Generalization of the Fourier transform-based method for calculating the response of a periodic railway track subject to a moving harmonic load

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Abstract A Fourier transform-based method has been developed for calculating the response of a railway track as an infinitely long uniform periodic structure subject to moving or stationary harmonic loads. The track may become a non-uniform periodic structure by, for example, rail dampers which are installed between sleepers to control rolling noise and roughness growth. The period of the structure may become greater than the sleeper spacing. For those new situations, the current version of the method cannot be directly applied; it must be generalized and this is the aim of this paper. Generalization is performed by applying periodic conditions to each type of support and summarizing contributions from all types of support. Responses of the rail, sleeper, and damper are all formulated as an inverse Fourier transform from wavenumber domain to spatial domain. The generalized method is applied to investigate dynamics of a typical track with rail dampers of particular design. It is found that the rail dampers can significantly suppress the pinned–pinned vibration of the original track, widen the stop bands and increase vibration decay rate along the rail. However, it is also found that a new pinned–pinned mode is created by the dampers and between about 450 and 1,300 Hz dampers vibrate stronger than the rail, making noise radiation from the dampers a potential issue. These concerns must be fully investigated in the future. The formulae presented in this paper provide a powerful tool to do that.

Keywords Rail damper · Track dynamics · Periodic structure

1 Introduction

A major concern for the railway industry is the growth of rail roughness, the formation of rail corrugation and the generation of wheel/rail noise. It is now well known that these unwanted phenomena are generated from dynamical interactions between moving wheels and rails. Due to operations of high-speed trains and roughness of short wavelengths on the wheel/rail rolling surfaces, wheel/rail interactions are of high frequencies up to several thousand Hertz, and involve complex vibrational wave propagations and resonances in the track structure. For a track with sleepers, a major track type used worldwide, it is normally modeled to be a periodic structure consisting of an infinitely long uniform main structure (i.e., the rails) attached at a given spacing (i.e., the sleeper spacing) by an infinite number of supports (i.e., the railpad/sleeper/ballast).

One of the important aspects in dealing with dynamics of a periodic structure such as a railway track is to calculate the free vibration characteristics, i.e., propagation constants and associated modes. These free vibration characteristics are utilized in a number of methods dealing with forced vibration of the periodic structure, and also helpful to develop understandings of mechanisms involved. In [1], Mead gives an extensive review on the subject, with focus mainly on work contributed from Southampton before 1996. Two methods are often used to calculate propagation constants: the receptance (or Green's function) method and the transfer matrix method. The first method is normally applied in an analytical manner and therefore restricted to simple main structures and supports [2]. With the finite element method (FEM) involved, the transfer matrix method can account for more complex structures but leads to heavy computations because of the necessary longitudinal discretization. In addition to that the eigenvalue

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equation may become ill-conditioned due to the presence of evanescent waves with high vibration decay rates, as noted by Gry and Gontier [3, 4] when dealing with vibrations of a rail on periodic supports at acoustic frequencies. One of the measures to overcome these shortcomings is given in these two references in which displacement variation of the rail in the longitudinal direction is synthesized using some sort of modes.

The periodic structures dealt with by Gry and Gontier are uniform, as defined by some researchers, i.e., no attachment is presented between the sleepers. Brown and Byrne propose a method to deal with the so-called non-uniform periodic structure [5]. The basis of the method presented in that paper, sub-structuring using wave shape coordinate reduction, is to divide the non-uniform periodic structure into a number of uniform substructures along the periodic axis and to reduce the number of coordinates in each substructure.

Another important aspect in dealing with track dynamics is to calculate the response of a railway track as a periodic structure subject to a moving or stationary harmonic load. Results from such calculations can not only further reveal the dynamical characteristics of the track, but can also provide a basis, either in the time-domain or in the frequency-domain, for dealing with wheel/rail interactions. Different approaches have been developed to analyze the response of a track, as a uniform periodic structure, to fast moving harmonic loads of high frequency.

Responses of a discretely supported rail to moving loads can be modeled in the time-domain (e.g., [6, 7]) by solving differential equations as an initial-value problem. Time-domain approaches require the track to be truncated into a finite length. To minimize the effect of wave reflections from the truncations and to be able to account for high frequency vibration, the track model must be sufficiently long. In fact, when dealing with interactions between a high-speed train and a railway track, the entire train should be taken into account. This is not only because inter-vehicle couplings in a high-speed train are much stronger [7], therefore having a significant effect on dynamics of both the train and the track, but also due to the long-distance propagating vibration waves induced in the track by the high speed. Therefore, the track model must be much longer than the train, up to 450 m, and the rail must be modeled using either the FEM [6] or the modal superposition method [7]. This would generate a large number of differential equations of time-varying coefficients. It is time-consuming to solve these equations, due not only to the large number of equations, but also to the very small time-steps required for high frequencies. For a periodic excitation, extra time is also required to allow the steady-state solution to achieve.

Computational efficiency and accuracy can be significantly increased using methods based on the periodic structure theory. Vibration of an infinite and periodically supported (by springs) beam subject to a moving harmonic load has been investigated in [8]. In this study, the author employs the Euler beam theory, which is only valid for frequencies below 250 Hz, on a single segment, and combines the periodic conditions to produce the steady-state response of the periodically supported beam. The periodic conditions are also used in [9] to investigate the steady-state responses of different periodic structures, including a railway track, to a moving load. Nordborg [10] also used a periodically supported Euler beam to represent a railway track subject to a moving load. However, the varying stiffness of the track is calculated using a quasi-static approach. This quasi-static approach has also been used by other researchers, e.g., [11].

Another method which has been used to deal with forced vibrations of, and wheel interactions with, a track as an infinitely long periodic structure is the Green function method [12–14], which is based on the Duhamel integration and working in the time-domain. The Green function of the track is defined as the response of the track at a location due to a unit impulsive force (a Dirac delta function) applied at the same or another location. To account for multiple and moving loads, Green functions for a large number of different locations (to simulate a wheel travels over a sleeper bay) are required. Green functions are normally computed as the inverse Fourier transform of the corresponding frequency response function, which is the response of the track to a unit harmonic load at different frequencies. Therefore, it is essential to be able to calculate the response of a periodic railway track subject to a harmonic load of high frequency.

A Fourier transform-based method is developed in [15] by the current author. With this method one can efficiently calculate track vibrations excited by a harmonic load of high frequency and moving at high speed. In the method, the rail can be described using either a multiple-beam model as done in [16, 17] or a two-and-half dimensional (2.5D) finite element model [18], and the supports may have arbitrary degrees of freedom, either translational or rotational. The response of the track is expressed as Fourier transform from the wavenumber (in the track direction) domain to the spatial domain. It is shown in [15] that the quasi-static approach mentioned above is not capable of dealing with vibrations of the track around the pinned-pinned frequency (around 1,000 Hz for a modern ballasted track). Eigen-value equations for determining propagation constants of the track as a periodic structure is established straightforwardly from equations presented in [15], as explored in [19]. Based on [15] and assuming railhead roughness to be periodic in the track direction and the

period is equal to the length of one or more sleeper bays, the so-called Fourier-series approach is developed in [20]. According to the approach, wheel/rail forces generated from railhead roughness as well as from the parametric excitation of the moving wheels can be calculated by solving a set of linear algebraic equations.

In all the work mentioned above, the track (rail, sleeper, ballast, etc.) is assumed to rest on a rigid foundation, without taking into account of the elasticity of the ground. For track dynamics of high frequency (e.g., higher than 250 Hz), such a simplification is reasonable. However, when load frequency is low, as in the case of rail traffic-induced ground vibration, interactions between the track and the ground, and/or a tunnel may become significant. The track/tunnel/ground system may be simplified as a structure which is periodic in the track direction. In recent years, the Floquet transform has been employed to analyze the response of track/tunnel/ground systems as a periodic structure to moving harmonic loads [21–23]. With this transform, only a single segment has to be considered. The segment is modeled in a hybrid manner, i.e., using the 3D FEM for the track and tunnel structure, and the boundary element method for the surrounding ground.

Now it is well understood that the pinned–pinned vibration of the rail has an important impact on noise radiation and roughness growth. Rail vibration dampers (or rail vibration absorbers) [24, 25] are thus designed and installed between sleepers in order to suppress the pinned–pinned vibration. The rail damper proposed in [24] are tuned, damped mass–spring absorber systems, with either a single mass or two masses enclosed in an elastomeric material. These rail dampers have been installed at several sites in Europe, with some variations in design. To evaluate the effect of rail dampers on wheel–rail interaction forces and rail roughness growth [26], based on [6], uses the FEM to model the track. The track is truncated to include 50 sleeper bays (30 m in length) and the rail is modeled using four Timoshenko beam elements per sleeper bay. With such a track model being used, differential equations have to be solved in the time-domain at a low computational efficiency and possibly at a low accuracy. This is an obvious disadvantage of the FEM modeling approach, since the prediction of roughness growth requires a very large number of repetitious wheel/rail interaction calculations. Wu [27] outlines guidelines for designing rail dampers. When the damper is not short compared to the sleeper spacing, bending of the beam in a damper may play a role. However, it is demonstrated in [28] that it is the rigid body motion of the beam of the absorber that leads to energy dissipation of rail vibration, whereas the bending deformation of the beam is a minor factor. Therefore, a simple mass–spring model in which the mass is allowed to vibrate

translationally and rotationally, instead of an advanced beam–spring model, is enough to represent the rail damper.

With the addition of the rail dampers, the original uniform periodic track structure becomes a non-uniform one, and the version of the Fourier transform-based method presented in [15] cannot be applied directly to evaluate its response to a fast moving harmonic load of high frequency. To fully explore the usefulness of the method, generalization of the method should be performed. This is the aim of the current paper. In Sect. 2, the problem to be solved is described and the associated differential equations are established. Solutions to the differential equations are presented in Sect. 3 and in the Appendix. Using the generalized method, the effect of the rail damper of a particular design is evaluated in Sect. 4. And finally, in Sect. 5, the paper is concluded.

2 Differential equation of a railway track as a non-uniform periodic structure

2.1 Differential equation of the rail

The 2.5D finite element presentation is employed here to describe the vibration of the rail. A unique discretization is made for every cross-section of the rail, and nodal lines parallel to the x -axis along the track direction are formed by nodes in a cross-section and corresponding counterparts in other cross-sections. The displacements of the n nodes on the x cross-section are denoted by a vector having $3n$ elements,

$$\mathbf{q}(x, t) = (u_1, v_1, w_1, \dots, u_n, v_n, w_n)^T, \quad (1)$$

where u , v and w are displacement components in the longitudinal (x -), lateral (y -) and vertical (z -) directions. According to the 2.5D FEM [18, 29], the differential equation of motion of the rail is given by

$$\mathbf{M}\ddot{\mathbf{q}}(x, t) + \mathbf{K}_0\mathbf{q}(x, t) + \mathbf{K}_1\frac{\partial}{\partial x}\mathbf{q}(x, t) - \mathbf{K}_2\frac{\partial^2}{\partial x^2}\mathbf{q}(x, t) = \mathbf{f}(x, t), \quad (2)$$

where $\mathbf{M}$, $\mathbf{K}_0$ and $\mathbf{K}_2$ are $3n \times 3n$ symmetric matrices, and $\mathbf{K}_1$ is an anti-symmetric matrix; $\mathbf{f}(x, t)$ denotes the nodal force vector, in units N/m, consisting of two parts, one being the externally applied loads, and other being those provided by the supports.

2.2 The externally applied loads

The externally applied loads on the rail are assumed to be harmonic with radian frequency Ω and moving in the

x -direction at speed c . At $t = 0$, the loads are applied at the x_0 cross-section. The corresponding nodal force vector is given by

$$f_e(x, t) = \delta(x - x_0 - ct) p_0 e^{i\Omega t}, \quad (3)$$

where $i = \sqrt{-1}$, $\delta(\cdot)$ is the delta-function, and p_0 denotes the amplitude vector of the loads. In case of a moving constant load such as an axle load, $\Omega = 0$.

2.3 Loads applied by the supports

It is assumed that the track structure is periodic in the x -direction with period equal to L , where L may be equal to, or greater than, the sleeper spacing L_0 . In other words, at every length L in the x -direction, the track repeats all the details found in the interval $[0, L]$, which is termed the 0th bay. The track consists of an infinite number of identical bays of length L , and the j th bay is located from $x = jL$ to $x = (j + 1)L$, where $j = -\infty, \dots, -1, 0, 1, 2, \dots, +\infty$. Within each, say the 0th bay, there are a number, S , of supports (including attachments such as rail dampers) having arbitrary degrees of freedom. The s th support in the 0th bay is located at $x = x_s$, where $0 \leq x_s < L$. The s th support in the j th bay is located at $x = jL + x_s$. The supports may produce not only point forces to part of the nodes of the main structure, but also torques. The point forces produced by the s th support in the j th bay are denoted by a force vector $f_{js}(t)$ which is applied at $x = jL + x_s$ and assumed to contain N_s components. A torque applied by the support can be one of the following: that in the cross-sectional (yz) plane (see Fig. 1), that in a longitudinally vertical plane (parallel with the xz plane) and that in a horizontal plane (parallel with the xy plane). A torque in the cross-sectional plane can be replaced by two vertical (or two lateral) point forces and therefore they are not treated as a torque. Remaining torques are denoted by a torque vector $m_{js}(t)$ consisting M_s components. This torque vector may be represented by two force vectors, $m_{js}(t)/\Delta x$ and $-m_{js}(t)/\Delta x$, applied, respectively, at two cross-sections separated by a distance Δx : $x = jL + x_s + \Delta x$ and $x = jL + x_s$; see Fig. 1 for illustration. In Fig. 1, a torque, M_y , is applied in the vertical plane containing the k th nodal

Fig. 1 Coordinate system and torques generated by a support

line. This torque is replaced by two forces F_z and F'_z , where $F_z = -\frac{M_y}{\Delta x}$ and $F'_z = \frac{M_y}{\Delta x}$. Another torque, M_z , is applied in the horizontal plane containing the l th nodal line. This torque is replaced by two forces F_y and F'_y , where $F_y = \frac{M_z}{\Delta x}$ and $F'_y = -\frac{M_z}{\Delta x}$.

The nodal force vector provided to the rail by all the supports is given by

$$f_c(x, t) = \sum_{j=-\infty}^{\infty} \sum_{s=1}^S \delta(x - x_s - jL) U_s f_{js}(t) + \frac{1}{\Delta x} \sum_{j=-\infty}^{\infty} \sum_{s=1}^S [\delta(x - x_s - jL - \Delta x) - \delta(x - x_s - jL)] V_s m_{js}(t), \quad (4)$$

where U_s is a matrix of order $3n \times N_s$, $N_s \leq 3n$, with elements being either 1, -1 , or 0 and such that $U_s^T U_s$ is a unit matrix of order $N_s \times N_s$. U_s describes the connectivity between the s th support and the rail. As an element of U_s , $U_s(i, k) = 1$ if the k th element of $f_{js}(t)$ acts at the i th degree of freedom of the rail and in the same direction. $U_s(i, k) = -1$ if the k th element of $f_{js}(t)$ acts at the i th degree of freedom of the rail but in the opposite direction. $U_s(i, k) = 0$ if otherwise.

V_s is similar to U_s but is of order $3n \times M_s$, where $M_s \leq 3n$. Again $V_s^T V_s$ is a unit matrix of order $M_s \times M_s$. If the equivalent force at the $x = jL + x_s + \Delta x$ cross-section of the k th element of $m_{js}(t)$ acts at the i th degree of freedom of the rail and in the same direction, then $V_s(i, k) = 1$. If the equivalent force at the $x = jL + x_s + \Delta x$ cross-section of the k th element of $m_{js}(t)$ acts at the i th degree of freedom of the rail but in the opposite direction, then $V_s(i, k) = -1$. Otherwise $V_s(i, k) = 0$.

U_s and V_s may be termed connectivity matrices of the s th support.

2.4 Receptance matrix of a support

The dynamics of the s th support is described by the receptance matrix of the support observed at the degrees of freedom corresponding to $f_{js}(t)$ and $m_{js}(t)$, and this matrix is denoted by $H_s(\omega)$, where ω is angular frequency. This is a symmetric matrix of order $(N_s + M_s) \times (N_s + M_s)$. It may be decomposed into four sub-matrices as below:

$$H_s(\omega) = \begin{bmatrix} [H_{11}(\omega)]_s & [H_{12}(\omega)]_s \\ [H_{21}(\omega)]_s & [H_{22}(\omega)]_s \end{bmatrix}, \quad (5)$$

with the upper left sub-matrix being of order $N_s \times N_s$, the upper right sub-matrix being of order $N_s \times M_s$, the lower left sub-matrix being of order $M_s \times N_s$, and the lower right sub-matrix being of order $M_s \times M_s$.

2.5 Differential equation of the track

Substitution of Eqs. (3) and (4) into Eq. (2) yields the differential equation of the track structure:

$$\begin{aligned} M\ddot{\mathbf{q}}(x, t) + \mathbf{K}_0\mathbf{q}(x, t) + \mathbf{K}_1\frac{\partial}{\partial x}\mathbf{q}(x, t) - \mathbf{K}_2\frac{\partial^2}{\partial x^2}\mathbf{q}(x, t) \\ = \mathbf{f}_e(x, t) + \mathbf{f}_c(x, t) = \delta(x - x_0 - ct)\mathbf{p}_0e^{i\Omega t} \\ + \sum_{j=-\infty}^{\infty} \sum_{s=1}^S \delta(x - x_s - jL) \mathbf{U}_s \mathbf{f}_{js}(t) \\ + \frac{1}{\Delta x} \sum_{j=-\infty}^{\infty} \sum_{s=1}^S [\delta(x - x_s - jL - \Delta x) \\ - \delta(x - x_s - jL)] \mathbf{V}_s \mathbf{m}_{js}(t). \end{aligned} \quad (6)$$

3 Solution

As shown in Eq. (6), excitations to the rail consist of two parts, one being the externally applied loads, and the other being interaction forces between the rail and supports. Therefore, the nodal displacement vector $\mathbf{q}(x, t)$ can also be divided into two parts, i.e.,

$$\mathbf{q}(x, t) = \mathbf{q}_e(x, t) + \mathbf{q}_c(x, t), \quad (7)$$

where $\mathbf{q}_e(x, t)$ is due to the externally applied loads, satisfying

$$\begin{aligned} M\ddot{\mathbf{q}}_e(x, t) + \mathbf{K}_0\mathbf{q}_e(x, t) + \mathbf{K}_1\frac{\partial}{\partial x}\mathbf{q}_e(x, t) \\ - \mathbf{K}_2\frac{\partial^2}{\partial x^2}\mathbf{q}_e(x, t) = \mathbf{f}_e(x, t) \\ = \delta(x - x_0 - ct)\mathbf{p}_0e^{i\Omega t}, \end{aligned} \quad (8)$$

and $\mathbf{q}_c(x, t)$ satisfies

$$\begin{aligned} M\ddot{\mathbf{q}}_c(x, t) + \mathbf{K}_0\mathbf{q}_c(x, t) + \mathbf{K}_1\frac{\partial}{\partial x}\mathbf{q}_c(x, t) \\ - \mathbf{K}_2\frac{\partial^2}{\partial x^2}\mathbf{q}_c(x, t) = \mathbf{f}_c(x, t) = \sum_{j=-\infty}^{\infty} \sum_{s=1}^S \delta(x - x_s - jL) \mathbf{U}_s \mathbf{f}_{js}(t) \\ + \frac{1}{\Delta x} \sum_{j=-\infty}^{\infty} \sum_{s=1}^S [\delta(x - x_s - jL - \Delta x) - \delta(x - x_s - jL)] \mathbf{V}_s \mathbf{m}_{js}(t). \end{aligned} \quad (9)$$

3.1 Solution for $\mathbf{q}_e(x, t)$

Equation (8) is identical to Eq. (8) in [15] and the solution of Eq. (8) for $\mathbf{q}_e(x, t)$ must be the same as that presented in that reference:

$$\mathbf{q}_e(x, t) = \mathbf{Q}_e(x - x_0 - ct)\mathbf{p}_0e^{i\Omega t}, \quad (10)$$

where, by denoting

$$\omega = \Omega - \beta c, \quad (11)$$

$$\mathbf{D}(\beta, \omega) = -\omega^2 \mathbf{M} + \mathbf{K}_0 + i\beta \mathbf{K}_1 + \beta^2 \mathbf{K}_2, \quad (12)$$

the $3n \times 3n$ matrix $\mathbf{Q}_e(x)$ is determined by the following inverse Fourier transform:

$$\mathbf{Q}_e(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta x} d\beta, \quad (13)$$

where β is the wavenumber in the x -direction in units rad/m. Since $\mathbf{K}_1$ is an anti-symmetric matrix, according to Eq. (12), $\mathbf{D}(\beta, \omega)$ is a Hermitian matrix; i.e., it is identical to its conjugate transpose.

3.2 Solution for $\mathbf{q}_c(x, t)$

It can be seen that Eq. (9) is different from Eq. (9) in [15] due to the extra supports in a bay. However, the procedure to find $\mathbf{q}_c(x, t)$ is similar to that presented in [15] and details are given in the Appendix for readers' convenience. The key point is that since the track is periodic in the track direction and the moving loads are of the same frequency Ω , the force vector of a support in a bay is identical to that of the corresponding support in another bay apart from a time lag, i.e.,

$$\begin{aligned} \mathbf{f}_{js}(t + \frac{jL}{c}) = \mathbf{f}_{0s}(t) e^{i\Omega jL/c}, \quad j = -\infty, \dots, 0, \dots, \infty; \\ s = 1, 2, \dots, S, \end{aligned} \quad (14)$$

$$\begin{aligned} \mathbf{m}_{js}(t + \frac{jL}{c}) = \mathbf{m}_{0s}(t) e^{i\Omega jL/c}, \quad j = -\infty, \dots, 0, \dots, \infty; \\ s = 1, 2, \dots, S. \end{aligned} \quad (15)$$

Equations (14) and (15) have been termed the periodic conditions in the literature. $\mathbf{q}_c(x, t)$ is given by (see the Appendix),

$$\begin{aligned} \mathbf{q}_c(x, t) = \left[\sum_{j=-\infty}^{\infty} \left(-\frac{1}{2\pi L} e^{-i2\pi jx/L} \int_{-\infty}^{\infty} [\mathbf{D}(\beta_j, \omega)]^{-1} \right. \right. \\ \times \mathbf{C}(\beta_j) [\mathbf{A}(\beta)]^{-1} \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta(x-x_0-ct)} d\beta \Big) \\ \times \mathbf{p}_0 e^{i\Omega t}, \text{ (see (78))}, \end{aligned} \quad (16)$$

where ω is given by Eq. (11), and

$$\beta_j = \beta - \frac{2\pi j}{L}, \quad (17)$$

$$\begin{aligned} \mathbf{C}(\beta_j) = (\mathbf{e}^{-i\beta_j x_1} \mathbf{U}_1, \dots, \mathbf{e}^{-i\beta_j x_S} \mathbf{U}_S, -i\beta_j \mathbf{e}^{-i\beta_j x_1} \\ \times \mathbf{V}_1, \dots, -i\beta_j \mathbf{e}^{-i\beta_j x_S} \mathbf{V}_S), \text{ (see (77))}, \end{aligned} \quad (18)$$

$$\begin{aligned} \mathbf{B}(\beta) = (\mathbf{e}^{i\beta x_1} \mathbf{U}_1, \dots, \mathbf{e}^{i\beta x_S} \mathbf{U}_S, i\beta \mathbf{e}^{i\beta x_1} \mathbf{V}_1, \dots, i\beta \mathbf{e}^{i\beta x_S} \mathbf{V}_S)^T \\ \text{(see (72), (73))}, \end{aligned} \quad (19)$$

$$\mathbf{A}(\beta) = \begin{bmatrix} \mathbf{A}_{11}(\beta) & \mathbf{A}_{12}(\beta) \\ \mathbf{A}_{21}(\beta) & \mathbf{A}_{22}(\beta) \end{bmatrix}, \quad (\text{see (75)}), \quad (20)$$

where

$$\mathbf{A}_{11}(\beta) = ([\mathbf{A}_{11}(\beta)]_{rs})_{r,s=1,\dots,S}, \quad (21a)$$

$$[\mathbf{A}_{11}(\beta)]_{rs} = \mathbf{U}_r^T \left(\frac{1}{L} \sum_{j=-\infty}^{\infty} [\mathbf{D}(\beta_j, \omega)]^{-1} e^{i\beta_j(x_r-x_s)} \right) \mathbf{U}_s + \delta_{rs}[\mathbf{H}_{11}(\omega)]_r, \quad (\text{see (65)}), \quad (21b)$$

$$\mathbf{A}_{12}(\beta) = ([\mathbf{A}_{12}(\beta)]_{rs})_{r,s=1,\dots,S}, \quad (21c)$$

$$[\mathbf{A}_{12}(\beta)]_{rs} = -\mathbf{U}_r^T \left(\frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j) [\mathbf{D}(\beta_j, \omega)]^{-1} e^{i\beta_j(x_r-x_s)} \right) \mathbf{V}_s + \delta_{rs}[\mathbf{H}_{12}(\omega)]_r, \quad (\text{see (67)}), \quad (21d)$$

$$\mathbf{A}_{21}(\beta) = ([\mathbf{A}_{21}(\beta)]_{rs})_{r,s=1,\dots,S}, \quad (21e)$$

$$[\mathbf{A}_{21}(\beta)]_{rs} = \mathbf{V}_r^T \left(\frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j) [\mathbf{D}(\beta_j, \omega)]^{-1} e^{i\beta_j(x_r-x_s)} \right) \mathbf{U}_s + \delta_{rs}[\mathbf{H}_{21}(\omega)]_r, \quad (\text{see (69)}), \quad (21f)$$

$$\mathbf{A}_{22}(\beta) = ([\mathbf{A}_{22}(\beta)]_{rs})_{r,s=1,\dots,S}, \quad (21g)$$

$$[\mathbf{A}_{22}(\beta)]_{rs} = -\mathbf{V}_r^T \left(\frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j)^2 [\mathbf{D}(\beta_j, \omega)]^{-1} e^{i\beta_j(x_r-x_s)} \right) \mathbf{V}_s + \delta_{rs}[\mathbf{H}_{22}(\omega)]_r, \quad (\text{see (71)}), \quad (21h)$$

where δ_{rs} is the Dirac-delta.

3.3 Receptance matrix of the rail

Equation (16) combined with Eq. (10) gives the total response of the rail. If observation is made from a reference frame moving with the loads, then the displacements of the structure are given by Eqs. (10) and (16) by setting $x = x' + x_0 + ct$, i.e.,

$$\mathbf{q}(x', t) = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta x'} d\beta + \sum_{j=-\infty}^{\infty} \left(-\frac{1}{2\pi L} e^{-i2\pi j(x' + x_0)/L} \int_{-\infty}^{\infty} [\mathbf{D}(\beta_j, \omega)]^{-1} \times \mathbf{C}(\beta_j) [\mathbf{A}(\beta)]^{-1} \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta x'} d\beta \right) e^{-i2\pi jct/L} \right] \times \mathbf{p}_0 e^{i\Omega t}, \quad (22)$$

where x' is the coordinate relative to the moving frame of reference. As in the case of stationary harmonic loads, the matrix in Eq. (22) before the moving load vector is also termed receptance matrix. It can be seen that the receptance matrix of the rail is not temporally constant, but instead, it is a periodic function of time t with period equal to L/c . This periodic matrix is given by

$$\mathbf{Q}(x', t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta x'} d\beta + \sum_{j=-\infty}^{\infty} \left(-\frac{1}{2\pi L} e^{-i2\pi j(x' + x_0)/L} \int_{-\infty}^{\infty} [\mathbf{D}(\beta_j, \omega)]^{-1} \times \mathbf{C}(\beta_j) [\mathbf{A}(\beta)]^{-1} \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta x'} d\beta \right) e^{-i2\pi jct/L}. \quad (23)$$

It is in the Fourier series form, that is

$$\mathbf{Q}(x', t) = \sum_{j=-\infty}^{\infty} \tilde{\mathbf{Q}}_j(x') e^{-i2\pi jct/L}, \quad (24)$$

with the constant term (i.e., the term with $j = 0$) being given by

$$\tilde{\mathbf{Q}}_0(x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta x'} d\beta - \frac{1}{2\pi L} \int_{-\infty}^{\infty} [\mathbf{D}(\beta, \omega)]^{-1} \times \mathbf{C}(\beta) [\mathbf{A}(\beta)]^{-1} \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta x'} d\beta. \quad (25)$$

and the j th term being given by

$$\tilde{\mathbf{Q}}_j(x') = -\frac{1}{2\pi L} e^{-i2\pi j(x' + x_0)/L} \int_{-\infty}^{\infty} [\mathbf{D}(\beta_j, \omega)]^{-1} \mathbf{C}(\beta_j) [\mathbf{A}(\beta)]^{-1} \times \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} e^{i\beta x'} d\beta. \quad (26)$$

It can be seen from Eqs. (25) and (26) that each term is expressed as an inverse Fourier transform of a particular matrix in the wavenumber (β -) domain. An inverse Fourier transform may be performed using the FFT technique. According to FFT, responses within $|x'| \leq x'_{\max}$ will be available, where, $x'_{\max} = \frac{2\pi}{2\Delta\beta}$, with $\Delta\beta$ being the wavenumber resolution. For example, if $\Delta\beta = 2\pi \times 0.0025$ rad/m, then $x'_{\max} = 200$ m. In other words, rail responses within a total length of 400 m will be available. This makes it comfortable to deal with interactions between a whole high-speed train and a railway track.

It is also worthy of pointing out that if the dynamic stiffness of all the supports and dampers are evenly distributed along the track, then the track becomes a continuously supported and damped structure. In this situation, all the terms in Eq. (24) vanish except for that with $j = 0$.

In other words, Eq. (24) becomes $\mathbf{Q}(x', t) = \tilde{\mathbf{Q}}_0(x')$; i.e., the rail receptance matrix is time-invariant.

3.4 Forces and responses of supports in the 0th bay

The spectra of the support force vector in the 0th bay are given in Eq. (74). It can be shown that the support force time-history is given by

$$\begin{pmatrix} \mathbf{f}_{01}(t) \\ \vdots \\ \mathbf{f}_{0S}(t) \\ \mathbf{m}_{01}(t) \\ \vdots \\ \mathbf{m}_{0S}(t) \end{pmatrix} = - \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathbf{A}(\beta)]^{-1} \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} \times e^{-i\beta(x_0+ct)} d\beta \right) \mathbf{p}_0 e^{i\Omega t}. \quad (27)$$

Displacements of all or part of the degrees of freedom of all the supports in the 0th bay is denoted by a vector $\mathbf{g}_0(t)$. The spectrum of it is denoted by $\hat{\mathbf{g}}_0(f)$, where f is spectral frequency. A receptance matrix, denoted by $\mathbf{G}(2\pi f)$, may be defined for the supports such that

$$\hat{\mathbf{g}}(f) = -\mathbf{G}(2\pi f) \begin{Bmatrix} \hat{\mathbf{f}}_0(f) \\ \hat{\mathbf{m}}_0(f) \end{Bmatrix}, \quad (28)$$

where $\hat{\mathbf{f}}_0(f)$ and $\hat{\mathbf{m}}_0(f)$ are defined in Eq. (63), and the minus sign indicates that forces and moments are opposite in direction to associated displacement vectors. It can be shown that

$$\mathbf{g}_0(t) = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{G}(\omega) [\mathbf{A}(\beta)]^{-1} \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} e^{-i\beta(x_0+ct)} d\beta \right) \times \mathbf{p}_0 e^{i\Omega t}. \quad (29)$$

In Eqs. (27) and (29), ω is given by Eq. (11). The terms in the bracket are Fourier transforms. These transforms are from the wavenumber (β -) domain to the space ($x = x_0 + ct$) domain. Equations (27) and (29) indicate that the support force and displacement are oscillating at frequency Ω but their amplitudes decay as the loads move away along the track.

3.5 Forces and responses of supports in the j th bay

Support forces in the j th bay can be worked out according to Eqs. (14) and (15),

$$\begin{pmatrix} \mathbf{f}_{j1}(t) \\ \vdots \\ \mathbf{f}_{jS}(t) \\ \mathbf{m}_{j1}(t) \\ \vdots \\ \mathbf{m}_{jS}(t) \end{pmatrix} = - \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathbf{A}(\beta)]^{-1} \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} \times e^{-i\beta(x_0+ct-jL)} d\beta \right) e^{i\Omega jL/c} \mathbf{p}_0 e^{i\Omega t}, \quad (30)$$

and displacements of the supports are given by

$$\mathbf{g}_j(t) = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{G}(\omega) [\mathbf{A}(\beta)]^{-1} \mathbf{B}(\beta) [\mathbf{D}(\beta, \omega)]^{-1} \times e^{-i\beta(x_0+ct-jL)} d\beta \right) e^{i\Omega jL/c} \mathbf{p}_0 e^{i\Omega t}. \quad (31)$$

3.6 Propagation constant equation

Characteristic free vibration at frequency ω of a periodic structure satisfies

$$\mathbf{q}(x+L, t) = \mathbf{q}(x, t) e^{i\bar{\beta}L}, \quad (32)$$

where the non-dimensional quantity $\bar{\beta}L$ is termed propagation constant at the given frequency [1]. As shown in [15], $\bar{\beta}$ is the root of the following equation

$$\det(\mathbf{A}(\beta, \omega)) = 0, \quad (33)$$

where since ω is independent of β [see Eq. (11) with the load speed being vanishing], matrix in Eq. (20) has been denoted alternatively by $\mathbf{A}(\beta, \omega)$.

According to Eq. (21), $\mathbf{A}(\beta, \omega)$ is a periodic function of β with period equal to $2\pi/L$. Thus, if $\bar{\beta}$ is a root of Eq. (33), then $\bar{\beta} \pm \frac{j2\pi}{L}$ ($j = \pm 1, \pm 2, \dots$) are roots of Eq. (33) as well.

4 Application to a railway track with rail dampers

Formulae derived above are now applied to a conventional ballasted railway track with two rail dampers installed either side of the rail at the mid-span of each sleeper bay (Fig. 2). This application is to investigate the effect of the dampers on the dynamics of the track, including vibration propagations along, resonances of, and attenuations along the track. A set of typical parameters for the track and dampers are given in Sect. 4.1. Since the formulae involve sums of infinite terms and integrations along the entire wavenumber-axis, truncations must be made. That how many terms should be included and how to choose the integration limits must be examined and this is presented in Sect. 4.2. Propagation constants of the track/damper system are computed and discussed in Sect. 4.3.

Fig. 2 Track model with rail dampers between sleepers

Table 1 Parameters for the vertical dynamics of a track

Density of the rail	$\rho = 7,850 \text{ kg/m}^3$
Young's modulus of the rail	$E = 2.1 \times 10^{11} \text{ N/m}^2$
Shear modulus of the rail	$G = 0.81 \times 10^{11} \text{ N/m}^2$
Cross-sectional area of the rail	$A = 7.69 \times 10^{-3} \text{ m}^2$
Bending moment of inertia of the rail	$I = 30.55 \times 10^{-6} \text{ m}^4$
Shear coefficient of the rail cross-section	$\kappa = 0.4$
Vertical rail pad stiffness	$k_P = 3.5 \times 10^8 \text{ N/m}$
Rail pad loss factor	$\eta_P = 0.25$
Mass of half a sleeper	$m_S = 162 \text{ kg}$
Sleeper spacing	$L = 0.6 \text{ m}$
Vertical ballast stiffness	$k_B = 50 \times 10^6 \text{ N/m}$
Loss factor of ballast	$\eta_B = 1.0$

Vibration resonance characteristics of the track/damper system are shown in Sect. 4.4, and vibration attenuation characteristics are presented in Sect. 4.5.

4.1 Parameters of the track/damper system and associated matrices

A set of typical parameters for the track structure are listed in Table 1. These parameters are for half the railway structure and correspond to a track with concrete sleepers and moderately stiff rail pads. The rail damper is constructed with a metal block on a layer of elastomeric material, and allowed to have translational vibration in the vertical direction as well as rotational vibration in the longitudinally vertical (i.e., x - z) plane. Parameters used for the dampers are listed in Table 2, similar to those used in [26].

With the addition of the rail dampers, the period of the track structure is still equal to the sleeper spacing, but within each period, there are two supports, one being the railpad/sleeper/ballast system and the other being two rail dampers. For the vertical dynamics of a railway track up to 3,000 Hz, the Timoshenko beam model can be employed to model the rail. According to the Timoshenko beam theory, the differential equation for the rail subject to a unit vertical moving harmonic load is given by

Table 2 Parameters for the damper

Tuned frequency (vertical/rotational)	1,000/928 Hz
Damper length	$l_D = 0.2 \text{ m}$
Mass of two dampers	$m_D = 15 \text{ kg}$
Inertia moment of two dampers	$J_D = 0.058 \text{ kg m}^2$
Vertical stiffness of two damper springs	$k_{DW} = 5.92 \times 10^8 \text{ N/m}$
Rotational stiffness of the two dampers	$k_{D\psi} = 1.97 \times 10^6 \text{ Nm}$
Loss factor of dampers	$\eta_D = 0.35$

$$\rho A \frac{\partial^2 w}{\partial t^2} - \kappa A G \frac{\partial^2 w}{\partial x^2} + \kappa A G \frac{\partial \psi}{\partial x} = \delta(x - x_0 - ct) e^{i\Omega t} + \sum_{j=-\infty}^{\infty} \sum_{s=1}^S \delta(x - x_s - jL) F_{js}(t), \quad (34)$$

$$\rho I \frac{\partial^2 \psi}{\partial t^2} - EI \frac{\partial^2 \psi}{\partial x^2} - \kappa A G \frac{\partial w}{\partial x} + \kappa A G \psi = \sum_{j=-\infty}^{\infty} \sum_{s=1}^S \delta(x - x_s - jL) M_{js}(t), \quad (35)$$

where w is the vertical displacement (directed downwards) of the rail and ψ is the rotation angle (directed clockwise) of the cross-section due to the bending moment only; $F_{j1}(t)$ is the vertical force applied on the rail by the j th sleeper and $F_{j2}(t)$ the force applied by the two dampers in the j th bay; $M_{j1}(t)$ is the torque exerted on the rail in the longitudinally vertical plane by the j th sleeper and $M_{j2}(t)$ the torque from the two dampers in the j th bay; and finally, $S = 2$, $x_1 = 0$ and $x_2 = L/2$. Comparing Eqs. (34) and (35) with Eq. (6), it follows that

$$\mathbf{q} = \begin{Bmatrix} w \\ \psi \end{Bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix}, \quad \mathbf{K}_0 = \begin{bmatrix} 0 & 0 \\ 0 & \kappa A G \end{bmatrix},$$

$$\mathbf{K}_1 = \begin{bmatrix} 0 & \kappa A G \\ -\kappa A G & 0 \end{bmatrix},$$

$$\mathbf{K}_2 = \begin{bmatrix} \kappa A G & 0 \\ 0 & EI \end{bmatrix}, \quad \mathbf{p}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix},$$

$$\mathbf{U}_1 = \mathbf{U}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}_1 = \mathbf{V}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

That the connectivity matrix $V_1 = V_2 = 0$ is the result of the Timoshenko beam model of the rail in which the rotation angle of the cross-section due to the bending moment only is chosen to be one of the degree of freedom.

The receptance matrix of the support including a railpad, a sleeper and the ballast, is given by

$$H_1(\omega) = \begin{bmatrix} \frac{k_B + k_P - m_S \omega^2}{k_P [k_B - m_S \omega^2]} & 0 \\ 0 & \frac{12}{b_S^3 k_P} \end{bmatrix}, \quad (36)$$

where k_P and k_B are complex stiffness of the railpad and ballast, m_S is half the sleeper mass and $b_S = 0.25$ m is the width of the sleeper. Sleepers are assumed to be rigid and vibrate in the vertical direction only.

The receptance matrix of the two dampers is given by

$$H_2(\omega) = \begin{bmatrix} -\frac{k_{D\psi} - m_D \omega^2}{m_D k_{D\psi} \omega^2} & 0 \\ 0 & -\frac{k_{D\psi} - J_D \omega^2}{J_D k_{D\psi} \omega^2} \end{bmatrix}, \quad (37)$$

and the meanings of symbols appearing in Eq. (37) are defined in Table 2. Each rail damper has two natural frequencies, one for vertical vibration and the other for rotational vibration, as shown in Table 2. These frequencies are all close to the first pinned–pinned frequency of the original track, which is around 1,070 Hz [19].

Figure 3 shows the driving point receptances of the supports at the rail/support connecting points. At the natural frequency of vertical vibration, the corresponding receptance of a rail damper is the minimum (zero if there is no damping; Fig. 3, dashed line), providing effective constraints to the rail at the mid-span. It is expected that the dampers will damp the pinned–pinned vibration of the rail. Since in the pinned–pinned vibration mode, rail rotation is least at mid-span, the rotational resonance of the rail damper will have an insignificant effect. For frequencies well below the first pinned–pinned frequency, the receptances of the rail dampers are

Fig. 3 Receptances of the supports

much higher than those of the sleepers (Fig. 3, solid line). It thus can be reasoned that, at those frequencies, the addition of the dampers has an insignificant effect. However, for frequencies well above the first pinned–pinned frequency, the receptances of the rail dampers are still smaller than those of the sleepers. It thus can be reasoned that, at those frequencies, the addition of the dampers will have some effect.

4.2 Terms which should be included and integration limits

4.2.1 Terms which should be included in Eq. (21)

Equation (21) for calculating the matrix $A(\beta)$ involves a sum, defined by $\sum_{j=-\infty}^{\infty} [D(\beta_j, \omega)]^{-1}$. The first diagonal element of this expression is given by

$$g_{11}(\beta) = \sum_{j=-\infty}^{\infty} d_{11}(\beta_j, \omega), \quad (38)$$

where $d_{11}(\beta_j, \omega)$ is the first element of $[D(\beta_j, \omega)]^{-1}$. For a given wavenumber β , $\omega = \Omega - \beta c$, and $\beta_j = \beta - 2\pi j/L$. For a load of 3,000 Hz moving at 100 m/s, $d_{11}(\beta_j, \omega)$ are calculated for index $j = -50, \dots, 0, \dots, 50$, and for wavenumber β ranging from -25 to 25 rad/m at spacing $2\pi \times 0.0025$ rad/m. By computing $\alpha(\beta_j, \omega) = 20 \log(|d_{11}(\beta_j, \omega)|/10^{-12})$ (in dB), a contour plot can be produced as shown below in Fig. 4. The difference between the maximum level and the minimum level of the contour plot is 120 dB. It can be seen that for $|\beta| \leq 25$, $\alpha(\beta_j, \omega)$ are all fall below the minimum level when $|j| > 10$.

4.2.2 Terms which should be included in Eq. (22) and choice of integration limits

For a load of 3,000 Hz moving at 100 m/s, the first element, denoted by $\tilde{w}_R(\beta, j)$, of the integrand matrix in

Fig. 4 Contour plot of $\alpha(\beta_j, \omega)$

Fig. 5 Contour plot of $\tilde{W}_R(\beta, j)$

Eq. (25) and that in Eq. (26) are calculated, for index $j = -20, \dots, 0, \dots, 20$, and for wavenumber β ranging from -25 to 25 rad/m at spacing $2\pi \times 0.0025$ rad/m. $\tilde{w}_R(\beta, j)$ is a function of β and j . By computing $\tilde{W}_R(\beta, j) = 20 \log(|\tilde{w}_R(\beta, j)|/10^{-12})$, a contour plot can be produced as shown below in Fig. 5. The difference between the maximum level and the minimum level of the contour plot is 55 dB. It can be seen that for $|j| > 10$ and $|\beta| > 25$, $\tilde{W}_R(\beta, j)$ are all fall below the minimum level.

It is also seen from Fig. 5 that there are two peaks which occur at $j = 0$, and $\beta = 10.2$ and -11 , approximately corresponding to the two intersections of the dispersion curve of the free rail and the straight line defined by $\omega = \Omega - \beta c$ (see Fig. 6). The wavenumber of free vibration of the rail at frequency ω is given by the rail's so-called dispersion equation, that is

$$\det(\mathbf{D}(\beta, \omega)) = \det(-\omega^2 \mathbf{M} + \mathbf{K}_0 + i\beta \mathbf{K}_1 + \beta^2 \mathbf{K}_2) = 0, \quad (39)$$

or,

$$(\kappa GEI)\beta^4 - \rho I(\kappa G + E)\omega^2 \beta^2 + \omega^4 \rho^2 I - \omega^2 \rho A \kappa G = 0. \quad (40)$$

For the rail parameters listed in Table 1, the dispersion curve can be produced according to Eq. (40), as shown in Fig. 6. It can be seen that, within 5,000 Hz, the wavenumber is less than 20 rad/m.

From above, it now can be decided how many terms should be included in sums appearing in the formulae and how to choose the upper and lower limits for integrals. In what follows, the index j ranges from -50 to 50 , and β ranges from -25 to 25 rad/m at spacing of $2\pi \times 0.0025$ rad/m.

Fig. 6 Dispersion curves of the free rail

4.3 Propagation constants

Propagation constants (or wavenumber) of the track/damper system can be produced by solving Eq. (33). Alternatively, a contour plot of $1/|\det(\mathbf{A}(\beta, \omega))|$ on the frequency–wavenumber plane, as shown in Fig. 7, can also be employed to show propagation wavenumbers for given frequencies, which are displayed as yellow curves. These yellow curves are also called dispersion curves of the periodic structure. A point on a dispersion curve defines a wavenumber and a frequency. If there is no damping in the track, the periodically supported rail allows free vibration at that frequency to propagate, without attenuation, along the rail at that wavenumber. For comparison, propagation wavenumbers of the original track are also shown here on the left (from [19]). It can be seen that the presence of the dampers generates more bounding frequencies [19], but the most important effect of the dampers is that they convert Pass Band 2 of the original track into two stop bands, and pushes the lower bounding frequency of Pass Band 3 to a much higher value, thus significantly increasing the total width of stop frequency bands.

4.4 Quasi-receptances

When the load is moving, the displacement amplitude of the rail at the loading point is not, due to the discrete supports, constant but instead a periodic function of time. The variation of the displacement amplitude of the loading point with time t varying over $[0, L/c]$ is equivalent to the variation due to the initial loading position, x_0 , varying over $[0, L]$ at $t = 0$.

The vertical displacement amplitude of a sleeper to a unit vertical force applied at the top of the railpad is given by $1/(k_B - m_S \omega^2)$, and that due to a torque in the

Fig. 7 Propagation wavenumbers of the track. **a** Without damper and **b** with damper

longitudinally vertical plane is zero. The vertical displacement amplitude of the mass of a damper to a unit vertical force at its bottom is given by $-1/(m_D\omega^2)$, and that due to a torque in the longitudinally vertical plane is zero. Thus, the matrix, $G(\omega)$, in Eqs. (29) and (31) for calculating the vertical displacements of the sleeper and the damper mass is given by

$$G(\omega) = \begin{bmatrix} \frac{1}{k_B - m_S\omega^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{m_D\omega^2} & 0 \end{bmatrix}. \quad (41)$$

Figure 8b shows rail receptance at the loading point to a stationary load applied above a sleeper (in solid line) and at the mid-span (in dashed line; i.e., above a damper). For comparison, rail receptance of the original track is also shown in the figure on the left (Fig. 8a).

It can be seen that the pinned–pinned frequency at 1,070 Hz (indicated by letter A) of the original track disappears, as expected, thanks to the dampers. However, the installation of the dampers generates two new peaks (indicated by letters B and C). Peak B corresponds to the bounding frequency B shown in Fig. 7b. At this frequency, *sleepers behave like pins* and the point at the mid-way (that is the position of the damper) between two neighboring sleepers has the maximum response across the span defined by these two sleepers. In other words, the pinned–pinned frequency of the original track is shifted down to a much lower value by the dampers. Peak C, at about 1,250 Hz, corresponds to the bounding frequency C shown in Fig. 7b. At this frequency, *dampers behave like pins* and the point at the mid-way (that is the position of the sleeper) between two neighboring dampers has the maximum response. Thus this is a pinned–pinned frequency created by the dampers. The effect of these two

pinned–pinned frequencies on rail roughness growth is unknown at the moment, and to be investigated in the future.

Rail receptance at the loading point to a load initially being above a sleeper (in solid line) and at the mid-span (in dashed line) is shown in Fig. 9, with the left plot for a stationary load and the right plot for a load moving at 100 m/s. It can be seen that Peak B is flattened by the load speed, and Peak C is split into two sub-peaks.

Figure 10 shows receptance (at the initial moment) of a sleeper to a load initially being above the sleeper (in solid line) and at the mid-span (in dashed line) next to the sleeper, with the left plot for a stationary load and the right one for a load moving at 100 m/s. It can be seen that the effect of the initial loading point within the associated sleeper bay on the response of the sleeper is quite small and this effect becomes even smaller as the load moves fast.

Figure 11 shows receptance (at the initial moment) of a damper to a load initially being above the left neighboring sleeper (in solid line) and at the mid-span (i.e., above this damper; in dashed line), with the left plot for a stationary load and the right one for a load moving at 100 m/s. Peak B is due to the pinned–pinned vibration of the rail: in this vibration mode, the mid-span of the rail, and the damper attached here, have peak responses. Load speed splits this peak into two sub-peaks.

Comparison between the rail, sleeper, and damper is shown in Fig. 12 for receptance magnitude and in Fig. 13 for phase angle. It can be seen from Fig. 12 that, for frequencies higher than 500 Hz, rail vibration is largely absorbed by the railpad, with a small fraction transmitted to the sleeper. The phase angles corresponding to the peaks at around 500 Hz are close to -90° . Between 450 and 1,300 Hz, the damper vibrates

Fig. 8 Rail receptances at the loading point to a stationary load applied above a sleeper and at the mid-span. **a** For the original track and **b** for the track with dampers

Fig. 9 Rail receptances at the loading point to a load initially being above a sleeper and at the mid-span. **a** The load is stationary and **b** the load is moving at 100 m/s

more strongly than the rail, but not in phase. The effect of damper vibration on total noise radiation from the track is to be investigated in the future.

4.5 Vibration decay along the rail

As a harmonic load moves along the rail, vibration wave are generated in the rail ahead and behind the load. Due to the load speed, the wave ahead the load is different from that behind the load, not only in amplitude, but also in

wavelength and decay rate. The wave behind the load exhibits, more or less, higher amplitude, longer wavelength and less decay rate than the wave ahead the load.

Figure 14 shows the waves generated in the rail by a unit load of 2,000 Hz moving at 100 m/s along the rail. Due to damping in the track, vibration attenuates with distance from the load. However, significant vibration is still observed even 20 m away from the load.

Figure 15 shows the displacement of a damper generated by the same load as a function of distance between the

Fig. 10 Receptances of a sleeper to a load initially being above the sleeper and at the mid-span next to the sleeper. **a** The load is stationary and **b** the load is moving at 100 m/s

Fig. 11 Receptances of a damper to a load initially being above the left neighboring sleeper and above the damper. **a** The load is stationary and **b** the load is moving at 100 m/s

damper and the load. Initially the load is above the left sleeper of the damper, i.e., the initial distance between the damper and the load is -0.3 m. The damper's response increases as the load approaches the damper, and then decreases as the load moves away. However, the decay is not purely exponentially. Amplitude modulation at the sleeper passing frequency is evident in Fig. 15: the response of the damper is a little smaller when the load is at a sleeper than those when the load is just before and after the sleeper.

The rate of vibration decay along the rail may be worked out by plotting the vibration level against distance from the load, as shown in Fig. 16. Vibration level is defined as $20\log(|w|)$. Two straight lines can be seen in the figure. The slope of the left straight line is the decay rate, in dB/m, of vibration waves behind the load, and that of the right straight line is the decay rate of vibration waves ahead the load. Decay rates determined by this way is different from those numerically computed from the so-called dispersion equation [or propagation constant equation, i.e., Eq. (33)],

Fig. 12 Receptances of rail, sleeper and damper to a load initially being above the damper and moving at 100 m/s

Fig. 13 Phase angles of rail, damper and difference between them

since the former includes the effect of the load speed, but the latter does not.

Figure 17 shows the decay rate of the rail vibration wave ahead a load moving at 100 m/s. The curve in solid line is for the track with dampers, and that in dashed line is for the original track. Load frequencies range from 50 to 2,000 Hz at a step of 50 Hz.

It can be seen that, for frequencies higher than about 500 Hz, the decay rate of the original track is quite small. Even for frequencies within the stop band (from 1,070 to 1,300 Hz, see [19]) between the second and the third pass bands (see Fig. 7), the decay rate still remains small (less than 2 dB/m). This is because that the stop band is too narrow to avoid the effect of the load speed: propagating waves are still generated by the load moving at 100 m/s.

However, with the dampers, decay rates for frequencies above 500 Hz are significantly increased. This is caused by

Fig. 14 Waves generated in the rail by a unit load of 2,000 Hz moving at 100 m/s along the rail

Fig. 15 Displacement of a damper generated by a unit load of 2,000 Hz moving at 100 m/s along the rail. Initially the load is above the left sleeper of the damper

the combined effect of wider stop bands (see Fig. 7b) and extra damping in the dampers.

Thus, decay rate for a stationary load may not be sufficient to assess the acoustic behavior of a track; it is important to consider the effect of train speed.

5 Conclusion

The Fourier transform-based method, developed in [15] for calculating the response of a railway track subject to moving or stationary harmonic loads, is extended in this

Fig. 16 Rail vibration level for a unit load of 2,000 Hz moving at 100 m/s along the rail

Fig. 17 Decay rate of rail vibration wave ahead a load moving at 100 m/s

paper, so that it can be applied not only for conventional ballasted tracks as a uniform periodic structure, but also for tracks which, for various reasons, have been made to be a non-uniform periodic structure.

The renewed formulae are applied to investigate the effect of rail dampers of particular design which are installed between sleepers. The rail dampers are tuned to the first pinned–pinned frequency of the original track, and allowed to vibrate vertically and rotationally. It is found that the rail dampers can significantly widen the width of stop bands and increase the rate of vibration decay along the rail.

Having said that, the effects of the dampers on wheel/rail interactions, rail roughness growth and noise radiation from the track are not totally known at the moment (for frequencies between about 450 and 1,300 Hz, the damper

vibrates stronger than the rail at the same position, and noise radiation from the damper may become an issue), although there are claims in the literature that properly designed dampers are beneficial to part, or all, of the above three aspects. This must be fully investigated in the future. The formulae presented in the paper provide a powerful tool to deal with these issues.

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Appendix: solving Eq. (9) for $q_c(x, t)$

To solve Eq. (9), the Fourier transform with respect to x (i.e., from x to wavenumber β) is performed:

$$\begin{aligned} M\ddot{\bar{q}}_c(\beta, t) + K_0\bar{q}_c(\beta, t) + i\beta K_1\bar{q}_c(\beta, t) + \beta^2 K_2\bar{q}_c(\beta, t) \\ = \sum_{j=-\infty}^{\infty} \sum_{s=1}^S U_s f_{js}(t) e^{-i\beta(x_s+jL)} + \sum_{j=-\infty}^{\infty} \sum_{s=1}^S V_s m_{js}(t) \frac{1}{\Delta x} \\ \times \left(e^{-i\beta(x_s+jL+\Delta x)} - e^{-i\beta(x_s+jL)} \right) \\ = \sum_{j=-\infty}^{\infty} \sum_{s=1}^S U_s f_{js}(t) e^{-i\beta(x_s+jL)} - i\beta \sum_{j=-\infty}^{\infty} \sum_{s=1}^S V_s m_{js}(t) e^{-i\beta(x_s+jL)}. \end{aligned} \quad (42)$$

For this equation, the Fourier transform is further performed but with respect to time t :

$$\begin{aligned} \left(-(2\pi f)^2 M + K_0 + i\beta K_1 + \beta^2 K_2 \right) \hat{\bar{q}}_c(\beta, f) \\ = \sum_{j=-\infty}^{\infty} \sum_{s=1}^S U_s \hat{f}_{js}(f) e^{-i\beta(x_s+jL)} - i\beta \sum_{j=-\infty}^{\infty} \sum_{s=1}^S V_s \hat{m}_{js}(f) e^{-i\beta(x_s+jL)}, \end{aligned} \quad (43)$$

where $\bar{q}_c(\beta, t) = \int_{-\infty}^{\infty} q_c(x, t) e^{-i\beta x} dx$, $\hat{f}_{js}(f) = \int_{-\infty}^{\infty} f_{js}(t) \times e^{-i2\pi f t} dt$, and f is spectral frequency. From Eq. (43), we have

$$\begin{aligned} \hat{\bar{q}}_c(\beta, f) = [D(\beta, 2\pi f)]^{-1} \left\{ \sum_{j=-\infty}^{\infty} \sum_{s=1}^S U_s \hat{f}_{js}(f) e^{-i\beta(x_s+jL)} \right. \\ \left. - \beta \sum_{j=-\infty}^{\infty} \sum_{s=1}^S V_s \hat{m}_{js}(f) e^{-i\beta(x_s+jL)} \right\}, \end{aligned} \quad (44)$$

where matrix D is defined in Eq. (12).

Fourier transforms of Eqs. (14) and (15) with respect to t are given by

$$\hat{f}_{js}(f) = \hat{f}_{0s}(f) e^{i(\Omega - 2\pi f)jL/c} = \hat{f}_{0s}(f) e^{i\beta^* jL}, \quad (45)$$

$$\hat{m}_{js}(f) = \hat{m}_{0s}(f)e^{i(\Omega-2\pi f)L/c} = \hat{m}_{0s}(f)e^{i\beta^*jL}, \quad (46)$$

where

$$\beta^* = (\Omega - 2\pi f)/c, \quad (47)$$

is a wavenumber. Inserting Eqs. (45) and (46) into Eq. (44), gives

$$\begin{aligned} \hat{q}_c(\beta, f) &= [\mathbf{D}(\beta, 2\pi f)]^{-1} \left\{ \sum_{j=-\infty}^{\infty} \sum_{s=1}^S \mathbf{U}_s \hat{f}_{0s}(f) e^{-i\beta x_s} e^{-i(\beta-\beta^*)jL} - i\beta \sum_{j=-\infty}^{\infty} \sum_{s=1}^S \mathbf{V}_s \hat{m}_{0s}(f) e^{-i\beta x_s} e^{-i(\beta-\beta^*)jL} \right\} \\ &= [\mathbf{D}(\beta, 2\pi f)]^{-1} \left\{ \sum_{s=1}^S \mathbf{U}_s \hat{f}_{0s}(f) e^{-i\beta x_s} - i\beta \sum_{s=1}^S \mathbf{V}_s \hat{m}_{0s}(f) e^{-i\beta x_s} \right\} \sum_{j=-\infty}^{\infty} e^{-i(\beta-\beta^*)jL}. \end{aligned} \quad (48)$$

It can be shown that

$$\sum_{j=-\infty}^{\infty} e^{-i(\beta-\beta^*)jL} = 2\pi \sum_{j=-\infty}^{\infty} \delta((\beta^* - \beta)L - 2\pi j). \quad (49)$$

In fact, the term on the right hand side is a periodic function of β with period equal to $2\pi/L$, and the term on the left hand side is the Fourier series of that periodic function. Thus

$$\begin{aligned} \hat{q}_c(\beta, f) &= 2\pi [\mathbf{D}(\beta, 2\pi f)]^{-1} \\ &\times \left\{ \sum_{s=1}^S \mathbf{U}_s \hat{f}_{0s}(f) e^{-i\beta x_s} - i\beta \sum_{s=1}^S \mathbf{V}_s \hat{m}_{0s}(f) e^{-i\beta x_s} \right\} \\ &\times \sum_{j=-\infty}^{\infty} \delta((\beta^* - \beta)L - 2\pi j). \end{aligned} \quad (50)$$

The inverse Fourier transform of Eq. (50) with respect to β therefore is given by

$$\begin{aligned} \hat{q}_c(x, f) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{q}_c(\beta, f) e^{i\beta x} d\beta = \frac{1}{L} \sum_{j=-\infty}^{\infty} \left[[\mathbf{D}(\beta_j, 2\pi f)]^{-1} \right. \\ &\times \left. \left\{ \sum_{s=1}^S \mathbf{U}_s \hat{f}_{0s}(f) e^{-i\beta_j x_s} - i\beta_j \sum_{s=1}^S \mathbf{V}_s \hat{m}_{0s}(f) e^{-i\beta_j x_s} \right\} \right] e^{i\beta_j x}, \end{aligned} \quad (51)$$

where,

$$\beta_j = \beta^* - \frac{2\pi j}{L} = \frac{\Omega - 2\pi f}{c} - \frac{2\pi j}{L}. \quad (52)$$

The displacement at the interface between the r th support in the k th bay and the rail must be continuous. This requires that

$$\begin{aligned} \mathbf{U}_r^T (\hat{q}_c(x_r + kL, f) + \hat{q}_e(x_r + kL, f)) \\ = -[\mathbf{H}_{11}(f)]_r \hat{f}_{kr}(f) - [\mathbf{H}_{12}(f)]_r \hat{m}_{kr}(f), \end{aligned} \quad (53)$$

$$\begin{aligned} \mathbf{V}_r^T \frac{\partial}{\partial x} [\hat{q}_c(x_r + kL, f) + \hat{q}_e(x_r + kL, f)] \\ = -[\mathbf{H}_{21}(f)]_r \hat{f}_{kr}(f) - [\mathbf{H}_{22}(f)]_r \hat{m}_{kr}(f), \end{aligned} \quad (54)$$

where $[\mathbf{H}_{11}(f)]_r$, etc., are defined in Eq. (5), and $\hat{q}_e(x, f)$ is the frequency spectrum of the rail displacement due to the moving external loads only [15], i.e.,

$$\begin{aligned} \hat{q}_e(x, f) &= \int_{-\infty}^{\infty} \mathbf{q}_e(x, t) e^{-i2\pi ft} dt \\ &= \frac{1}{c} e^{i\beta^*(x-x_0)} [\mathbf{D}(\beta^*, 2\pi f)]^{-1} \mathbf{p}_0. \end{aligned} \quad (55)$$

According to Eqs. (45) and (46), Eqs. (53) and (54) become

$$\begin{aligned} \mathbf{U}_r^T [\hat{q}_c(x_r + kL, f) + \hat{q}_e(x_r + kL, f)] \\ = -[\mathbf{H}_{11}(f)]_r \hat{f}_{0r}(f) e^{i\beta^* kL} - [\mathbf{H}_{12}(f)]_r \hat{m}_{0r}(f) e^{i\beta^* kL}, \end{aligned} \quad (56)$$

$$\begin{aligned} \mathbf{V}_r^T \frac{\partial}{\partial x} [\hat{q}_c(x_r + kL, f) + \hat{q}_e(x_r + kL, f)] \\ = -[\mathbf{H}_{21}(f)]_r \hat{f}_{0r}(f) e^{i\beta^* kL} - [\mathbf{H}_{22}(f)]_r \hat{m}_{0r}(f) e^{i\beta^* kL}. \end{aligned} \quad (57)$$

Equations (56) and (57), combined with Eqs. (51) and (55), gives

$$\begin{aligned} \frac{1}{L} \sum_{j=-\infty}^{\infty} \mathbf{U}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \sum_{s=1}^S \mathbf{U}_s \hat{f}_{0s}(f) e^{-i\beta_j x_s} e^{i\beta_j (x_r + kL)} \\ - \frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j) \mathbf{U}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \sum_{s=1}^S \mathbf{V}_s \hat{m}_{0s}(f) e^{-i\beta_j x_s} e^{i\beta_j (x_r + kL)} \\ + [\mathbf{H}_{11}(f)]_r \hat{f}_{0r}(f) e^{i\beta^* kL} + [\mathbf{H}_{12}(f)]_r \hat{m}_{0r}(f) e^{i\beta^* kL} \\ = -\frac{1}{c} e^{i\beta^*(x_r + kL - x_0)} \mathbf{U}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \mathbf{p}_0, \end{aligned} \quad (58)$$

$$\begin{aligned} \frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j) \mathbf{V}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \sum_{s=1}^S \mathbf{U}_s \hat{f}_{0s}(f) e^{-i\beta_j x_s} e^{i\beta_j (x_r + kL)} \\ - \frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j)^2 \mathbf{V}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \sum_{s=1}^S \mathbf{V}_s \hat{m}_{0s}(f) e^{-i\beta_j x_s} e^{i\beta_j (x_r + kL)} \\ + [\mathbf{H}_{21}(f)]_r \hat{f}_{0r}(f) e^{i\beta^* kL} + [\mathbf{H}_{22}(f)]_r \hat{m}_{0r}(f) e^{i\beta^* kL} \\ = -\frac{i\beta^*}{c} e^{i\beta^*(x_r + kL - x_0)} \mathbf{V}_r^T [\mathbf{D}(\beta^*, 2\pi f)]^{-1} \mathbf{p}_0. \end{aligned} \quad (59)$$

Since $e^{i\beta_j kL} = e^{i(\beta^* - 2\pi j/L)kL} = e^{i\beta^* kL}$, Eqs. (58) and (59) are equivalent to

$$\begin{aligned}
& \frac{1}{L} \sum_{j=-\infty}^{\infty} \mathbf{U}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \sum_{s=1}^S \mathbf{U}_s \hat{\mathbf{f}}_{0s}(f) e^{i\beta_j(x_r-x_s)} \\
& - \frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j) \mathbf{U}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \sum_{s=1}^S \mathbf{V}_s \hat{\mathbf{m}}_{0s}(f) e^{i\beta_j(x_r-x_s)} \\
& + [\mathbf{H}_{11}(f)]_r \hat{\mathbf{f}}_{0r}(f) + [\mathbf{H}_{12}(f)]_r \hat{\mathbf{m}}_{0r}(f) \\
& = -\frac{1}{c} e^{i\beta^*(x_r-x_0)} \mathbf{U}_r^T [\mathbf{D}(\beta^*, 2\pi f)]^{-1} \mathbf{p}_0 \quad (r = 1, 2, \dots, S),
\end{aligned} \quad (60)$$

$$\begin{aligned}
& \frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j) \mathbf{V}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \sum_{s=1}^S \mathbf{U}_s \hat{\mathbf{f}}_{0s}(f) e^{i\beta_j(x_r-x_s)} \\
& - \frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j)^2 \mathbf{V}_r^T [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \sum_{s=1}^S \mathbf{V}_s \hat{\mathbf{m}}_{0s}(f) e^{i\beta_j(x_r-x_s)} \\
& + [\mathbf{H}_{21}(f)]_r \hat{\mathbf{f}}_{0r}(f) + [\mathbf{H}_{22}(f)]_r \hat{\mathbf{m}}_{0r}(f) \\
& = -\frac{i\beta^*}{c} e^{i\beta^*(x_r-x_0)} \mathbf{V}_r^T [\mathbf{D}(\beta^*, 2\pi f)]^{-1} \mathbf{p}_0 \quad (r = 1, 2, \dots, S).
\end{aligned} \quad (61)$$

Equations (60) and (61) can be rewritten in a more compact form:

$$\begin{aligned}
& \begin{bmatrix} \mathbf{A}_{11}(f) & \mathbf{A}_{12}(f) \\ \mathbf{A}_{21}(f) & \mathbf{A}_{22}(f) \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{f}}_0(f) \\ \hat{\mathbf{m}}_0(f) \end{Bmatrix} \\
& = -\frac{1}{c} e^{-i\beta^* x_0} \begin{bmatrix} \mathbf{B}_1(\beta^*) \\ \mathbf{B}_2(\beta^*) \end{bmatrix} [\mathbf{D}(\beta^*, 2\pi f)]^{-1} \mathbf{p}_0,
\end{aligned} \quad (62)$$

where,

$$\hat{\mathbf{f}}_0(f) = \begin{Bmatrix} \hat{\mathbf{f}}_{01}(f) \\ \hat{\mathbf{f}}_{02}(f) \\ \vdots \\ \hat{\mathbf{f}}_{0S}(f) \end{Bmatrix}, \quad \hat{\mathbf{m}}_0(f) = \begin{Bmatrix} \hat{\mathbf{m}}_{01}(f) \\ \hat{\mathbf{m}}_{02}(f) \\ \vdots \\ \hat{\mathbf{m}}_{0S}(f) \end{Bmatrix}, \quad (63)$$

$$\mathbf{A}_{11}(f) = ([\mathbf{A}_{11}(f)]_{rs})_{r,s=1,\dots,S}, \quad (64)$$

$$\begin{aligned}
[\mathbf{A}_{11}(f)]_{rs} &= \mathbf{U}_r^T \left(\frac{1}{L} \sum_{j=-\infty}^{\infty} [\mathbf{D}(\beta_j, 2\pi f)]^{-1} e^{i\beta_j(x_r-x_s)} \right) \mathbf{U}_s \\
&+ \delta_{rs} [\mathbf{H}_{11}(f)]_r,
\end{aligned} \quad (65)$$

$$\mathbf{A}_{12}(f) = ([\mathbf{A}_{12}(f)]_{rs})_{r,s=1,\dots,S}, \quad (66)$$

$$\begin{aligned}
[\mathbf{A}_{12}(f)]_{rs} &= -\mathbf{U}_r^T \left(\frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j) [\mathbf{D}(\beta_j, 2\pi f)]^{-1} e^{i\beta_j(x_r-x_s)} \right) \mathbf{V}_s \\
&+ \delta_{rs} [\mathbf{H}_{12}(f)]_r,
\end{aligned} \quad (67)$$

$$\mathbf{A}_{21}(f) = ([\mathbf{A}_{21}(f)]_{rs})_{r,s=1,\dots,S}, \quad (68)$$

$$\begin{aligned}
[\mathbf{A}_{21}(f)]_{rs} &= \mathbf{V}_r^T \left(\frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j) [\mathbf{D}(\beta_j, 2\pi f)]^{-1} e^{i\beta_j(x_r-x_s)} \right) \mathbf{U}_s \\
&+ \delta_{rs} [\mathbf{H}_{21}(f)]_r,
\end{aligned} \quad (69)$$

$$\mathbf{A}_{22}(f) = ([\mathbf{A}_{22}(f)]_{rs})_{r,s=1,\dots,S}, \quad (70)$$

$$\begin{aligned}
[\mathbf{A}_{22}(f)]_{rs} &= -\mathbf{V}_r^T \left(\frac{1}{L} \sum_{j=-\infty}^{\infty} (i\beta_j)^2 [\mathbf{D}(\beta_j, 2\pi f)]^{-1} e^{i\beta_j(x_r-x_s)} \right) \mathbf{V}_s \\
&+ \delta_{rs} [\mathbf{H}_{22}(f)]_r,
\end{aligned} \quad (71)$$

$$\mathbf{B}_1(\beta^*) = \begin{Bmatrix} e^{i\beta^* x_1} \mathbf{U}_1^T \\ e^{i\beta^* x_2} \mathbf{U}_2^T \\ \vdots \\ e^{i\beta^* x_S} \mathbf{U}_S^T \end{Bmatrix}, \quad (72)$$

$$\mathbf{B}_2(\beta^*) = \begin{Bmatrix} i\beta^* e^{i\beta^* x_1} \mathbf{V}_1^T \\ i\beta^* e^{i\beta^* x_2} \mathbf{V}_2^T \\ \vdots \\ i\beta^* e^{i\beta^* x_S} \mathbf{V}_S^T \end{Bmatrix}. \quad (73)$$

Thus the spectrum of the supporting force vectors at the 0th bay can be worked out as

$$\begin{aligned}
\begin{Bmatrix} \hat{\mathbf{f}}_0(f) \\ \hat{\mathbf{m}}_0(f) \end{Bmatrix} &= -\frac{1}{c} e^{-i\beta^* x_0} \begin{bmatrix} \mathbf{A}_{11}(f) & \mathbf{A}_{12}(f) \\ \mathbf{A}_{21}(f) & \mathbf{A}_{22}(f) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_1(\beta^*) \\ \mathbf{B}_2(\beta^*) \end{bmatrix} \\
&\times [\mathbf{D}(\beta^*, 2\pi f)]^{-1} \mathbf{p}_0 \\
&= -\frac{1}{c} e^{-i\beta^* x_0} [\mathbf{A}(f)]^{-1} \mathbf{B}(\beta^*) [\mathbf{D}(\beta^*, 2\pi f)]^{-1} \mathbf{p}_0,
\end{aligned} \quad (74)$$

where,

$$\mathbf{A}(f) = \begin{bmatrix} \mathbf{A}_{11}(f) & \mathbf{A}_{12}(f) \\ \mathbf{A}_{21}(f) & \mathbf{A}_{22}(f) \end{bmatrix}, \quad \mathbf{B}(\beta^*) = \begin{bmatrix} \mathbf{B}_1(\beta^*) \\ \mathbf{B}_2(\beta^*) \end{bmatrix}. \quad (75)$$

The displacement vector $\mathbf{q}_c(x, t)$ is determined by performing an inverse Fourier transform on Eq. (51). That is

$$\begin{aligned}
\mathbf{q}_c(x, t) &= \int_{-\infty}^{\infty} \hat{\mathbf{q}}_c(x, f) e^{i2\pi ft} df = \frac{1}{L} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \\
&\times \left(\sum_{s=1}^S \mathbf{U}_s \hat{\mathbf{f}}_{0s}(f) e^{-i\beta_j x_s} \right) e^{i\beta_j x} e^{i2\pi ft} df \\
&- \frac{1}{L} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} (i\beta_j) [\mathbf{D}(\beta_j, 2\pi f)]^{-1} \\
&\times \left(\sum_{s=1}^S \mathbf{V}_s \hat{\mathbf{m}}_{0s}(f) e^{-i\beta_j x_s} \right) e^{i\beta_j x} e^{i2\pi ft} df,
\end{aligned}$$

or,

$$\mathbf{q}_c(x, t) = \frac{1}{L} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} [D(\beta_j, 2\pi f)]^{-1} \times C(\beta_j) \left\{ \begin{matrix} \hat{\mathbf{f}}_0(f) \\ \hat{\mathbf{m}}_0(f) \end{matrix} \right\} e^{i\beta_j x} e^{i2\pi f t} df, \quad (76)$$

where $C(\beta_j)$ is a matrix of order $3n \times \sum_{s=1}^S (N_s + M_s)$, given by

$$C(\beta_j) = [e^{-i\beta_j x_1} \mathbf{U}_1 \quad \dots \quad e^{-i\beta_j x_S} \mathbf{U}_S \quad -i\beta_j e^{-i\beta_j x_1} \mathbf{V}_1 \quad \dots \quad -i\beta_j e^{-i\beta_j x_S} \mathbf{V}_S]. \quad (77)$$

Insertion of Eq. (74) into (76), gives

$$\mathbf{q}_c(x, t) = -\frac{1}{cL} \left(\sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} [D(\beta_j, 2\pi f)]^{-1} C(\beta_j) [\mathbf{A}(f)]^{-1} \times \mathbf{B}(\beta^*) [D(\beta^*, 2\pi f)]^{-1} e^{-i\beta^* x_0} e^{i\beta_j x} e^{i2\pi f t} df \right) \mathbf{p}_0. \quad (78)$$

Equation (78) is expressed in terms of an infinite integral with respect to the spectral frequency f . As explained in [15], it is more computationally convenient to express them in terms of the wavenumber β in the x -direction. The transform from spectral frequency f to wavenumber β is realized through Eq. (47), and the results are listed in Sect. 3.

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