New scenarios for classical and quantum mechanical systems with position-dependent mass

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Abstract An inhomogeneous Kaluza–Klein compactification to four dimensions, followed by a conformal transformation, results in a system with position-dependent mass (PDM). This origin of a PDM is quite different from the condensed matter one. A substantial generalization of a previously studied nonlinear oscillator with variable mass is obtained, wherein the position dependence of the mass of a nonrelativistic particle is due to a dilatonic coupling function emerging from the extra dimension. Previously obtained solutions for such systems can be extended and reinterpreted as nonrelativistic particles interacting with dilaton fields, which, themselves, can have interesting structures. An application is presented for the nonlinear oscillator, where within the new scenario the particle is coupled to a dilatonic string.

Keywords Position-dependent mass · Nonlinear quantum oscillator · Kaluza–Klein theory · Dimensional reduction

Mathematics Subject Classification 70Gxx · 70Hxx · 70Sxx · 81Sxx

1 Introduction

Quantum mechanical systems involving nonrelativistic particles with position-dependent masses have been studied by numerous authors (see, e.g., [1–7]). Such systems can arise in condensed matter settings, as pointed out, for example, in Ref. [2], but are of mathematical interests in their own right. An example of such a system is a previously studied prototype [1, 2] that derives from a classical lagrangian

$$L = \frac{1}{2} \frac{m_0}{(\lambda x^2 + 1)} \left(\dot{x}^2 - \alpha^2 x^2 \right) \quad (1)$$

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which can also be extended to higher dimensions [3, 6–9]. The particle described by (1) has an effective position-dependent mass (PDM)

$$m(x) = \frac{m_0}{(\lambda x^2 + 1)} \quad (2)$$

and is subject to an effective “spring constant” $k(x) = m(x)\alpha^2$. This problem has been studied at both the classical and quantum mechanical levels, and in various space dimensions. (See, e.g., [2, 5–7] for exact solutions of the quantum mechanical problem.)

One problem that arises for systems with a PDM is the question of how to proceed from a classical description to a quantum one, since an ordering ambiguity for momentum operators becomes apparent. The quantization can be accomplished with several approaches [2, 4, 10]. One approach, that has been used in several studies involving quantum mechanical particles with position-dependent masses, is one where the classical kinetic energy is factorized into two symmetric parts, each involving a canonical momentum. The classical momentum is then canonically replaced with its quantum counterpart, $\mathbf{p} \rightarrow -i\hbar\nabla$. A classical kinetic energy of the form $T = \frac{1}{2m_0} f^{-1}(\mathbf{x}) \mathbf{p} \cdot \mathbf{p} = \frac{1}{2m_0} (f^{-1/2}(\mathbf{x}) \mathbf{p}) \cdot (f^{-1/2}(\mathbf{x}) \mathbf{p})$ then takes its quantum form [2, 5, 7]

$$T \rightarrow \hat{T} = -\frac{\hbar^2}{2m_0} \left[f^{-1/2}(\mathbf{x}) \nabla \right] \cdot \left[f^{-1/2}(\mathbf{x}) \nabla \right] \quad (3)$$

where m_0 is a constant mass related to an effective position-dependent mass $m(\mathbf{x})$ by $m(\mathbf{x}) = f(\mathbf{x})m_0$. Other approaches exist as well, and different quantization prescriptions can result in different Hamiltonians [4, 10].

We can generalize the system in (1) by writing a lagrangian in the form (in cartesian coordinates)

$$L = \frac{1}{2}m(x) \left[\delta_{ij} u^i u^j - U_0(\mathbf{x}) \right] \quad (4)$$

where $m(x)U_0(\mathbf{x})$ represents a nonlinear potential energy, in general, and the summation convention is used. It is to be demonstrated, here, that this form of lagrangian emerges naturally from an inhomogeneous Kaluza–Klein compactification of an extra spacetime dimension to yield an effective 4D theory. The extra space dimension, when compactified, gives rise to an effective mass $m(x)$ in the effective four-dimensional theory. An inhomogeneous compactification is manifest as a spacetime-dependent scale factor $b(x^\mu)$ for the extra dimension, so that there can be regions of spacetime where the size of the extra dimension becomes larger or smaller. This can result in some interesting physical effects and objects, for example, gravitational bags [11], dimension bubbles [12], and scattering from dimensional boundaries [13]. However, interest here is focused upon the emergence and treatment of a low-energy quantum mechanical system where a position-dependent mass is involved. In the 4D theory, the function $b(x^\mu)$ is related to a dilaton coupling function which determines how strongly the dilaton field couples to matter particles.

We begin with an action describing a classical particle moving in a five-dimensional spacetime with pure Einstein gravity and the inclusion of a possible cosmological constant Λ . We then dimensionally reduce this 5D theory to an effective 4D one, where the extra-dimensional scale factor $b(x)$ is related to a scalar field $\phi(x)$ that is nonminimally coupled to the 4D Ricci scalar $\tilde{R}[\tilde{g}_{\mu\nu}]$. A conformal transformation from the 4D Jordan frame with metric $\tilde{g}_{\mu\nu}$ to an Einstein frame with metric $g_{\mu\nu}$ results in a representation where the action contains an ordinary 4D Einstein–Hilbert term, along with a scalar field ϕ derived from the scale factor b and a classical matter action. In the Einstein frame representation of the theory, the scale factor $b(x)$ is no longer coupled to the Ricci scalar, but now becomes coupled to the matter sector of the theory [16, 17].

Passing to the flat 4D spacetime limit with nonrelativistic matter, the 4D action can be recast in the form of a low-energy theory for nonrelativistic matter, along with a relativistic scalar field with a potential $V \sim \Lambda b^{-1}$. The

equations of motion for the system are obtained, and explicit, exact solutions for the scalar field are obtainable in certain cases. The classical matter particle has an attendant variable mass $m(\mathbf{x})$ in a flat spacetime.

A quantization of the Hamiltonian then yields a Schrödinger-like equation for the system. We use the quantization method described above, where the kinetic energy operator is given by (3),

$$\hat{H}\psi(\mathbf{x}) = \left[\hat{T} + \mathcal{U}_0(\mathbf{x}) \right] \psi(\mathbf{x}) = E\psi(\mathbf{x}); \quad \mathcal{U}_0(\mathbf{x}) = \frac{1}{2}m(\mathbf{x})U_0(\mathbf{x}) \quad (5)$$

which is obtained from a quantization procedure for a classical nonrelativistic particle characterized by a Lagrangian in a flat space representation of the form given by (4). An application of this method using a particular solution of the dilaton field equation is seen to produce the system of (1) and (2), generalized to two or three spatial dimensions, with $x \rightarrow r = \sqrt{x^2 + y^2}$ being a radial coordinate. In addition, there is a nontrivial dilatonic field configuration which couples to the particle oscillator. This problem has been examined previously, but without any connection of the position-dependent mass to a dilaton field. In this example, the dilaton field has a structure resembling a global cosmic string. Another difference is that the mass function $m(\mathbf{x})$ is not an arbitrary input function, but is determined by the field equation for the dilaton scalar ϕ .

2 The model

2.1 The 5D action and dimensional reduction

We begin by assuming a 5D spacetime equipped with a metric

$$d\tilde{s}_D^2 = \tilde{g}_{MN}(x^\mu, y) dx^M dx^N = \tilde{g}_{\mu\nu}(x^\alpha) dx^\mu dx^\nu - b^2(x^\alpha) dy^2 \quad (6)$$

where the 4D metric $\tilde{g}_{\mu\nu}(x)$ has a negative signature $(+, -, -, -)$. We use $\mu, \nu = 0, 1, 2, 3$ for the ordinary spacetime indices and $M, N = 0, 1, 2, 3, 5$. Absolute values of metric determinants are denoted by $\tilde{g}_5 = |\det \tilde{g}_{MN}|$, and $\tilde{g} = |\det \tilde{g}_{\mu\nu}|$, so that $\tilde{g}_5 = \tilde{g}b^2$. The action for the 5D theory that includes gravitation and matter is

$$S = \int d^5x \sqrt{\tilde{g}_5} \left\{ \frac{1}{2\kappa_5^2} [\tilde{R}_5[\tilde{g}_{MN}] - 2\Lambda] \right\} + S_m[\tilde{g}_{MN}, \dots] \quad (7)$$

where Λ is a cosmological constant, and S_m is the matter action which depends upon particle velocities and metric $\tilde{g}_{MN}$. For instance, the matter action for a free classical particle is given by

$$S_m = - \int m_0 d\tilde{s}_5 = - \int m_0 \sqrt{\tilde{g}_{MN} \tilde{u}^M \tilde{u}^N} d\tilde{s}_5 \quad (8)$$

where m_0 is a constant mass parameter in the D -dimensional theory, and $\tilde{u}^M = dx^M/d\tilde{s}_5$, and we have $\sqrt{\tilde{g}_{MN} \tilde{u}^M \tilde{u}^N} = 1$ when evaluated on the particle worldline where $d\tilde{s}_5^2 = \tilde{g}_{MN} dx^M dx^N$ holds. We have a 5D gravitational constant denoted by G_5 with $\kappa_5^2 = 8\pi G_5$. It is related to the 4D gravitational constant G by $G_5 = V_y G$, and hence $\kappa_5^2 = V_y \kappa^2$, where V_y is the “coordinate volume” of internal space, $V_y = \int dy = 2\pi R_0$. We assume that particle trajectories and fields are y independent, i.e., there are no Kaluza–Klein (KK) modes. Specifically, we assume that the classical particle’s trajectory is confined to the 4D spacetime, with $dy = 0$ along the particle’s path. Therefore, along the particle’s trajectory we have $d\tilde{s}_5^2|_{\text{path}} = d\tilde{s}^2 = \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu$ with $\tilde{u}^5 = 0$.

We dimensionally reduce the 5D theory to an effective 4D theory by performing an integration over the internal space with $d^5x = d^4x dy$ and using $\sqrt{\tilde{g}_5} = \sqrt{\tilde{g}}b(x)$. The 5D Ricci scalar $\tilde{R}[\tilde{g}_{MN}] = \tilde{g}^{MN} \tilde{R}_{MN}[\tilde{g}_{MN}]$, with $\tilde{R}_{MN}$

the Ricci tensor, can be broken into a 4D Ricci scalar $R[\tilde{g}_{\mu\nu}]$ plus terms involving the scale factor $b(x)$ and its derivatives. The result is (see, e.g., [12, 14, 15])

$$S = \int d^4x \sqrt{\tilde{g}} \left\{ \frac{1}{2\kappa^2} \left[b \tilde{R}[\tilde{g}_{\mu\nu}] + 2\tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu b - b\Lambda \right] \right\} + S_m \quad (9)$$

where $\tilde{\nabla}_\mu$ represents a covariant derivative with respect to the metric $\tilde{g}_{\mu\nu}$. The matter action for a particle of mass m_0 can be written as

$$S_m = - \int m_0 \sqrt{\tilde{g}_{\mu\nu} \tilde{u}^\mu \tilde{u}^\nu} d\tilde{s} + S_{int} \quad (10)$$

where S_{int} represents an action describing the interaction of the particle with any nongravitational sources in its environment.

Dropping a total divergence in the action, the action can be rewritten as

$$S = \int d^4x \sqrt{\tilde{g}} b \left\{ \frac{1}{2\kappa^2} \tilde{R}[\tilde{g}_{\mu\nu}] - \frac{\Lambda}{\kappa^2} \right\} + S_m. \quad (11)$$

Notice that the action of (11) contains a scalar field b that is nonminimally coupled to the Ricci scalar as appears in the term $b\tilde{R}[\tilde{g}_{\mu\nu}]$, so that we have a 4D Jordan frame representation of the theory. We can perform a conformal transformation of the metric to recast the theory in the Einstein frame representation, where the scalar field decouples from the curvature scalar, but then becomes coupled to the matter sector [16].

2.2 Conformal transformation to the Einstein frame

Let us now define the Einstein frame metric $g_{\mu\nu}$ in terms of the Jordan frame metric $\tilde{g}_{\mu\nu}$:

$$g_{\mu\nu} = b\tilde{g}_{\mu\nu}, \quad g^{\mu\nu} = b^{-1}\tilde{g}^{\mu\nu}, \quad \sqrt{g} = b^2\sqrt{\tilde{g}} \quad (12)$$

The action in the Einstein frame representation now takes the form [13, 14]

$$S = \int d^4x \sqrt{g} \left\{ \frac{1}{2\kappa^2} \left[R[g_{\mu\nu}] + \frac{3}{2} b^{-2} g^{\mu\nu} (\nabla_\mu b)(\nabla_\nu b) \right] - b^{-1} \frac{\Lambda}{\kappa^2} \right\} + S_m[b^{-1}g_{\mu\nu}, \dots]. \quad (13)$$

Now define the scalar field $\phi(x)$ by

$$b = e^{a\phi}, \quad \phi = \frac{1}{a} \ln b, \quad a = \sqrt{\frac{2}{3}} \kappa \quad (14)$$

where a is a constant that is determined by requiring the kinetic term in the action to take the canonical form $\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$. Then, the scalar field ϕ is explicitly related to the extra-dimensional scale factor b by

$$\phi(x) = \sqrt{\frac{3}{2}} \frac{1}{\kappa} \ln b(x), \quad b(x) = \exp \left[\sqrt{\frac{2}{3}} \kappa \phi(x) \right] \quad (15)$$

and the action takes the form

$$S = \int d^4x \sqrt{g} \left\{ \frac{1}{2\kappa^2} R[g_{\mu\nu}] + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - e^{-a\phi} \frac{\Lambda}{\kappa^2} \right\} + S_m[e^{-a\phi} g_{\mu\nu}, \dots] \quad (16a)$$

$$= S_R + S_\phi + S_m \quad (16b)$$

with $e^{-a\phi} = b^{-1}$, and we define

$$S_R = \int d^4x \sqrt{g} \frac{1}{2\kappa^2} R[g_{\mu\nu}], \quad S_\phi = \int d^4x \sqrt{g} \left[\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right], \quad V(\phi) = \frac{\Lambda}{\kappa^2} e^{-a\phi}. \quad (17)$$

The scalar field ϕ has canonical mass dimension 1, and $\kappa\phi$ is dimensionless.

The Einstein frame action contains an Einstein–Hilbert term for gravity, along with an action for the scalar field ϕ . In addition, there is a matter action where matter fields have an anomalous coupling to the scalar field. In passing from the Jordan frame to the Einstein frame, the coupling of the scalar to the gravitational sector has been shifted to a coupling of the scalar to the matter sector.

3 Nonrelativistic matter

Before obtaining an appropriate Schrödinger equation for a nonrelativistic matter particle, we first focus on the nonrelativistic limit of the action for a classical particle. The matter action given by (10) can be rewritten in the Einstein frame. The free particle portion of the action $S_m = S_{\text{free}} + S_{\text{int}}$ is given by

$$S_{\text{free}} = - \int m_0 d\tilde{s} = - \int m_0 b^{-1/2} ds = - \int m ds = - \int m_0 b^{-1/2} \sqrt{g_{\mu\nu} u^\mu u^\nu} ds. \quad (18)$$

This shows that the particle mass in the Einstein frame representation is given by [16]

$$m(x) = m_0 b^{-1/2}. \quad (19)$$

Furthermore, in the nonrelativistic limit $ds = dt$, $u^i \ll u^0 = 1$ and we identify the nonrelativistic portion of the classical Lagrangian as

$$L_{\text{free}} = -m_0 b^{-1/2} (g_{\mu\nu} u^\mu u^\nu)^{1/2} = -m_0 b^{-1/2} (u_0 u^0 + g_{ij} u^i u^j)^{1/2} = m_0 b^{-1/2} \left[\frac{1}{2} (\mathbf{u} \cdot \mathbf{u}) - 1 \right] \quad (20)$$

where use has been made of $-g_{ij} u^i u^j = \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \ll 1$.

Next, we write the nonrelativistic limit of the interaction term in S_m as

$$S_{\text{int}} = \int L_{\text{int}} d\tilde{s} = - \int U(\mathbf{x}) d\tilde{s} = - \int U(\mathbf{x}) b^{-1/2} ds \approx - \int b^{-1/2} U(\mathbf{x}) dt \quad (21)$$

and $L_{\text{int}} = b^{-1/2} U(\mathbf{x})$, where $U(\mathbf{x})$ is an arbitrary potential energy function defined in the Jordan frame, describing interactions of the particle with nongravitational forces. The nonrelativistic limit of the matter action is then

$$S_m = \int L dt = \int dt b^{-1/2} \left\{ \left[\frac{1}{2} m_0 \mathbf{u} \cdot \mathbf{u} \right] - [U(\mathbf{x}) + m_0] \right\}. \quad (22)$$

The potential $U(\mathbf{x})$ can be redefined to absorb the constant m_0 , i.e., $U(\mathbf{x}) + m_0 \rightarrow U(\mathbf{x})$, and we can therefore write the Lagrangian for the nonrelativistic particle in the Einstein frame representation as

$$L = b^{-1/2} \left[\frac{1}{2} m_0 \mathbf{u} \cdot \mathbf{u} - U(\mathbf{x}) \right] \equiv b^{-1/2} L_0 \quad (23)$$

where $L_0 \equiv L|_{b=1}$. If we define $U_0(\mathbf{x}) = 2U(\mathbf{x})/m_0$, then (23) takes the form, in cartesian coordinates,

$$L = \frac{1}{2} m(x) \left[\delta_{ij} u^i u^j - U_0(\mathbf{x}) \right] \quad (24)$$

which coincides with that in (4), in the limit of a flat spacetime where $g_{00} = 1$, $g_{ij} = -\delta_{ij}$ (for cartesian coordinates).

4 The flat space nonrelativistic theory

We want to focus on a laboratory type of setting for a nonrelativistic particle, i.e., a setting where the explicit curvature of spacetime can be ignored. In other words, we consider a small enough region of space where it becomes reasonable to approximate the spacetime as Minkowskian, with $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat space Minkowski metric, appropriately expressed in terms of the coordinate system used. (For cartesian coordinates we have $\eta_{00} = \eta^{00} = 1$, $\eta_{ij} = \eta^{ij} = -\delta_{ij} = -\delta^{ij}$). In this approximation, the curvature action S_R given by (17) is set to zero, and the action S of (16) for the system reduces to

$$S = S_\phi + S_m = \int d^4x \sqrt{\eta} \left[\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - b^{-1} \frac{\Lambda}{\kappa^2} \right] + S_m \quad (25a)$$

$$= \int d^4x \sqrt{\eta} \left[\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - e^{-a\phi} \frac{\Lambda}{\kappa^2} \right] + \int dt e^{-a\phi/2} \left[\frac{1}{2} m_0 (-\eta_{ij}) u^i u^j - U(\mathbf{x}) \right] \quad (25b)$$

where (23) has been used for the matter Lagrangian, and the metric $\eta_{\mu\nu}$ has been left explicitly in the expressions, with a reminder that the metric is one for flat spacetime, with an arbitrary choice for cartesian or curvilinear coordinates. (We have $\eta_{00} = 1$, and η_{ij} is just the negative of the metric for a Euclidean space, with $-\eta_{ij} > 0$ for $i \neq j$, and $\eta = |\det \eta_{\mu\nu}|$.) The parameterization $b = e^{a\phi}$, as given by (14), has also been used. A lagrangian density $\mathcal{L}_m$ for the matter particle can be defined by writing $\mathcal{L}_m = \sqrt{\eta} L \delta^{(3)}(\mathbf{x} - \mathbf{x}_p)$ where $\mathbf{x}_p$ locates the instantaneous position of the particle. Then, (25) can be expressed as

$$S = \int d^4x \sqrt{\eta} \left[\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - e^{-a\phi} \frac{\Lambda}{\kappa^2} \right] + \int d^4x \sqrt{\eta} e^{-a\phi/2} \left[\frac{1}{2} m_0 (-\eta_{ij}) u^i u^j - U(\mathbf{x}) \right] \delta^{(3)}(\mathbf{x} - \mathbf{x}_p). \quad (26)$$

From this action, the equation of motion for the field ϕ can be obtained:

$$\square \phi + \frac{\partial V(\phi)}{\partial \phi} - \sigma = 0; \quad V(\phi) = e^{-a\phi} \frac{\Lambda}{\kappa^2}, \quad \sigma = \frac{\partial \mathcal{L}_m}{\partial \phi} = \left(-\frac{a}{2} \right) \sqrt{\eta} L \delta^{(3)}(\mathbf{x} - \mathbf{x}_p) \quad (27)$$

where $\square = \nabla_\mu \partial^\mu = \partial_t^2 - \nabla^2$. (The field ϕ is referred to as a 4D “dilaton” field.) Since $\mathcal{L}_m \propto \delta^{(3)}(\mathbf{x} - \mathbf{x}_p)$, we take the σ term to vanish at all $\mathbf{x} \neq \mathbf{x}_p$. So, at least as a first approximation, we set $\sigma = 0$ in the equation of motion to obtain a background solution for $\phi(x)$ that can couple to the matter particle in L . The equation of motion for ϕ therefore reads

$$\square\phi - a\frac{\Lambda}{\kappa^2}e^{-a\phi} = 0. \quad (28)$$

Here, we focus attention on static solutions of (28), which satisfy

$$\nabla^2\phi = -a\frac{\Lambda}{\kappa^2}e^{-a\phi}. \quad (29)$$

Solutions to this equation can be found for certain specific cases.

As an example, we consider $\phi(x, y)$ to be a function of the two cartesian coordinates x and y , and (29) reads as

$$\left(\partial_x^2 + \partial_y^2\right)\phi = -a\frac{\Lambda}{\kappa^2}e^{-a\phi} = 2\tilde{\kappa}\frac{\Lambda}{\kappa^2}e^{2\tilde{\kappa}\phi} \quad (30)$$

where $\tilde{\kappa} \equiv -\frac{1}{2}a$. Then (30) is recognized as the 2D Liouville differential equation [18, 19] (also see, for example, [20, 21]). Upon defining $\zeta = x + iy$, the solutions are given by [19, 20]

$$b^{-1} = e^{2\tilde{\kappa}\phi} = \frac{2\kappa^2}{\tilde{\kappa}^2|\Lambda|} \frac{|f'(\zeta)|^2}{[|f(\zeta)|^2 + 1]^2}, \quad (\Lambda = -|\Lambda| < 0) \quad (31a)$$

$$b^{-1} = e^{2\tilde{\kappa}\phi} = \frac{2\kappa^2}{\tilde{\kappa}^2|\Lambda|} \frac{|f'(\zeta)|^2}{[|f(\zeta)|^2 - 1]^2}, \quad (\Lambda = +|\Lambda| > 0) \quad (31b)$$

where $f(\zeta)$ is a holomorphic function of ζ , and $f'(\zeta) = df/d\zeta$. The matter coupling function $b^{-\frac{1}{2}} = e^{\tilde{\kappa}\phi}$ that appears in the matter particle lagrangian L is

$$b^{-\frac{1}{2}} = e^{\tilde{\kappa}\phi} = \frac{\sqrt{2}\kappa}{|\tilde{\kappa}|\sqrt{|\Lambda|}} \frac{|f'(\zeta)|}{[|f(\zeta)|^2 \pm 1]}, \quad (\Lambda = \mp|\Lambda|). \quad (32)$$

Different choices for the function $f(\zeta)$ give different mathematical solutions to Liouville's equation. (See, e.g., [20, 22] for some applications to dilaton gravity and low-energy string theory.)

5 Quantization

The classical lagrangian for a particle in the flat space nonrelativistic limit is given by (23), for instance. In terms of cartesian coordinates, we can write the lagrangian as

$$L = b^{-1/2} \left[\frac{1}{2} m_0 \delta_{ij} u^i u^j - U(\mathbf{x}) \right] = T - \mathcal{U}; \quad \mathcal{U} = b^{-1/2} U(\mathbf{x}). \quad (33)$$

(For cartesian coordinates the spatial metric, with positive signature, is $-g_{ij} = \delta_{ij}$, but one could use other coordinates with an appropriate metric $\delta_{ij} \rightarrow \gamma_{ij}$).

As noted in (19), the effective mass of the particle in the Einstein frame representation is

$$m(\mathbf{x}) = m_0 b^{-1/2} = m_0 e^{-a\phi/2} \quad (34)$$

where we assume that $b = b(\mathbf{x})$ is time independent. The classical kinetic energy is

$$T = \frac{1}{2} m_0 b^{-1/2} \delta_{ij} u^i u^j = \frac{1}{2} m(\mathbf{x}) \delta_{ij} u^i u^j = \frac{1}{2} m(\mathbf{x}) \mathbf{u} \cdot \mathbf{u}. \quad (35)$$

The canonically conjugate momentum is

$$p^i = p_i = \frac{\partial L}{\partial u^i} = m_0 b^{-1/2} u^i = m(\mathbf{x}) u^i; \quad (36a)$$

$$u^i = \frac{p^i}{m(\mathbf{x})} = b^{1/2} \frac{p^i}{m_0}. \quad (36b)$$

Therefore, in terms of the canonical momentum, the kinetic energy is

$$T = b^{1/2} \delta_{ij} \frac{p^i p^j}{2m_0} = b^{1/2} \frac{\mathbf{p} \cdot \mathbf{p}}{2m_0}. \quad (37)$$

We can now implement the quantization procedure described in the Introduction, and used in previous works [2,4,5,7]. We write T in a symmetrical form

$$T = \frac{1}{2m_0} \left(b^{1/4} \mathbf{p} \right) \cdot \left(b^{1/4} \mathbf{p} \right) \quad (38)$$

and then make the canonical quantum replacement $\mathbf{p} \rightarrow -i\hbar \nabla$ to obtain the kinetic part of the quantum Hamiltonian $\hat{H}$:

$$T \rightarrow \hat{T} = -\frac{\hbar^2}{2m_0} \left(b^{1/4} \nabla \right) \cdot \left(b^{1/4} \nabla \right) = -\frac{\hbar^2}{2m_0} \left[b^{1/2} \nabla^2 + b^{1/4} \left(\nabla b^{1/4} \right) \cdot \nabla \right]. \quad (39)$$

The classical Hamiltonian

$$H = T + b^{-1/2} U(\mathbf{x}) = T + \mathcal{U}(\mathbf{x}) \quad (40)$$

is then replaced by a quantum Hamiltonian

$$\hat{H} = \hat{T} + \mathcal{U}(\mathbf{x}) = \hat{T} + b^{-1/2} U(\mathbf{x}) \quad (41)$$

where $\hat{T}$ is given by (39).

6 Application: a dilaton string background

6.1 The nonrelativistic quantum particle

As an example we consider a particle subjected to harmonic oscillation in the x - y plane with translation invariance in the z direction, or a particle subject to oscillation in a two-dimensional space of the x - y plane. We employ the quantization procedure described in the previous section. Consider now a solution of the 2D Liouville equation for which $f(\zeta) = A\zeta$, where $\zeta = x + iy$, and hence $f'(\zeta) = A$, $|f(\zeta)|^2 = A^2(x^2 + y^2) = A^2 r^2$. Then, (for $\Lambda = -|\Lambda|$) (32) gives a solution (see, for example, [20,22])

$$b^{-1/2} = \frac{C}{(1 + A^2 r^2)}, \quad C = \frac{\sqrt{2\kappa} A}{|\tilde{\kappa}| \sqrt{|\Lambda|}}; \quad b^{1/2} = \frac{(1 + A^2 r^2)}{C}; \quad b^{1/4} = \frac{(1 + A^2 r^2)^{1/2}}{\sqrt{C}}. \quad (42)$$

Using

$$\left(\partial_r b^{1/4}\right) = \frac{(1 + A^2 r^2)^{-1/2}}{\sqrt{C}} \cdot A^2 r \quad (43)$$

(39) yields

$$\hat{T} = -\frac{\hbar^2}{2m_0} \left[\frac{(1 + A^2 r^2)}{C} \nabla^2 + \frac{A^2 r}{C} \partial_r \right]. \quad (44)$$

Interestingly, this is the same form of the kinetic Hamiltonian that has been studied by various authors for a nonlinear quantum oscillator with position-dependent mass [2, 4, 5, 7]. In fact, if we choose a potential $U(r) = \frac{1}{2} K r^2$, we have a potential term in the Hamiltonian $\hat{H}$ given by

$$\mathcal{U}(r) = b^{-1/2} U(r) = \frac{CK}{2} \frac{r^2}{(1 + A^2 r^2)}. \quad (45)$$

Upon choosing $C = 1$, $A^2 = \lambda > 0$, and $K = m_0 \alpha^2$, the Hamiltonian is [23]

$$\hat{H} = -\frac{\hbar^2}{2m_0} \left[(1 + \lambda r^2) \nabla^2 + \lambda r \partial_r \right] + \frac{1}{2} \frac{m_0 \alpha^2 r^2}{(1 + \lambda r^2)}. \quad (46)$$

This system and related systems have been studied extensively for the cases of 1, 2, 3, and n dimensions of space [2–7]. See, e.g., [2, 5–7] for exact expressions for the normalizable, orthogonal eigenfunctions, along with the spectrum of energy eigenvalues for a class of Hamiltonians of this type. In those works, however, the mass function $m(\mathbf{x})$ is an assumed input function, whereas here it emerges as a solution of the 2D Liouville equation associated with the dilaton field $\phi(r)$.

Previous studies of this system have focused largely on its mathematical properties, with some mention that such systems may achieve physical realizations in condensed matter settings, as pointed out in [2], for instance. Here, however, we see this type of system being realized in the context of an inhomogeneous compactification of extra space dimensions, resulting in a quantum particle in “ordinary” space having an effective position-dependent mass, whose variation depends upon a variation of the dilaton field, or equivalently, a variation in the size of the extra dimensions. The effective mass is

$$m(r) = m_0 b^{-1/2}(r) = \frac{m_0 C}{(1 + A^2 r^2)} \quad (47)$$

in an effective potential

$$\mathcal{U}(r) = b^{-1/2}(r) U(r) \propto \frac{r^2}{(1 + A^2 r^2)}. \quad (48)$$

There is a discrete spectrum of bound states [7], where we could consider nonrelativistic particles as becoming trapped within the “dilaton” string-like core described by the dilaton coupling function $b^{-1/2}$, for which the size of the extra dimensions is

$$b(r) \sim (1 + A^2 r^2)^2. \quad (49)$$

So, the coupling function $b^{-1/2} = e^{\tilde{\kappa}\phi}$ has a maximum in the string core where $b \sim 1$, and outside the core the coupling function $b^{-1/2}(r)$ decreases, and the extra-dimensional scale factor $b(r)$ increases (i.e., the extra dimensions get bigger). Bound state particles are then localized within the dilatonic string core, where the extra dimensions are small, and the effective particle mass is large.

6.2 The dilaton string

We choose a negative cosmological constant, $\Lambda = -|\Lambda|$, so that from (27) the dilaton string has a potential given by

$$V(\phi) = -\frac{|\Lambda|}{\kappa^2} e^{-a\phi} = -\frac{|\Lambda|}{\kappa^2} e^{2\tilde{\kappa}\phi} = -\frac{|\Lambda|}{\kappa^2} b^{-1}. \quad (50)$$

The energy density of the scalar field ϕ is

$$\mathcal{H} = \mathcal{H}_{\text{kin}} + \mathcal{H}_{\text{pot}} = \frac{1}{2}(\partial_r \phi)^2 + V(\phi) \quad (51)$$

with $r = \sqrt{x^2 + y^2}$ being the radial distance from the center of the vortex (2D space dimensions) or the center of the string (which is centered on the z -axis in a 3D space). We now adopt the settings $C = 1$ and $A^2 = \lambda > 0$. We then have

$$b = e^{a\phi} = \left(\lambda r^2 + 1\right)^2, \quad \phi = -\frac{1}{\tilde{\kappa}} \ln\left(\lambda r^2 + 1\right) = -\frac{1}{\tilde{\kappa}} \ln(\xi + 1), \quad \xi \equiv \lambda r^2 \quad (52)$$

where $\xi = \lambda r^2$ is a dimensionless distance parameter. The energy per unit length of the string, i.e., the string tension, is (assuming, for simplicity, z independence)

$$\mu = \mu_{\text{kin}} + \mu_{\text{pot}} = 2\pi \int_0^{r_C} (\mathcal{H}_{\text{kin}} + \mathcal{H}_{\text{pot}}) r dr = 2\pi \int_0^{r_C} \mathcal{H}(r) r dr = \frac{\pi}{\lambda} \int_0^{\xi_C} \mathcal{H}(\xi) d\xi \quad (53)$$

where r_C and ξ_C are large distance cutoffs, as the tension diverges logarithmically, as with a global cosmic string [24, 25]. (This dilaton string resembles the string studied in [20, 22], with the cosmological constant replacing a constant magnetic H field.) Using (50)–(53), some calculation yields

$$\frac{\mu}{2\pi} = \frac{1}{\tilde{\kappa}^2} \left[\ln(\xi_C + 1) + \frac{1}{(\xi_C + 1)} - 1 \right] - \frac{|\Lambda|}{2\lambda\kappa^2} \left[1 - \frac{1}{(\xi_C + 1)} \right]. \quad (54)$$

For $\xi_C \gg 1$ ($r_C \gg 1/\sqrt{\lambda}$) this simplifies to

$$\frac{\mu}{2\pi} \approx \frac{1}{\tilde{\kappa}^2} (\ln \xi_C - 1) - \frac{|\Lambda|}{2\lambda\kappa^2}. \quad (55)$$

We note that since the potential is negative, but the kinetic term is positive, one can find

$$\left| \frac{\mathcal{H}_{\text{kin}}}{\mathcal{H}_{\text{pot}}} \right| = \frac{\frac{1}{2}(\partial_r \phi)^2}{|V(\phi)|} = \frac{2\lambda^2\kappa^2}{|\Lambda|\tilde{\kappa}^2} r^2 \quad (56)$$

so that the energy density becomes negative within a certain critical radius:

$$\mathcal{H} \leq 0 \quad \text{for} \quad r \leq r_{crit} = \frac{|\Lambda|^{1/2} |\tilde{\kappa}|}{\sqrt{2\lambda\kappa}}, \quad \xi_{crit} = \frac{|\Lambda| \tilde{\kappa}^2}{2\lambda\kappa^2} \quad (57)$$

so that inside this portion of the string core, the weak energy condition is violated. However, the string tension is positive, $\mu > 0$, provided that $\xi_C \gg 1$, and the condition

$$\ln(\xi_C + 1) > 1 + \xi_{crit} \quad (58)$$

is satisfied.

7 Summary

Beginning with a classical particle propagating in a 4D subspace of a 5D spacetime, where the internal space dimension gets inhomogeneously, toroidally, compactified, we arrive at an effective 4D theory for a classical particle. When a conformal transformation is made from the resulting 4D Jordan frame to a 4D Einstein frame, the particle acquires a position-dependent mass. The classical nonrelativistic system in a flat Einstein frame spacetime can be quantized using existing quantization procedures. The result is the emergence of a quite general class of quantum systems that have position-dependent mass and somewhat arbitrary potentials. Systems of this type, e.g., the nonlinear oscillator with a PDM, emerging from different physical circumstances, have been studied already. Here, we show another way in which this quantum system and generalizations of it can arise. It is also seen that within the type of scenario considered here there is a somewhat different physical interpretation involving a particle interacting with a dilaton field, which is associated with the size of the extra dimension.

The hope is that a possible physical realization of such systems, in the context of higher-dimensional physics, will provide motivation for their further study, along with new physical interpretations of results materializing from these systems. We have presented an application of this formalism to obtain a previously studied system describing a nonlinear quantum oscillator with a position-dependent mass. In the scenario presented here, however, the effective position-dependent particle mass depends explicitly upon the scalar dilaton field (i.e., the size of the extra dimension), which itself may have nontrivial, interesting structure.

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