partial \beta^2} \right) & E \left(\frac{\partial^2 \ell}{\partial \beta \partial \lambda} \right) \\ E \left(\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} \right) & E \left(\frac{\partial^2 \ell}{\partial \lambda \partial \beta} \right) & E \left(\frac{\partial^2 \ell}{\partial \lambda^2} \right) \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}. \tag{24}$$

By using

$$\int_0^\infty x^{r-1}(1 + \gamma x)^{-\delta} dx = \gamma^{-r} B(r, \delta - r) \tag{25}$$

for $0 < r < \delta$, where $B(x, y)$ denotes the beta function, we obtain

$$I_{11} = \frac{n}{\alpha^2}, \tag{26}$$

$$I_{12} = I_{21} = 0, \tag{27}$$

$$I_{22} = \frac{m}{\beta^2}, \tag{28}$$

$$\begin{aligned} I_{13} = I_{31} &= \sum_{i=1}^n (1 + r_i) E \left(\frac{X_i}{1 + \lambda X_i} \right) \\ &= \frac{\alpha}{\lambda} \sum_{i=1}^n (1 + r_i) \sum_{k=1}^i a_{k,i} c_{i-1} B(2, \alpha \gamma_k). \end{aligned} \tag{29}$$

Since

$$E \left(\frac{X_i}{1 + \lambda X_i} \right) = \int_0^\infty \frac{x}{1 + \lambda x} c_{i-1} \sum_{k=1}^i a_{k,i} \alpha \lambda (1 + \lambda x)^{-(\alpha \gamma_k + 1)} dx \tag{30}$$

$$= \sum_{k=1}^i \alpha \lambda a_{k,i} c_{i-1} \int_0^\infty x (1 + \lambda x)^{-(\alpha \gamma_k + 2)} dx \tag{31}$$

$$= \sum_{k=1}^i \alpha \lambda a_{k,i} c_{i-1} \lambda^{-2} B(2, \alpha \gamma_k), \tag{32}$$

we obtain

$$I_{23} = I_{32} = \frac{\beta}{\lambda} \sum_{j=1}^m (1 + r'_j) c_{i-1} \sum_{k=1}^i a_{k,i} B(2, \beta \gamma_k), \tag{33}$$

$$I_{33} = \frac{1}{\lambda^2} \left\{ (n + m) - \alpha \sum_{i=1}^n [\alpha (1 + r_i) + 1] c_{i-1} \sum_{k=1}^i a_{k,i} B(3, \alpha \gamma_k) \right. \tag{34}$$

$$\left. - \beta \sum_{j=1}^m [\beta (1 + r'_j) + 1] c_{i-1} \sum_{k=1}^i a_{k,i} B(3, \beta \gamma_k) \right\}. \tag{35}$$

Theorem 1 now follows.

Theorem 1 As $n \rightarrow \infty$ and $m \rightarrow \infty$,

$$[\sqrt{n}(\hat{\alpha} - \alpha), \sqrt{m}(\hat{\beta} - \beta), \sqrt{n}(\hat{\lambda} - \lambda)] \rightarrow N_3(\mathbf{0}, \mathbf{A}^{-1}(\alpha, \beta, \lambda)), \tag{36}$$

where

$$\mathbf{A}(\alpha, \beta, \lambda) = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (37)$$

with

$$a_{11} = \frac{1}{n} I_{11}, \quad (38)$$

$$a_{22} = \frac{1}{m} I_{22}, \quad (39)$$

$$a_{13} = a_{31} = \frac{1}{n} I_{13}, \quad (40)$$

$$a_{23} = a_{32} = \frac{1}{\sqrt{nm}} I_{23}, \quad (41)$$

$$a_{33} = \frac{1}{n} I_{33}. \quad (42)$$

Theorem 2 As $n \rightarrow \infty$, $m \rightarrow \infty$ and $\frac{n}{m} \rightarrow p$,

$$\sqrt{n}(\widehat{R} - R) \rightarrow N(0, B), \quad (43)$$

where

$$B = \frac{1}{c(\alpha + \beta)^4} \left[\beta^2 (a_{22}a_{33} - a_{23}^2) - 2\alpha\beta\sqrt{p}a_{23}a_{31} + \alpha^2 p (a_{11}a_{33} - a_{13}^2) \right], \quad (44)$$

and $c = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$.

4 Estimation of R When λ is Known

Without loss of generality we can assume that $\lambda = 1$. It is clear that

$$R = P(Y < X) = \frac{\beta}{\alpha + \beta}. \quad (45)$$

4.1 Maximum Likelihood Estimator of R

Let $X_{1:N}, X_{2:N}, \dots, X_{n:N}$ be a progressively type II censored sample from $GP(\alpha, 1)$ and $Y_{1:M}, Y_{2:M}, \dots, Y_{m:M}$ be a progressively type II censored sample from $GP(\beta, 1)$. The MLEs of α and β are

$$\widehat{\alpha} = \frac{n}{\sum_{i=1}^n (1 + r_i) \log(1 + \lambda x_i)}, \quad (46)$$

$$\widehat{\beta} = \frac{m}{\sum_{j=1}^m (1 + r'_j) \log(1 + \lambda y_j)}. \quad (47)$$

It is easy to show that $(1 + r_i) \log(1 + \lambda x_i)$ is an exponential random variable with mean $\frac{1+r_i}{\alpha}$. So,

$$\sum_{i=1}^n (1 + r_i) \log(1 + \lambda X_i) \sim \Gamma\left(n, \frac{\alpha}{1 + r_i}\right), \quad (48)$$

and

$$\sum_{j=1}^m (1 + r'_j) \log(1 + \lambda Y_j) \sim \Gamma\left(n, \frac{\beta}{1 + r'_j}\right). \quad (49)$$

Also we can write

$$2\alpha \sum_{i=1}^n (1 + r_i) \log(1 + \lambda X_i) \sim \chi^2(2n), \quad (50)$$

$$2\beta \sum_{j=1}^m (1 + r'_j) \log(1 + \lambda Y_j) \sim \chi^2(2m). \quad (51)$$

So,

$$\widehat{R} = \frac{1}{\frac{\alpha}{\beta} F + 1} \quad (52)$$

with

$$\frac{R}{1 - R} \times \frac{1 - \widehat{R}}{\widehat{R}} \sim F, \quad (53)$$

an F random variable with $2m$ and $2n$ degrees of freedom.

4.2 UMVUE of R

In Sect. 4.1, we saw that $T^x = \log(1 + \lambda X)$ is an exponential random variable with mean $\frac{1}{\alpha}$. The joint pdf of $(T_{1:N}^x, T_{2:N}^x, \dots, T_{n:N}^x)$ is

$$f_{T_{1:N}^x, \dots, T_{n:N}^x}(t_{1:N}^x, \dots, t_{n:N}^x) = K_1 \alpha^n \exp\left(-\alpha \left(\sum_{i=1}^n (r_i + 1) t_{i:N}^x\right)\right) \quad (54)$$

for $0 < t_1^x < \dots < t_n^x < \infty$, where K_1 is as defined in (6). Consider the following transformation

$$Z_1^x = NT_{1:N}^x, \tag{55}$$

$$Z_i^x = (n - r_1 - \dots - r_{n-1} - i + 1) (T_{i:N}^x - T_{i-1:N}^x) \tag{56}$$

for $i = 1, 2, \dots, n$. [25] have shown that Z_i s are independent exponential random variables with mean $\frac{1}{\alpha}$. Furthermore,

$$\sum_{i=1}^n (r_i + 1) T_{i:n}^x = \sum_{i=1}^n Z_i^x \tag{57}$$

and (54) shows that the left hand side is a complete sufficient statistic for α .

Theorem 3 *The uniformly minimum variance unbiased estimator of R is*

$$\widehat{R} = \begin{cases} 1 - \int_0^1 (n - 1) (1 - u_1)^{n-2} (1 - u_1 W)^{m-1} du_1, & \text{if } W \leq 1, \\ 1 - \int_0^{1/W} (n - 1) (1 - u_1)^{n-2} (1 - u_1 W)^{m-1} du_1, & \text{if } W > 1. \end{cases} \tag{58}$$

Proof An unbiased estimator for R is

$$\Phi(T_1, T_2) = \begin{cases} 1, & \text{if } MT_{1:M}^y > NT_{2:N}^x, \\ 0, & \text{if } MT_{1:M}^y < NT_{2:N}^x. \end{cases} \tag{59}$$

Since $\widehat{R} = E(\Phi(T_1, T_2) | S) = P(MT_{j:n}^y < NT_{i:n}^x | S)$,

$$S = (S_1, S_2) = \left(\sum_{i=1}^n (r_i + 1) T_{i:n}^x, \sum_{j=1}^m (r'_j + 1) T_{j:n}^y \right) \tag{60}$$

is a jointly completely sufficient statistic. Define

$$U_1 = \frac{Z_1^x}{\sum_{i=1}^n Z_i^x}, \tag{61}$$

$$U_2 = \frac{Z_1^y}{\sum_{j=1}^m Z_j^y}, \tag{62}$$

$$W = \frac{\sum_{i=1}^n Z_i^x}{\sum_{j=1}^m Z_j^y}. \tag{63}$$

So,

$$\widehat{R} = P(U_2 < U_1 W | S). \tag{64}$$

It can be shown that U_1 and U_2 are independent beta random variables with parameters $(1, n - 1)$ and $(1, m - 1)$, respectively, see [5]. Now

$$f_{U_1, U_2 | S}(u_1, u_2 | S) = (n - 1)(m - 1)(1 - u_1)^{n-2}(1 - u_2)^{m-2} \tag{65}$$

for $0 < u_1, u_2 < 1$. The theorem follows from (64) and (65). □

4.3 Bayes Estimation of R

Here, we assume that α and β are random variables. In Bayesian approach, the loss function is important and most studies use the squared error loss (SEL) function, see [7] and [6]. We suppose α and β have the gamma priors, i.e., $\pi(\alpha) \sim \Gamma(a_1, b_1)$ and $\pi(\beta) \sim \Gamma(a_2, b_2)$, where a_1, a_2, b_1 and b_2 are positive constants. Then the posterior pdfs of α and β are

$$\pi(\alpha | \text{data}) \sim \Gamma\left(a_1 + nb_1 + \sum_{i=1}^n (1 + r_i) \log(1 + \lambda x_i)\right), \tag{66}$$

and

$$\pi(\beta | \text{data}) \sim \Gamma\left(a_2 + mb_2 + \sum_{j=1}^m (1 + r'_j) \log(1 + \lambda y_j)\right). \tag{67}$$

Using (66) and (67), the posterior pdf of R becomes

$$f_R(r) = k \frac{r^{a_2+m-1} (1-r)^{a_1+n-1}}{\{r(b_2 + S_2) + (1-r)(b_1 + S_1)\}^{(n+m+a_1+a_2)}} \tag{68}$$

for $0 < r < 1$, where

$$k = \frac{\Gamma(n + m + a_1 + a_2)}{\Gamma(n + a_1) \Gamma(m + a_2)} (b_1 + S_1)^{a_1+n} (b_2 + S_2)^{a_2+m}. \tag{69}$$

The mean or median of (68) cannot be obtained in closed form. The mode of (68) is the root of

$$\frac{\partial f_R(r)}{\partial r} = \frac{r^{c-1} (1-r)^{c'-1} \{c'd + 2r^2(d' - d) + r(2d_1 - 2d' - c'd - cd')\}}{\{d(1-r) + d'r\}^{c+c'+3}} = 0, \tag{70}$$

where $c = a_1 + n - 1$, $c' = a_2 + m - 1$, $d = b_1 + S_1$ and $d' = b_2 + S_2$. The mode of the posterior is a Bayes estimator of R with respect to the 0-1 loss function. Using [14]’s approximation, a Bayes estimator for R under the SEL is

$$\widehat{R}_{Lindley} = \widetilde{R} \left\{ 1 + \frac{\widetilde{\alpha} \widetilde{R}^2 [\widetilde{\alpha} (n + a_1 - 1) - \widetilde{\beta} (m + a_2 - 2)]}{\widetilde{\beta}^2 (n + a_1 - 1) (m + a_2 - 1)} \right\}, \tag{71}$$

where $\widetilde{R} = \frac{\widetilde{\beta}}{\widetilde{\alpha} + \widetilde{\beta}}$, $\widetilde{\alpha} = \frac{n + a_1 - 1}{b_1 + S_1}$, and $\widetilde{\beta} = \frac{m + a_2 - 2}{b_2 + S_2}$.

4.4 Credible Interval and Confidence Interval of R

Based on asymptotic properties of R , we can deduce asymptotic confidence intervals for R . Percentile bootstrap and bootstrap- t confidence intervals can also be derived, see Sect. 6.

The credible set for R is given by

$$P(R \leq L | \text{data}) = \frac{\gamma}{2} \quad \text{and} \quad P(R \leq U | \text{data}) = 1 - \frac{\gamma}{2}, \tag{72}$$

where L and U are the lower and upper bounds. The following algorithm can be used to calculate the credible interval

- Find the mode of the posterior pdf by solving (70), say r^m .
- generate k random numbers from (68), using the acceptance rejection principle, say r_1, r_2, \dots, r_k , see [8].
- sort r_1, r_2, \dots, r_k and compute the $\frac{\gamma}{2}$ th and $(1 - \frac{\gamma}{2})$ th percentiles.

We refer the readers to [23] for other algorithms.

5 Estimation of R in the General Case

Let $X \sim GP(\alpha, \lambda_1)$ and $Y \sim GP(\beta, \lambda_2)$ be independent random variables. Then

$$R = 1 - \alpha \left(\frac{\lambda_2}{\lambda_1} \right)^{-\beta} \int_0^\infty (1 + t)^{-(\alpha+1)} \left(\frac{\lambda_1}{\lambda_2} + t \right)^{-\beta} dt. \tag{73}$$

By using

$$\int_0^\infty x^{v-1} (\eta + x)^{-\mu} (x + \gamma)^{-Q} dx = \eta^{-\mu} \gamma^{v-Q} B(v, \mu - v + Q) {}_2F_1\left(\mu, v; \mu + Q; 1 - \frac{\gamma}{\eta}\right) \tag{74}$$

for $\nu > 0$ and $\mu > \nu - Q$, where ${}_2F_1$ denotes the Gauss hypergeometric function, we can express (73) as

$$R = 1 - \frac{\alpha}{\alpha + \beta} \frac{\lambda_1}{\lambda_2} {}_2F_1 \left(\alpha + 1, 1; \alpha + \beta + 1; 1 - \frac{\lambda_1}{\lambda_2} \right). \tag{75}$$

5.1 Maximum Likelihood Estimator of R

Let $X_{1:N}, X_{2:N}, \dots, X_{n:N}$ and $Y_{1:M}, Y_{2:M}, \dots, Y_{m:M}$ be progressively type II censored samples from $GP(\alpha, \lambda_1)$ and $GP(\beta, \lambda_2)$, respectively. The likelihood function of the parameters is

$$L(\alpha, \beta, \lambda_1, \lambda_2) = K_1 K_2 \lambda_1^n \lambda_2^m \alpha^n \beta^m \prod_{i=1}^n (1 + \lambda_1 x_i)^{-(\alpha(1+r_i)+1)} \prod_{j=1}^m (1 + \lambda_2 y_j)^{-(\beta(1+r'_j)+1)}. \tag{76}$$

The log likelihood function is

$$\ell(\alpha, \beta, \lambda) = \log K_1 + \log K_2 + n \log \lambda_1 + m \log \lambda_2 + n \log \alpha + m \log \beta \tag{77}$$

$$- \sum_{i=1}^n (\alpha(1+r_i) + 1) \log(1 + \lambda_1 x_i) - \sum_{j=1}^m (\beta(1+r'_j) + 1) \log(1 + \lambda_2 y_j). \tag{78}$$

The MLEs of α and β are

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^n (1+r_i) \log(1 + \lambda_1 x_i)}, \tag{79}$$

$$\hat{\beta} = \frac{m}{\sum_{j=1}^m (1+r'_j) \log(1 + \lambda_2 y_j)}. \tag{80}$$

The MLEs of λ_1 and λ_2 are the solutions of

$$\frac{n}{\lambda_1} - \frac{n}{\sum_{i=1}^n (1+r_i) \log(1 + \lambda_1 x_i)} \sum_{i=1}^n \frac{(1+r_i) x_i}{1 + \lambda_1 x_i} - \sum_{i=1}^n \frac{x_i}{1 + \lambda_1 x_i} = 0, \tag{81}$$

$$\frac{m}{\lambda_2} - \frac{m}{\sum_{i=1}^n (1 + r'_i) \log(1 + \lambda_2 y_i)} \sum_{i=1}^m \frac{(1 + r'_i) y_i}{1 + \lambda_2 y_i} - \sum_{i=1}^m \frac{y_i}{1 + \lambda_2 y_i} = 0. \tag{82}$$

By the invariant property, the MLE of R is

$$\widehat{R} = 1 - \frac{\widehat{\alpha} \widehat{\lambda}_1}{\widehat{\alpha} + \widehat{\beta} \widehat{\lambda}_2} {}_2F_1 \left(\widehat{\alpha} + 1, 1; \widehat{\alpha} + \widehat{\beta} + 1; 1 - \frac{\widehat{\lambda}_1}{\widehat{\lambda}_2} \right). \tag{83}$$

5.2 Asymptotic Distribution

Here, we compute the asymptotic distribution of $\theta = (\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}_1, \widehat{\lambda}_2)$. This requires the expected Fisher information matrix, i.e.,

$$I(\theta) = - \begin{bmatrix} E \left(\frac{\partial^2 \ell}{\partial \alpha^2} \right) & E \left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta} \right) & E \left(\frac{\partial^2 \ell}{\partial \alpha \partial \lambda_1} \right) & E \left(\frac{\partial^2 \ell}{\partial \alpha \partial \lambda_2} \right) \\ E \left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta} \right) & E \left(\frac{\partial^2 \ell}{\partial \beta^2} \right) & E \left(\frac{\partial^2 \ell}{\partial \beta \partial \lambda_1} \right) & E \left(\frac{\partial^2 \ell}{\partial \beta \partial \lambda_2} \right) \\ E \left(\frac{\partial^2 \ell}{\partial \lambda_1 \partial \alpha} \right) & E \left(\frac{\partial^2 \ell}{\partial \lambda_1 \partial \beta} \right) & E \left(\frac{\partial^2 \ell}{\partial \lambda_1^2} \right) & E \left(\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} \right) \\ E \left(\frac{\partial^2 \ell}{\partial \lambda_2 \partial \alpha} \right) & E \left(\frac{\partial^2 \ell}{\partial \lambda_2 \partial \beta} \right) & E \left(\frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_1} \right) & E \left(\frac{\partial^2 \ell}{\partial \lambda_2^2} \right) \end{bmatrix} \\ = \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}. \tag{84}$$

By using

$$\int_0^\infty x^{r-1} (1 + \gamma x)^{-\delta} dx = \gamma^{-r} B(r, \delta - r) \tag{85}$$

for $0 < r < \delta$, we obtain

$$I_{11} = \frac{n}{\alpha^2}, \tag{86}$$

$$I_{12} = I_{21} = 0, \tag{87}$$

$$I_{22} = \frac{m}{\beta^2}, \tag{88}$$

$$I_{13} = I_{31} = \frac{\alpha}{\lambda_1} \sum_{i=1}^n (1 + r_i) \sum_{k=1}^i a_{k,i} c_{i-1} B(2, \alpha \gamma_k), \tag{89}$$

$$I_{14} = I_{41} = 0, \tag{90}$$

$$I_{22} = \frac{m}{\beta^2}, \tag{91}$$

$$I_{23} = I_{32} = 0, \tag{92}$$

$$I_{24} = I_{42} = \frac{\beta}{\lambda_2} \sum_{j=1}^m (1 + r'_j) c_{i-1} \sum_{k=1}^i a_{k,i} B(2, \beta\gamma_k), \tag{93}$$

$$I_{33} = \frac{1}{\lambda_1^2} \left[n - \alpha \sum_{i=1}^n (\alpha (1 + r_i) + 1) c_{i-1} \sum_{k=1}^i a_{k,i} B(3, \alpha\gamma_k) \right], \tag{94}$$

$$I_{44} = \frac{1}{\lambda_2^2} \left[m - \beta \sum_{j=1}^m (\beta (1 + r'_j) + 1) c_{i-1} \sum_{k=1}^i a_{k,i} B(3, \beta\gamma_k) \right], \tag{95}$$

$$I_{34} = I_{43} = 0. \tag{96}$$

Based on the Fisher information matrix, the percentile bootstrap and bootstrap-*t* confidence intervals can be derived, see Sect. 6.

5.3 Bayes Estimation of *R*

Here, we consider gamma priors for α, β, λ_1 and λ_2 . We take $\pi(\alpha) \sim \Gamma(a_1, b_1), \pi(\beta) \sim \Gamma(a_2, b_2), \pi(\lambda_1) \sim \Gamma(a_3, b_3)$ and $\pi(\lambda_2) \sim \Gamma(a_4, b_4)$, where a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are all positive constants. Some routine algebra shows that the posterior pdfs of α and β are

$$\pi(\alpha|\text{data}) \sim \Gamma\left(a_1 + n, b_1 + \sum_{i=1}^n (1 + r_i) \log(1 + \lambda_1 x_i)\right), \tag{97}$$

and

$$\pi(\beta|\text{data}) \sim \Gamma\left(a_2 + mb_2 + \sum_{j=1}^m (1 + r'_j) \log(1 + \lambda_2 y_j)\right). \tag{98}$$

For λ_1 and λ_2 , we see

$$\pi(\lambda_1|\text{data}) \sim \lambda_1^{n+a_3-1} e^{-\lambda_1 b_3} \left[\prod_{i=1}^n (1 + \lambda_1 x_i)^{-(\alpha+1)} (1 + \lambda_1 x_i)^{-\alpha r_i} \right], \tag{99}$$

and

Table 1 Censoring schemes

	(n, N)	Censoring scheme
r_1	(10, 30)	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
r_2	(10, 30)	(20, 0, 0, 0, 0, 0, 0, 0, 0, 0)
r_3	(10, 30)	(2, 2, 2, 2, 2, 2, 2, 2, 2, 2)

Table 2 Information in the case of constant mean

	a_i	b_i	$E(\alpha)$	$Var(\alpha)$
Least informative	1	2	0.5	0.25
Informative	5	10	0.5	0.05
Most informative	10	20	0.5	0.005

Table 3 Estimation of parameters when λ is unknown

(α, β, λ)	C.S.	P.E.		I.E.					
		Bias	MSE	Boot-p	C.P	Boot-t	C.P.	C.I.	C.P.
(5,5,1)	(r_1, r_1)	0.0250	0.0108	0.392	0.939	0.398	0.955	0.380	0.927
	(r_1, r_2)	0.0201	0.0060	0.390	0.946	0.395	0.950	0.375	0.925
	(r_1, r_3)	0.0172	0.0890	0.376	0.951	0.387	0.962	0.364	0.928
	(r_2, r_2)	0.0071	0.0053	0.373	0.951	0.379	0.960	0.351	0.949
	(r_2, r_3)	0.0064	0.0061	0.364	0.941	0.369	0.955	0.351	0.936
	(r_3, r_3)	0.0016	0.0058	0.392	0.936	0.394	0.952	0.380	0.933
(3,3,2)	(r_1, r_1)	0.0261	0.0110	0.346	0.951	0.350	0.952	0.330	0.939
	(r_1, r_2)	0.0193	0.0091	0.330	0.945	0.340	0.953	0.321	0.949
	(r_1, r_3)	0.0163	0.0070	0.324	0.949	0.340	0.956	0.310	0.945
	(r_2, r_2)	0.0088	0.0058	0.318	0.941	0.327	0.953	0.302	0.940
	(r_2, r_3)	0.0070	0.0050	0.311	0.947	0.322	0.950	0.294	0.940
(7,7,2/3)	(r_1, r_1)	0.0183	0.0103	0.340	0.951	0.348	0.955	0.326	0.942
	(r_1, r_2)	0.0151	0.0106	0.276	0.952	0.288	0.951	0.261	0.940
	(r_1, r_3)	0.0117	0.0066	0.256	0.945	0.263	0.956	0.247	0.953
	(r_2, r_2)	0.0092	0.0041	0.240	0.948	0.250	0.952	0.229	0.950
	(r_2, r_3)	0.0089	0.0046	0.221	0.950	0.233	0.960	0.204	0.943
	(r_3, r_3)	0.0070	0.0063	0.207	0.947	0.215	0.960	0.202	0.938
	(r_3, r_3)	0.0039	0.0090	0.260	0.944	0.271	0.949	0.239	0.938

$$\pi(\lambda_2 | \text{data}) \sim \lambda_2^{m+a_4-1} e^{-\lambda_2 b_4} \left[\prod_{j=1}^m (1 + \lambda_2 y_j)^{-(\beta+1)} (1 + \lambda_2 y_j)^{-\beta r'_j} \right]. \quad (100)$$

The Metropolis-Hastings simulation method can be used to obtain the posterior pdfs of λ_1 and λ_2 . We refer the readers to [23] for other algorithms.

Table 4 Bayes estimation of parameters when $(\alpha = 3, \beta = 3, \lambda = 1)$

Information	C.S.	E.R.	H.P.D.	C.P.
Least informative	(r_1, r_1)	0.0119	0.355	0.948
	(r_1, r_2)	0.0091	0.353	0.941
	(r_1, r_3)	0.0129	0.348	0.940
	(r_2, r_2)	0.0083	0.340	0.951
	(r_2, r_3)	0.0085	0.335	0.954
	(r_3, r_3)	0.0090	0.352	0.933
Informative	(r_1, r_1)	0.0102	0.351	0.947
	(r_1, r_2)	0.0086	0.346	0.940
	(r_1, r_3)	0.0120	0.343	0.939
	(r_2, r_2)	0.0081	0.334	0.954
	(r_2, r_3)	0.0081	0.331	0.942
	(r_3, r_3)	0.0088	0.344	0.930
Most informative	(r_1, r_1)	0.0093	0.334	0.951
	(r_1, r_2)	0.0076	0.332	0.948
	(r_1, r_3)	0.0114	0.338	0.939
	(r_2, r_2)	0.0075	0.330	0.950
	(r_2, r_3)	0.0080	0.319	0.940
	(r_3, r_3)	0.0077	0.334	0.930

Table 5 Bayes estimation of parameters when $(\alpha = 5, \beta = 5, \lambda = 1)$

Information	C.S.	E.R.	H.P.D.	C.P.
Least informative	(r_1, r_1)	0.0132	0.319	0.946
	(r_1, r_2)	0.0115	0.316	0.949
	(r_1, r_3)	0.0093	0.304	0.958
	(r_2, r_2)	0.0088	0.290	0.952
	(r_2, r_3)	0.0077	0.279	0.955
	(r_3, r_3)	0.0126	0.320	0.948
Informative	(r_1, r_1)	0.0120	0.317	0.945
	(r_1, r_2)	0.0110	0.310	0.949
	(r_1, r_3)	0.0080	0.301	0.954
	(r_2, r_2)	0.0081	0.279	0.948
	(r_2, r_3)	0.0068	0.280	0.951
	(r_3, r_3)	0.124	0.310	0.952
Most informative	(r_1, r_1)	0.0116	0.302	0.944
	(r_1, r_2)	0.0103	0.301	0.944
	(r_1, r_3)	0.0076	0.292	0.951
	(r_2, r_2)	0.0072	0.280	0.949
	(r_2, r_3)	0.0060	0.277	0.946
	(r_3, r_3)	0.0110	0.309	0.940

Table 6 Bayes estimation of parameters when $(\alpha = 7, \beta = 7, \lambda = 1)$

Information	C.S.	E.R.	H.P.D.	C.P.
Least informative	(r_1, r_1)	0.0133	0.254	0.952
	(r_1, r_2)	0.109	0.240	0.954
	(r_1, r_3)	0.0061	0.220	0.950
	(r_2, r_2)	0.0071	0.211	0.948
	(r_2, r_3)	0.0110	0.0203	0.942
	(r_3, r_3)	0.0130	0.239	0.951
Informative	(r_1, r_1)	0.0129	0.250	0.939
	(r_1, r_2)	0.0100	0.240	0.951
	(r_1, r_3)	0.0054	0.209	0.954
	(r_2, r_2)	0.0060	0.202	0.959
	(r_2, r_3)	0.0099	0.177	0.943
	(r_3, r_3)	0.0127	0.244	0.950
Most informative	(r_1, r_1)	0.0118	0.233	0.941
	(r_1, r_2)	0.0804	0.230	0.942
	(r_1, r_3)	0.0053	0.213	0.946
	(r_2, r_2)	0.0055	0.197	0.949
	(r_2, r_3)	0.0084	0.184	0.949
	(r_3, r_3)	0.0116	0.240	0.945

Table 7 Estimation of parameters in the exponential ($\mu = 0, \xi = 0, \sigma = 1$) and Pareto ($\frac{1}{\xi} = 2, \frac{\sigma}{\xi} = 4$) distributions

(μ, ξ, σ)	C.S.	Bias	MSE	ER	E.C.I	C.P.
Exponential	(r_1, r_1)	0.0018	0.0123	0.0109	0.399	0.948
	(r_1, r_2)	0.0015	0.0118	0.0110	0.399	0.949
	(r_1, r_3)	0.0010	0.0117	0.0098	0.400	0.955
	(r_2, r_2)	0.0021	0.0121	0.0108	0.399	0.943
	(r_2, r_3)	0.0029	0.0112	0.0100	0.403	0.957
	(r_3, r_3)	0.0037	0.0122	0.0109	0.401	0.951
Pareto	(r_1, r_1)	0.0030	0.0121	0.0112	0.401	0.947
	(r_1, r_2)	0.0024	0.0117	0.0113	0.402	0.947
	(r_1, r_3)	0.0011	0.0114	0.0094	0.402	0.947
	(r_2, r_2)	0.0071	0.0119	0.0103	0.404	0.953
	(r_2, r_3)	0.0025	0.0120	0.0101	0.401	0.945
	(r_3, r_3)	0.0027	0.0120	0.0107	0.401	0.943

6 Simulation Study

Here, a Monte Carlo simulation study comparing the performances of all the estimators in the preceding sections is presented. We use the three censoring schemes in Table 1, those used in [26].

The estimators are compared in terms of bias, MSE, coverage probability, coverage length and estimated risk. The priors for $(\alpha, \beta, \lambda_1, \lambda_2)$ were taken to be one of least informative, informative or most informative, see Table 2 (Tables 3, 4, 5, 6, 7).

7 Conclusions

In this paper, we have addressed the problem of estimating $R = P(Y < X)$ for the generalized Pareto distribution based on progressively censored samples. We have considered the cases when the common scale parameter is known or unknown and when the scale parameters are different. When the common scale parameter is unknown, the maximum likelihood estimator works quite well. We have used the asymptotic distribution of the maximum likelihood estimator to construct confidence intervals. These intervals work well even for small sample sizes. Based on simulation results, we recommend to use the parametric bootstrap percentile method when the sample size is very small.

We have observed that the Bayes estimator has the smallest estimated risk. The scheme (r_1, r_3) has smaller MSE compared to the other schemes. Also in most cases the confidence interval for the (r_1, r_3) scheme has the shortest length. The estimated risk decreases as the priors become more “informative”.

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