Decoherence of weak values in a pure dephasing process

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Abstract Decoherence of weak values is studied by means of an exactly solvable model of pure dephasing. Expressions of weak values are derived for an irreversibly evolved qubit from a pre-selected state to a post-selected one. The condition that weak values are out of a spectral range of a measured observable is found. The anomaly of weak values is suppressed when qubit-environment coupling strength and environmental temperature becomes large. The parameter region is found where the anomaly of weak values can be observed. Furthermore, it is shown that initial correlation between qubit and environment reduces the anomaly of weak values.

Keywords Weak value · Weak measurement · Qubit · Decoherence · Dephasing process

1 Introduction

A measurement process is indispensable for gaining information on a quantum system, during which a measured system probabilistically and irreversibly changes its quantum state. The von Neumann measurement or the projective measurement is widely used for investigating quantum mechanical properties [1,2]. Generalized measurement is also applied [3,4]. When an observable $\hat{A}$ with eigenvalue a and eigenstate $|a\rangle$ is measured, a measured system initially prepared in a quantum state $|\psi\rangle$ changes into the eigenstate $|a\rangle$ with probability $|\langle a|\psi\rangle|^2$. Obviously the average value $\langle\psi|\hat{A}|\psi\rangle$ derived from the measurement outcomes is real and upper (lower) bounded by the maximum (minimum) eigenvalue of the measured observable $\hat{A}$. But this is not all about quantum measurement. In 1988, Aharonov, Albert and Vaidman have proposed the weak measurement which is performed on a quantum system between pre- and post-selection [5,6]. When initial and final states of the measured system are $|\psi_i\rangle$ and $|\psi_f\rangle$ which are not orthogonal, the weak value $A_w = \langle\psi_f|\hat{A}|\psi_i\rangle/\langle\psi_f|\psi_i\rangle$ of an observable $\hat{A}$ can be obtained from the measurement outcomes yielded by a measuring device which is weakly coupled with the measured system. The weak value may be complex and the real and imaginary parts can exceed the spectral range of the observable. After

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the seminal proposal, many authors have investigated the properties of weak values [7–26]. Furthermore, weak measurement together with pre- and post-selection has been applied to the direct measurement of a complex wave function [27] and the amplification of a weak signal which has experimentally been realized [28–30].

In many of the studies on weak values, a system to be measured and measuring device are assumed to be isolated from an environment and thus the time evolution of the total system is unitary. However, since we cannot avoid an influence of an environment in a realistic situation [31], it is important to consider how an environment affects weak values. In this paper, to investigate decoherence of weak values caused by an environment, we apply an exactly solvable model of pure dephasing, where a system to be measured is a qubit (or a two-level state) and an environment consists of independent harmonic oscillators in thermal equilibrium [32, 33]. In particular, we will concentrate our attention on examining whether the weak-value amplification under the influence of the environment can be observed.

This paper is organized as follows. In Sect. 2, we provide an expression of a weak value under the influence of an environment. In Sect. 3, we briefly explain an exactly solvable model of pure dephasing, which includes qubit-environment initial correlation. In Sect. 4, we investigate the weak values of qubit observables in the dephasing process. In Sect. 5, we give numerical analysis and show the property of weak values. In Sect. 6, we summarize our results.

2 Weak value influenced by environment

In this section, we provide a general expression of a weak value which is influenced by an environment. Such a weak value has also been considered from a different point of view in Ref. [16]. Suppose that a system to be measured and an environment are initially in a quantum state $\hat{\rho}_{SE}$ at time t_i . The whole system evolves with time until a measurement is performed at time t_m . The system just before measurement is in the quantum state $\hat{W}(t_m) = \hat{U}(t_m - t_i)\hat{\rho}_{SE}\hat{U}^\dagger(t_m - t_i)$, where $\hat{U}(t)$ is a unitary time evolution operator of the system and environment. To measure a system observable $\hat{A}_S$, we prepare a measuring device in a quantum state $\hat{\rho}_M$. The interaction Hamiltonian between them is assumed to be $\hat{H}_{int} = g\delta(t - t_m)\hat{A}_S \otimes \hat{P}_M$ [1], where $\hat{P}_M$ is a momentum operator of the measuring device, canonically conjugate to a pointer position $\hat{Q}_M$, and the parameter g stands for a coupling constant. For the sake of simplicity, we set $\hbar = 1$ throughout this paper. After the interaction with the measuring device, the whole state becomes $\hat{W}'(t_m) = e^{-ig\hat{A}_S \otimes \hat{P}_M}\hat{W}(t_m) \otimes \hat{\rho}_M e^{ig\hat{A}_S \otimes \hat{P}_M}$, and then it evolves again with time until post-selection is performed at time t_f . Hence, the total quantum system just before post-selection is given by

$$\hat{W}(t_f) = \hat{U}(t_f - t_m)e^{-ig\hat{A}_S \otimes \hat{P}_M}\hat{W}(t_m) \otimes \hat{\rho}_M e^{ig\hat{A}_S \otimes \hat{P}_M}\hat{U}^\dagger(t_f - t_m). \quad (1)$$

Since the weak measurement is considered, we expand the quantum state $\hat{W}(t_f)$ up to the first order with respect to the coupling constant g [5, 16],

$$\begin{aligned} \hat{W}(t_f) \approx & \hat{U}(t_f - t_i)\hat{\rho}_{SE}\hat{U}^\dagger(t_f - t_i) \otimes \hat{\rho}_M - ig\hat{U}(t_f - t_m)\hat{A}_S\hat{U}(t_m - t_i)\hat{\rho}_{SE}\hat{U}^\dagger(t_f - t_i) \otimes \hat{P}_M\hat{\rho}_M \\ & + ig\hat{U}(t_f - t_i)\hat{\rho}_{SE}\hat{U}^\dagger(t_m - t_i)\hat{A}_S\hat{U}^\dagger(t_f - t_m) \otimes \hat{\rho}_M\hat{P}_M. \end{aligned} \quad (2)$$

The post-selection of the system can be described by positive operator-valued measure (POVM) which satisfies the relations $\hat{\pi}_f \geq 0$ and $\sum_f \hat{\pi}_f = \hat{I}_S$ [3, 4], where $\hat{I}_S$ is an identity operator of the system and the summation is taken over all possible selections. After the post-selection f is succeeded, the quantum state of the measuring device is given by $\hat{\rho}_M^f = \text{Tr}_{SE}[\hat{\pi}_f \hat{W}(t_f)] / \text{Tr}_{SE} \text{Tr}_M[\hat{\pi}_f \hat{W}(t_f)]$, where Tr_{SE} and Tr_M stand for trace operations of the system-environment and the measuring device. Since we can obtain from Eq. (2),

$$\begin{aligned} \text{Tr}_{SE}[\hat{\pi}_f \hat{W}(t_f)] \approx & \text{Tr}_{SE}[\hat{\pi}_f \hat{U}(t_f - t_i)\hat{\rho}_{SE}\hat{U}^\dagger(t_f - t_i)] \left(\hat{\rho}_M - igA_w\hat{P}_M\hat{\rho}_M + igA_w^*\hat{\rho}_M\hat{P}_M \right) \\ \approx & \text{Tr}_{SE}[\hat{\pi}_f \hat{U}(t_f - t_i)\hat{\rho}_{SE}\hat{U}^\dagger(t_f - t_i)] e^{-igA_w\hat{P}_M}\hat{\rho}_M e^{igA_w^*\hat{P}_M}, \end{aligned} \quad (3)$$

where A_w is the weak value of the observable $\hat{A}_S$,

$$A_w = \frac{\text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{U}(t_f - t_m) \hat{A}_S \hat{U}(t_m - t_i) \hat{\rho}_{\text{SE}} \hat{U}^\dagger(t_f - t_i)]}{\text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{U}(t_f - t_i) \hat{\rho}_{\text{SE}} \hat{U}^\dagger(t_f - t_i)]}, \quad (4)$$

the quantum state of the measuring device after the post-selection of the system becomes

$$\hat{\rho}_M^f = \frac{e^{-igA_w \hat{P}_M} \hat{\rho}_M e^{igA_w^* \hat{P}_M}}{\text{Tr}_M[e^{2g(\text{Im}A_w) \hat{P}_M} \hat{\rho}_M]}. \quad (5)$$

The average shift of the pointer observable $\hat{Q}_M$ caused by the interaction with the system is given by $\Delta Q_M = \text{Tr}_M[\hat{Q}_M(\hat{\rho}_M^f - \hat{\rho}_M)]$. Up to the first order with respect to the coupling constant g , we can obtain [14, 20]

$$\Delta Q_M = g(\text{Re}A_w) + g(\text{Im}A_w)\langle\{\Delta \hat{Q}_M, \Delta \hat{P}_M\}\rangle, \quad (6)$$

where we set $\langle \cdots \rangle = \text{Tr}_M[\cdots \hat{\rho}_M]$, $\Delta \hat{X} = \hat{X} - \langle \hat{X} \rangle$ and $\{\hat{X}, \hat{Y}\} = \hat{X}\hat{Y} + \hat{Y}\hat{X}$. If the probability density flux of the measuring device is zero, the equality $\langle q | \{\hat{P}_M, \hat{\rho}_M\} | q \rangle = 0$ with $\hat{Q}_M | q \rangle = q | q \rangle$ holds [12, 14], which yields $\langle \{\hat{Q}_M, \hat{P}_M\} \rangle = 0$ and $\langle \hat{P}_M \rangle = 0$. In this case, we obtain $\Delta Q_M = g(\text{Re}A_w)$. On the other hand, the average shift of the device momentum is given by $\Delta P_M = \text{Tr}_M[\hat{P}_M(\hat{\rho}_M^f - \hat{\rho}_M)] = 2g(\text{Im}A_w)\langle(\Delta \hat{P}_M)^2\rangle$ [14, 20]. Therefore, we can obtain the real part and the imaginary part of the weak value from the outcomes yielded by the measuring device. Although we have considered a one-dimensional system as a measuring device, we can also use a qubit device system for a weak measurement [17, 18]. Even in this case, we can obtain the weak value from the measurement outcomes though the relations between the weak value and average values of a device observable are different.

Before closing this section, we briefly consider the case that the system-device coupling is not weak [20–23]. In this case, we first expand Eq. (1) as

$$\begin{aligned} \hat{W}(t_f) &= \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} (-1)^k \hat{U}(t_f - t_m) \hat{A}^{n-k} \hat{U}(t_m - t_i) \hat{\rho}_{\text{SE}} \\ &\quad \times \hat{U}^\dagger(t_m - t_i) \hat{A}^k \hat{U}^\dagger(t_f - t_m) \otimes \hat{P}_M^{n-k} \hat{\rho}_M \hat{P}_M^k. \end{aligned} \quad (7)$$

After the post-selection of the system, the quantum state of the measuring device becomes $\hat{\rho}_M^f = \text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{W}(t_f)] / \text{Tr}_{\text{SE}} \text{Tr}_M[\hat{\pi}_f \hat{W}(t_f)]$. Here we define

$$A_w^{k,l} = \frac{\text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{U}(t_f - t_m) \hat{A}_S^k \hat{U}(t_m - t_i) \hat{\rho}_{\text{SE}} \hat{U}^\dagger(t_m - t_i) \hat{A}_S^l \hat{U}^\dagger(t_f - t_m)]}{\text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{U}(t_f - t_i) \hat{\rho}_{\text{SE}} \hat{U}^\dagger(t_f - t_i)]}, \quad (8)$$

where $(A_w^{k,l})^* = A_w^{l,k}$ and $A_w^{1,0} = A_w$. This is the time-dependent version of the quantity introduced in Ref. [20]. Using Eqs. (7) and (8), we can derive

$$\begin{aligned} \text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{W}(t_f)] &= \text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{U}(t_f - t_i) \hat{\rho}_{\text{SE}} \hat{U}^\dagger(t_f - t_i)] \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} (-1)^k A_w^{n-k,k} \hat{P}_M^{n-k} \hat{\rho}_M \hat{P}_M^k. \end{aligned} \quad (9)$$

Assuming that $\text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{U}(t_f - t_i) \hat{\rho}_{\text{SE}} \hat{U}^\dagger(t_f - t_i)] \neq 0$, we find the quantum state of the measuring device after the post-selection of the system,

$$\hat{\rho}_M^f = \frac{\sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} (-1)^k A_w^{n-k,k} \hat{P}_M^{n-k} \hat{\rho}_M \hat{P}_M^k}{\sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \langle \hat{P}_M^n \rangle \sum_{k=0}^n \frac{n!}{k!(n-k)!} (-1)^k A_w^{n-k,k}}, \quad (10)$$

which is equivalent to that derived in Ref. [20] if we can ignore the time-dependence of $A_w^{k,l}$. The moments of the device observables are given by $\hat{Q}_M$ and $\hat{P}_M$,

$$\langle \hat{Q}_M^m \rangle_f = \text{Tr}_M[\hat{Q}_M^m \hat{\rho}_M^f] = \langle \hat{Q}_M^m \rangle \frac{1 + \mathcal{F}_m(\hat{Q}_M)}{1 + \mathcal{F}_0(\hat{Q}_M)}, \quad (11)$$

$$\langle \hat{P}_M^m \rangle_f = \text{Tr}_M[\hat{P}_M^m \hat{\rho}_M^f] = \langle \hat{P}_M^m \rangle \frac{1 + \mathcal{F}_m(\hat{P}_M)}{1 + \mathcal{F}_0(\hat{P}_M)}, \quad (12)$$

where we set

$$\mathcal{F}_m(\hat{X}) = \frac{1}{\langle \hat{X}^m \rangle} \sum_{n=1}^{\infty} \frac{(-ig)^n}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} (-1)^k A_w^{n-k,k} \langle \hat{P}_M^k \hat{X}^m \hat{P}_M^{n-k} \rangle. \quad (13)$$

Although the general expressions (11) and (12) with Eqs. (8) and (13) have been derived, it is difficult to investigate the dynamics under the influence of the environment. In the following sections, we will investigate the influence of the environment on the weak value A_w .

3 Exactly solvable dephasing model

In the previous section, we have derived the general expression (4) of the weak value under the influence of the environment. In order to investigate the decoherence of the weak value, we use an exactly solvable model of pure dephasing, where the relevant system is a qubit and the environment is a set of independent harmonic oscillators. The Hamiltonian of the pure dephasing model [32,33] is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} = \hat{H}_S + \hat{H}_E + \hat{H}_{\text{int}}, \quad (14)$$

with

$$\hat{H}_S = \frac{1}{2} \omega_0 \hat{\sigma}_z, \quad \hat{H}_E = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k, \quad \hat{H}_{\text{int}} = \hat{\sigma}_z \sum_k (g_k \hat{b}_k^\dagger + g_k^* \hat{b}_k), \quad (15)$$

where $\hat{\sigma}_z$ is the Pauli matrix and $\hat{b}_k$ is a bosonic annihilation operator of the k th environmental oscillator. Then the time evolution operator can be written as

$$\hat{U}(t) = e^{-i\hat{H}_0 t} T_+ \exp \left[-i \int_0^t dt_1 \hat{H}_{\text{int}}(t_1) \right], \quad (16)$$

where T_+ stands for the time ordering that operators are placed from the right to the left in the chronological order and $\hat{H}_{\text{int}}(t)$ is given by

$$\hat{H}_{\text{int}}(t) = \hat{\sigma}_z \sum_k (g_k e^{i\omega_k t} \hat{b}_k^\dagger + g_k^* e^{-i\omega_k t} \hat{b}_k). \quad (17)$$

Since the commutation relation $[\hat{H}_{\text{int}}(t_1), \hat{H}_{\text{int}}(t_2)] = -2i \sum_k |g_k|^2 \sin \omega_k (t_1 - t_2)$ holds, we can resolve the time ordering on the right-hand side of Eq. (16) [34],

$$\hat{U}(t) = e^{i\varphi(t)} e^{-i\hat{H}_0 t} \exp \left[\frac{\hat{\sigma}_z}{2} \sum_k (\alpha_k(t) \hat{b}_k^\dagger - \alpha_k^*(t) \hat{b}_k) \right], \quad (18)$$

where the parameters $\varphi(t)$ and $\alpha_k(t)$ are given respectively by

$$\varphi(t) = \sum_k |g_k|^2 \frac{\omega_k t - \sin \omega_k t}{\omega_k^2}, \quad \alpha_k(t) = \frac{2g_k}{\omega_k} (1 - e^{i\omega_k t}). \quad (19)$$

We denote the eigenstates of $\hat{\sigma}_z$ as $|0\rangle$ and $|1\rangle$ such that $\hat{\sigma}_z|0\rangle = -|0\rangle$ and $\hat{\sigma}_z|1\rangle = |1\rangle$. Then the time evolution operator $\hat{U}(t)$ of the pure dephasing model can be written as

$$\hat{U}(t) = |0\rangle\langle 0| \otimes \hat{U}_0(t) + |1\rangle\langle 1| \otimes \hat{U}_1(t), \quad (20)$$

where the environmental operators $\hat{U}_0(t)$ and $\hat{U}_1(t)$ are given respectively by

$$\hat{U}_0(t) = e^{i\omega_0 t/2} e^{i\varphi(t)} e^{-i\hat{H}_E t} \exp \left[-\frac{1}{2} \sum_k (\alpha_k(t) \hat{b}_k^\dagger - \alpha_k^*(t) \hat{b}_k) \right], \quad (21)$$

$$\hat{U}_1(t) = e^{i\omega_0 t/2} e^{i\varphi(t)} e^{-i\hat{H}_E t} \exp \left[\frac{1}{2} \sum_k (\alpha_k(t) \hat{b}_k^\dagger - \alpha_k^*(t) \hat{b}_k) \right]. \quad (22)$$

4 Weak value of qubit observable in the pure dephasing process

Using the time evolution operator (20) with Eqs. (21) and (22), we calculate the weak value from Eq. (4). First we suppose that the qubit observable to be measured is $\hat{\sigma}_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. Since $[\hat{\sigma}_z, \hat{U}(t)] = 0$, we obtain the weak value,

$$\sigma_{z,w} = \frac{\text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{\sigma}_z \hat{U}(t_f - t_i) \hat{\rho}_{\text{SE}} \hat{U}^\dagger(t_f - t_i)]}{\text{Tr}_{\text{SE}}[\hat{\pi}_f \hat{U}(t_f - t_i) \hat{\rho}_{\text{SE}} \hat{U}^\dagger(t_f - t_i)]}, \quad (23)$$

which is independent of the measurement time t_m . Let us assume that the qubit and the environment at the initial time are in the state $\hat{\rho}_{\text{SE}} = \hat{\rho}_S \otimes \hat{\rho}_E$, where $\hat{\rho}_E = e^{-\beta \hat{H}_E} / \text{Tr}_E e^{-\beta \hat{H}_E}$ with $\beta = 1/k_B T$. Then the weak value A_w becomes

$$\sigma_{z,w} = \frac{m_{11}\rho_{11} - m_{00}\rho_{00} - 2i\text{Im}[m_{10}\rho_{01}\mathcal{G}_0]}{m_{00}\rho_{00} + m_{11}\rho_{11} + 2\text{Re}[m_{10}\rho_{01}\mathcal{G}_0]}, \quad (24)$$

the real part of which is

$$\text{Re}\sigma_{z,w} = \frac{m_{11}\rho_{11} - m_{00}\rho_{00}}{m_{00}\rho_{00} + m_{11}\rho_{11} + 2\text{Re}[m_{10}\rho_{01}\mathcal{G}_0]}, \quad (25)$$

where $m_{jk} = \langle j | \hat{\pi}_f | k \rangle$ and $\rho_{jk} = \langle j | \hat{\rho}_S | k \rangle$ ($j, k = 0, 1$). In these equations, the time-dependent parameter $\mathcal{G}_0$ which characterizes the decoherence is given by

$$\mathcal{G}_0 = \text{Tr}_E[\hat{U}_1^\dagger(t_f - t_i) \hat{U}_0(t_f - t_i) \hat{\rho}_E]. \quad (26)$$

Substituting Eqs. (21) and (22) into this equation, we obtain

$$\mathcal{G}_0 = e^{i\omega_0(t_f - t_i)} e^{-\gamma(t_f - t_i)}, \quad (27)$$

where the decoherence function $\gamma(t)$ is

$$\gamma(t) = \int_0^\infty d\omega J(\omega) \frac{1 - \cos \omega t}{\omega^2} \coth \left(\frac{\beta \omega}{2} \right). \quad (28)$$

The spectral density $J(\omega)$ of the environmental oscillators [32,33] is defined by the relation $\sum_k 4|g_k|^2 f(\omega_k) = \int_0^\infty d\omega J(\omega) f(\omega)$.

In the above discussion, we have ignored the initial correlation between the qubit and the environment. In order to investigate an effect of the initial correlation on the weak value, we assume that there is measurement-induced initial correlation between the qubit and the environment [32]. Suppose that the total system before the preparation of the initial qubit state is in thermal equilibrium $e^{-\beta \hat{H}} / \text{Tr} e^{-\beta \hat{H}}$ with $\hat{H}$ being given by Eq. (14). To prepare the

initial state, we perform a projective measurement on the qubit at the initial time t_i . Then, the total system just after the measurement becomes $\hat{\rho}_{SE} = |\psi\rangle\langle\psi|(e^{-\beta\hat{H}}/\text{Tr}e^{-\beta\hat{H}})|\psi\rangle\langle\psi|$, where $|\psi\rangle$ represents the initial qubit state. Since $\hat{H}$ includes the interaction between the qubit and the environment, the environmental state after the measurement depends on the qubit state $|\psi\rangle$ [32, 36]. Taking the measurement-induced initial correlation into account, we have the initial state $\hat{\rho}_{SE} = |\psi\rangle\langle\psi| \otimes \hat{\rho}_E(\psi)$ with

$$\hat{\rho}_E(\psi) = \frac{\rho_{00}e^{\beta\omega_0/2}e^{-\beta\hat{H}_E^{(-)}} + \rho_{11}e^{-\beta\omega_0/2}e^{-\beta\hat{H}_E^{(+)}}}{\rho_{00}e^{\beta\omega_0/2}\text{Tr}_E[e^{-\beta\hat{H}_E^{(-)}}] + \rho_{11}e^{-\beta\omega_0/2}\text{Tr}_E[e^{-\beta\hat{H}_E^{(+)}}]}, \quad (29)$$

where the modified Hamiltonians $\hat{H}_E^{(\pm)}$ of the environment is given by

$$\hat{H}_E^{(\pm)} = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k \pm \sum_k (g_k \hat{b}_k^\dagger + g_k^* \hat{b}_k). \quad (30)$$

In the presence of the initial correlation, replacing $\hat{\rho}_E$ in Eq. (26) with Eq. (29), we can obtain after some calculation [32],

$$\mathcal{G}_0 = e^{i\omega_0(t_f-t_i)} e^{-\gamma(t_f-t_i)} \left\{ \cos[\Phi(t_f-t_i)] + i \frac{\sinh(\beta\omega_0/2) - (\rho_{11} - \rho_{00}) \cosh(\beta\omega_0/2)}{\cosh(\beta\omega_0/2) - (\rho_{11} - \rho_{00}) \sinh(\beta\omega_0/2)} \sin[\Phi(t_f-t_i)] \right\}, \quad (31)$$

where $\gamma(t)$ is Eq. (28) and $\Phi(t)$ is given by

$$\Phi(t) = \int_0^\infty d\omega J(\omega) \frac{\sin \omega t}{\omega^2}. \quad (32)$$

The weak value is provided by Eq. (24) with Eq. (31).

Next we suppose that the qubit observable $\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$ is measured at the time t_m . In this case, the weak value depends on the measurement time. Substituting $\hat{A}_S = \hat{\sigma}_x$ into Eq. (4), we obtain the weak value $\sigma_{x,w}$,

$$\sigma_{x,w} = \frac{m_{00}\rho_{10}\mathcal{G}_1 + m_{11}\rho_{01}\mathcal{G}_1^* + e^{-i\Psi}(m_{10}\rho_{11}\mathcal{G}_2 + m_{01}\rho_{00}\mathcal{G}_2^*)}{m_{00}\rho_{00} + m_{11}\rho_{11} + 2\text{Re}[m_{10}\rho_{01}\mathcal{G}_0]}. \quad (33)$$

In the absence of the initial correlation between the qubit and the environment, the time-dependent parameters $\mathcal{G}_1$ and $\mathcal{G}_2$ are calculated to be

$$\mathcal{G}_1 = \text{Tr}_E[\hat{U}_0^\dagger(t_f-t_i)\hat{U}_0(t_f-t_m)\hat{U}_1(t_m-t_i)\hat{\rho}_E] = e^{-i\omega_0(t_m-t_i)}e^{-\gamma(t_m-t_i)}, \quad (34)$$

$$\mathcal{G}_2 = \text{Tr}_E[\hat{U}_1^\dagger(t_f-t_i)\hat{U}_0(t_f-t_m)\hat{U}_1(t_m-t_i)\hat{\rho}_E] = e^{i\omega_0(t_f-t_m)}e^{-\gamma(t_f-t_m)}. \quad (35)$$

The time-dependent phase Ψ in Eq. (33) is

$$\Psi = \Phi(t_m-t_i) - \Phi(t_f-t_i) + \Phi(t_f-t_m), \quad (36)$$

where $\Phi(t)$ is given by Eq. (32). On the other hand, when there is the initial correlation, $\mathcal{G}_1$ and $\mathcal{G}_2$ are provided by

$$\mathcal{G}_1 = e^{-i\omega_0(t_m-t_i)}e^{-\gamma(t_m-t_i)} \left\{ \cos[\Phi(t_m-t_i)] - i \frac{\sinh(\beta\omega_0/2) - (\rho_{11} - \rho_{00}) \cosh(\beta\omega_0/2)}{\cosh(\beta\omega_0/2) - (\rho_{11} - \rho_{00}) \sinh(\beta\omega_0/2)} \sin[\Phi(t_m-t_i)] \right\}, \quad (37)$$

$$\mathcal{G}_2 = e^{i\omega_0(t_f-t_m)} e^{-\gamma(t_f-t_m)} \left\{ \cos[\Phi(t_m-t_i) - \Phi(t_f-t_i)] - i \frac{\sinh(\beta\omega_0/2) - (\rho_{11} - \rho_{00}) \cosh(\beta\omega_0/2)}{\cosh(\beta\omega_0/2) - (\rho_{11} - \rho_{00}) \sinh(\beta\omega_0/2)} \sin[\Phi(t_m-t_i) - \Phi(t_f-t_i)] \right\}. \quad (38)$$

If we choose $\hat{\sigma}_y = i(|0\rangle\langle 1| - |1\rangle\langle 0|)$ as a qubit observable, we derive the weak value,

$$\sigma_{y,w} = i \frac{m_{00}\rho_{10}\mathcal{G}_1 - m_{11}\rho_{01}\mathcal{G}_1^* + e^{-i\Psi}(m_{10}\rho_{11}\mathcal{G}_2 - m_{01}\rho_{00}\mathcal{G}_2^*)}{m_{00}\rho_{00} + m_{11}\rho_{11} + 2\text{Re}[m_{10}\rho_{01}\mathcal{G}_0]}. \quad (39)$$

5 Numerical analysis and discussion

In this section, we numerically investigate decoherence of the weak values $\text{Re}\sigma_{z,w}$ and $\sigma_{x,w}$ in the dephasing process. For this purpose, we assume that the spectral density of the environmental oscillators is given by $J(\omega) = \lambda\Omega^{1-s}\omega^s e^{-\omega/\Omega}$, where λ represents a coupling strength between the qubit and the environment and Ω is a cut-off frequency [35]. The environment is called sub-Ohmic, Ohmic and super-Ohmic respectively for $s < 1$, $s = 1$ and $s > 1$. In the rest of this section, we set $s = 1/2$ and $s = 3$ for the sub-Ohmic and super-Ohmic environments. Then we can explicitly obtain the decoherence function $\gamma(t)$ and the phase function $\Phi(t)$ from Eqs. (28) and (32) [32],

$$\gamma(t) = -2\pi\lambda \left\{ 2(\Omega\beta)^{1/2} \left[\zeta\left(-\frac{1}{2}, 1 + \frac{1}{\Omega\beta}\right) - \frac{1}{2} \zeta\left(-\frac{1}{2}, 1 + \frac{1}{\Omega\beta} + i\frac{t}{\beta}\right) + \zeta\left(-\frac{1}{2}, 1 + \frac{1}{\Omega\beta} - i\frac{t}{\beta}\right) \right] + 1 - \frac{\cos\left[-\frac{1}{2}\tan^{-1}\Omega t\right]}{(1 + (\Omega t)^2)^{-1/4}} \right\}, \quad (40)$$

$$\Phi(t) = -\frac{2\pi\lambda}{(1 + (\Omega t)^2)^{-1/4}} \sin\left[-\frac{1}{2}\tan^{-1}\Omega t\right], \quad (41)$$

for the sub-Ohmic environment ($s = 1/2$),

$$\gamma(t) = \lambda \left\{ \frac{1}{2} \ln[1 + (\Omega t)^2] + 2 \ln \Gamma\left(1 + \frac{1}{\Omega\beta}\right) - \ln \left| \Gamma\left(1 + \frac{1}{\Omega\beta} + i\frac{t}{\beta}\right) \right|^2 \right\}, \quad (42)$$

$$\Phi(t) = \lambda \tan^{-1}\Omega t, \quad (43)$$

for the Ohmic environment ($s = 1$) and

$$\gamma(t) = \lambda \left\{ 2(\Omega\beta)^{-2} \left[\zeta\left(2, 1 + \frac{1}{\Omega\beta}\right) - \frac{1}{2} \zeta\left(2, 1 + \frac{1}{\Omega\beta} + i\frac{t}{\beta}\right) + \frac{1}{2} \zeta\left(2, 1 + \frac{1}{\Omega\beta} - i\frac{t}{\beta}\right) \right] + 1 - \frac{\cos[2\tan^{-1}\Omega t]}{1 + (\Omega t)^2} \right\}, \quad (44)$$

$$\Phi(t) = \frac{\lambda}{1 + (\Omega t)^2} \sin[2\tan^{-1}\Omega t], \quad (45)$$

for the super-Ohmic environment ($s = 3$). In Eqs. (40) and (44), $\zeta(z, a)$ is the generalized Riemann zeta function [32]. If $0 < s \leq 2$, the decoherence function $\gamma(t)$ becomes infinity in the limit $t \rightarrow \infty$ and thus $\lim_{t \rightarrow \infty} \mathcal{G}_j = 0$ ($j = 1, 2$). On the other hand, if $s > 2$, the decoherence function $\gamma(t)$ remains finite in the limit $t \rightarrow \infty$, which means that $\lim_{t \rightarrow \infty} \mathcal{G}_j > 0$. Therefore, the coherence (off diagonal element of the density matrix) survives in the long time limit only if $s > 2$ [32].

To proceed further, we assume that the post-selection is done by the von Neumann measurement, the projector of which is $\hat{\pi}_f = |f\rangle\langle f|$, where $|f\rangle = \cos\theta'|0\rangle + e^{i\phi'}\sin\theta'|1\rangle$ with $0 \leq \theta' \leq 90^\circ$ and $0 \leq \phi' \leq 360^\circ$. To get rid

of the effect of the free motion on the weak value, we replace $\hat{\pi}_f$ and $\hat{\sigma}_x$ with $\hat{\pi}'_f = e^{-i\hat{H}_S(t_f-t_i)}\hat{\pi}_f e^{i\hat{H}_S(t_f-t_i)}$ and $\hat{\sigma}'_x = e^{i\hat{H}_S(t_m-t_i)}\hat{\sigma}_x e^{-i\hat{H}_S(t_m-t_i)}$. Furthermore, we assume that the initial qubit state $\hat{\rho}_S$ is given by $\hat{\rho}_S = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is given by $|\psi\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle$ with $0 \leq \theta \leq 90^\circ$ and $0 \leq \phi \leq 360^\circ$. In the rest of this section, we set the initial time $t_i = 0$ for the sake of simplicity. The real part of the weak value can exceed the spectral range of the measured observable, which is called the anomaly of the weak value. The anomaly of the weak value is useful for an experiment which detects tiny signals [29,30]. So we investigate whether the inequalities $\text{Re}\sigma_{z,w} > 1$ and $\text{Re}\sigma_{x,w} > 1$ can be found in the dephasing process.

The dependence of the weak value $\text{Re}\sigma_{z,w}$ given by Eq. (25) on the post-selection time t_f is illustrated in Fig. 1. It is obvious from Fig. 1a,b that the weak value decreases when λ becomes larger or β becomes smaller. Furthermore, it is found from Fig. 1c that the weak value in the super-Ohmic case is larger than those in the sub-Ohmic and the Ohmic case in the long time region. The anomaly of the weak value can survive under the influence of the environment. For instance, when we set $\theta' = 45^\circ$, the condition that the inequality $\text{Re}\sigma_{z,w} > 1$ holds is given by

$$0 < \cos\theta < \frac{1}{\sqrt{2}}, \quad -1 \leq \text{Re}[e^{-i(\phi-\phi')}\mathcal{G}_0] < -\frac{1}{\tan\theta}. \quad (46)$$

In the case of $s \leq 2$, this inequality cannot be fulfilled in the limit $t_f \rightarrow \infty$ since $\lim_{t_f \rightarrow \infty} \mathcal{G}_0 = 0$. In the case of $s > 2$, however, since the decoherence function takes a finite value even in the limit $t_f \rightarrow \infty$, the inequality (46) can be satisfied at some values of λ and β if we appropriately choose the qubit parameters ϕ and θ . The insets of Fig. 1 show the success probability of the post-selection $P(f)$,

Fig. 1 The dependence of the weak value $\text{Re}\sigma_{z,w}$ on the post-selection time t_f , where the *inset* represents the success probability of the post-selection. The *solid* and *dashed* lines stand respectively for the uncorrelated and correlated initial states. We set $\Omega\beta = 5$, $s = 1$, *i* $\lambda = 0.1$, *ii* $\lambda = 0.3$, *iii* $\lambda = 1$ in panel **a**, $\lambda = 0.3$, $s = 1$, *i* $\Omega\beta = 1$, *ii* $\Omega\beta = 5$, *iii* $\Omega\beta = 15$ in panel **b**, and $\Omega\beta = 5$, $\lambda = 0.3$, *i* $s = 0.5$, *ii* $s = 1$, *iii* $s = 3$ in panel **c**. In all the panels, we set $\theta = 55^\circ$, $\theta' = 45^\circ$, $\phi - \phi' = 180^\circ$ and $\Omega = \omega_0$

Fig. 2 The threshold time τ for the post-selection with **a** $\Omega\beta = 5$ and **b** $\lambda = 0.3$, where the *solid lines* stand for the uncorrelated initial state and the *dashed lines* for correlated initial state. Each *line* stands for i $s = 0.5$, ii $s = 1$, iii $s = 3$. In the figure, we set $\theta = 55^\circ$, $\theta' = 45^\circ$, $\phi - \phi' = 180^\circ$ and $\Omega = \omega_0$

$$P(f) = \int_{-\infty}^{\infty} dq P(f, q) = m_{00}\rho_{00} + m_{11}\rho_{11} + 2\text{Re}[m_{10}\rho_{01}\mathcal{G}_0]. \quad (47)$$

It is well known that the larger weak value is obtained with less success probability. Furthermore, it is found from the figure that the initial correlation between the qubit and the environment enhances the decay of the weak value.

Here we consider a threshold time τ for the post-selection, during which the inequality $\text{Re}\sigma_{z,w} > 1$ holds, or equivalently Eq. (46) is satisfied. We plot the threshold time τ as a function of λ and β in Fig. 2. It is seen from the figure that when $s = 3$, the threshold time τ becomes infinity at the coupling strength λ^* and the inverse temperature β^* which are determined by

$$-\frac{1}{\tan \theta} = \lim_{t \rightarrow \infty} \text{Re}[e^{-i(\phi - \phi')} \mathcal{G}_0]. \quad (48)$$

For instance, when $\theta = 55^\circ$ and $\phi - \phi' = 180^\circ$ and $\Omega\beta = 5$, we obtain $\lambda^* \approx 0.332$. Similarly, we obtain $\Omega\beta^* \approx 3.198$ when $\theta = 55^\circ$, $\phi - \phi' = 0$ and $\lambda = 0.3$.

Next we consider the weak value of the qubit observable $\hat{\sigma}_x$, which depends on the measurement time t_m . When we perform the weak measurement at $t_m = 0$, the weak value is derived from Eq. (33),

$$\sigma_{x,w} = \frac{m_{00}\rho_{10} + m_{11}\rho_{01} + m_{10}\rho_{11}\mathcal{G}_0 + m_{01}\rho_{00}\mathcal{G}_0}{m_{00}\rho_{00} + m_{11}\rho_{11} + 2\text{Re}[m_{10}\rho_{01}\mathcal{G}_0]}. \quad (49)$$

Taking the limit $t_f \rightarrow \infty$ in this equation, we obtain

$$\sigma_{x,w} = \frac{m_{00}\rho_{10} + m_{11}\rho_{01}}{m_{00}\rho_{00} + m_{11}\rho_{11}} \quad (s \leq 2), \quad (50)$$

$$\sigma_{x,w} = \frac{m_{00}\rho_{10} + m_{11}\rho_{01} + C(\Omega\beta, \lambda)(m_{10}\rho_{11} + m_{01}\rho_{00})}{m_{00}\rho_{00} + m_{11}\rho_{11} + 2C(\Omega\beta, \lambda)\text{Re}[m_{10}\rho_{01}]} \quad (s = 3), \quad (51)$$

where $C(\Omega\beta, \lambda)$ is given by

$$C(\Omega\beta, \lambda) = \exp \left\{ -\lambda \left[\frac{2}{(\Omega\beta)^2} \zeta \left(2, 1 + \frac{1}{\Omega\beta} \right) + 1 \right] \right\}, \quad (52)$$

which is derived from Eq. (44). It is found from Eqs. (50) and (51) that the anomaly of the weak value can be observed if the pre- and post-selected states, the coupling strength and the inverse temperature are appropriately chosen. On the other hand, when we perform the weak measurement at $t_m = t_f$, the weak value $\sigma_{x,w}$ becomes

$$\sigma_{x,w} = \frac{m_{00}\rho_{10}\mathcal{G}_0 + m_{11}\rho_{01}\mathcal{G}_0 + m_{10}\rho_{11} + m_{01}\rho_{00}}{m_{00}\rho_{00} + m_{11}\rho_{11} + 2\text{Re}[m_{10}\rho_{01}\mathcal{G}_0]}, \quad (53)$$

which yields in the limit $t_f \rightarrow \infty$,

$$\sigma_{x,w} = \frac{m_{10}\rho_{11} + m_{01}\rho_{00}}{m_{00}\rho_{00} + m_{11}\rho_{11}} \quad (s \leq 2), \quad (54)$$

$$\sigma_{x,w} = \frac{C(\Omega\beta, \lambda)(m_{00}\rho_{10} + m_{11}\rho_{01}) + m_{10}\rho_{11} + m_{01}\rho_{00}}{m_{00}\rho_{00} + m_{11}\rho_{11} + 2C(\Omega\beta, \lambda)\text{Re}[m_{10}\rho_{01}]} \quad (s = 3). \quad (55)$$

Thus we can also observe the anomaly of the weak value if we appropriately choose the parameters. The reason why we can observe the anomaly of the weak value in the limit $t_f \rightarrow \infty$ is that we choose $\hat{\sigma}_x$ as a measured observables in the pure dephasing model. For instance, let us consider the case that $t_m = 0$ and $\hat{A}_S = \hat{\sigma}_x$. Noting that

$$\text{Tr}_E[\hat{U}(t_f)\hat{\sigma}_x\hat{\rho}_{SE}\hat{U}^\dagger(t_f)] = \begin{pmatrix} \rho_{01} & \rho_{00}\mathcal{G}_0^* \\ \rho_{11}\mathcal{G}_0 & \rho_{10} \end{pmatrix}, \quad (56)$$

we obtain the weak value $\sigma_{x,w} = \rho_{10}/\rho_{00} = e^{i\phi} \tan \theta$ if the post-selection is done by the projector $\hat{\pi}_f = |0\rangle\langle 0|$. This result means that we can pick up the component which is not influenced by the pure dephasing. On the other hand, when we choose $\hat{\sigma}_z$ as a measured observable, we can obtain the weak value in the limit $t_f \rightarrow \infty$

$$\text{Re}\sigma_{z,w} = \frac{m_{11}\rho_{11} - m_{00}\rho_{00}}{m_{00}\rho_{00} + m_{11}\rho_{11}}, \quad (57)$$

for $s \leq 2$. It is easy to see that the inequality $-1 \leq \text{Re}\sigma_{z,w} \leq 1$ is always satisfied.

In Fig. 3, we plot the dependence of $\text{Re}\sigma_{x,w}$ on the post-selection time t_f when the measurement time is $t_m = t_f/2$. The dependence of the threshold time τ for the post-selection on λ and β is illustrated in Fig. 4. Next, to see the dependence of the weak value on the measurement time t_m , we introduce the parameter m by $t_m = mt_f$ ($0 < m < 1$). In Fig. 5, we plot m -dependence of the threshold time τ . We can see from the figure that in the case of $s = 1/2$ and $s = 1$, the behavior of τ in Fig. 5a is quite different from that in Fig. 5b. This difference can be understood by considering the magnitude of the coherence. When the measurement t_m is close to the pre-selection time $t_i = 0$ (or the post-selection time t_f), the threshold time τ becomes larger if the inequality $|\rho_{01}| > |m_{01}|$ (or $|\rho_{01}| < |m_{01}|$) is satisfied. Hence, if the coherence of the pre-selected state is greater than that of the post-selected state, we can obtain the large weak value when we approach the measurement time t_m toward the pre-selection time ($t_i = 0$) because the amount of coherence destroyed by the environment is made smaller. In the case of $s = 3$, on the other hand, the threshold time τ becomes infinity when we approach the measurement time toward the pre-selection time and the post-selection time. From Fig. 5c, d, it is found that the threshold time τ becomes infinity when we approach the measurement time toward the pre-selection time and post-selection time even in the cases of $s = 1/2$ and $s = 1$ for the given values of θ and θ' . We note that the threshold time τ does not take a finite value for any m in the case of $s = 3$.

Now we consider the parameter regions, where we can observe the anomaly of the weak value $\sigma_{x,w}$ in the dephasing process. When we perform the weak measurement at $t_m = 0$, using Eqs. (50), (51), (54) and (55), we can obtain the condition that the absolute value $|\sigma_{x,w}|$ of the weak value is greater than unity in the limit $t_f \rightarrow \infty$

$$|m_{00}\rho_{10} + m_{11}\rho_{01}| > P(f) \quad (s \leq 2), \quad (58)$$

$$|m_{00}\rho_{10} + m_{11}\rho_{01} + C(\Omega\beta, \lambda)(m_{00}\rho_{11} + m_{01}\rho_{00})| > P(f) \quad (s = 3). \quad (59)$$

Fig. 3 The dependence of the weak value $\text{Re}\sigma_{x,w}$ on the post-selection time t_f in the case of $t_m = t_f/2$. The inset represents the success probability of the post-selection. The *solid* and *dashed* lines stand respectively for the uncorrelated and correlated initial states. We set $\Omega\beta = 5$, $s = 1$, *i* $\lambda = 0.1$, *ii* $\lambda = 0.3$, *iii* $\lambda = 1$ in panel **a**, $\lambda = 0.3$, $s = 1$, *i* $\Omega\beta = 1$, *ii* $\Omega\beta = 5$, *iii* $\Omega\beta = 15$ in panel **b** and $\Omega\beta = 5$, $\lambda = 0.3$, *i* $s = 0.5$, *ii* $s = 1$, *iii* $s = 3$ in panel **c**. In all the panels, we set $\theta = 33^\circ$, $\theta' = 77^\circ$, $\phi = 0$, $\phi' = 180^\circ$ and $\Omega = \omega_0$

Fig. 4 The dependence of the threshold time τ on **a** the coupling parameter λ with $\Omega\beta = 5$ and **b** the inverse temperature β with $\lambda = 0.3$ in the case of $t_m = t_f/2$, where the *solid* lines stand for the uncorrelated initial state and the *dashed* lines for correlated initial state. Each line stands for *i* $s = 0.5$, *ii* $s = 1$, *iii* $s = 3$. We set the parameters $\theta = 33^\circ$, $\theta' = 77^\circ$, $\phi = 0$ and $\phi' = 180^\circ$. In the case of $s = 3$, the threshold time τ becomes infinity at $\lambda^* = 0.293$ and at $\Omega\beta^* = 5.856$

Similarly we derive the condition for the weak measurement at $t_m = t_f$,

$$|m_{10}\rho_{11} + m_{01}\rho_{00}| > P(f), \quad (s \leq 2), \quad (60)$$

$$|m_{10}\rho_{11} + m_{01}\rho_{00} + C(\Omega\beta, \lambda)(m_{00}\rho_{10} + m_{11}\rho_{01})| > P(f) \quad (s = 3). \quad (61)$$

Fig. 5 The dependence of the threshold time τ on $m = t_m/t_f$, where the *solid* and *dashed* lines stand respectively for the uncorrelated and correlated initial states. We set $\theta = 36^\circ, \theta' = 75^\circ$ in panel **a**, $\theta = 75^\circ$ and $\theta' = 36^\circ$ in panel **b**, $\theta = 25^\circ$ and $\theta' = 66^\circ$ in panel **c**, and $\theta = 66^\circ$ and $\theta' = 25^\circ$ in panel **d**. Each line stands for i $s = 0.5$, ii $s = 1$, iii $s = 3$. In all the panels, we set $\Omega\beta = 5$, $\lambda = 0.3$, $\phi = \phi' = 0$ and $\Omega = \omega_0$

In the case of $s \leq 2$, if the measurement time is chosen in the range $0 < t_m < t_f$, the weak value $\sigma_{x,w}$ becomes zero in the limit $t_f \rightarrow \infty$ since $\lim_{t_f \rightarrow \infty} \mathcal{G}_j = 0$ for $s \leq 2$. On the other hand, if $s > 2$, we can observe the anomaly of the weak value when we appropriately choose the parameters θ, θ', λ and β because the decoherence function takes a finite value even in the limit $t_f \rightarrow \infty$. In particular, setting $s = 3$, we have the condition that the inequality $|\sigma_{x,w}| > 1$ holds for any measurement time,

$$C(\Omega\beta, \lambda)|m_{00}\rho_{10} + m_{10}\rho_{11} + m_{01}\rho_{00} + m_{11}\rho_{01}| > P(f). \quad (62)$$

In Fig. 6, we plot the region of the parameters θ and θ' that satisfies the conditions Eqs. (58)–(62). If values of θ and θ' are chosen in the red regions (i) and (ii) of Fig. 6a, b, the anomaly of the weak value can be observed in the limit $t_f \rightarrow \infty$ when $s \leq 2$ and $s = 3$. If the values are in the blue regions (iii) and (iv) of Fig. 6a, b, the anomaly can be observed in the limit $t_f \rightarrow \infty$ only in the case of $s = 3$. If we perform the weak measurement at any time t_m in the range $0 < t_m < t_f$, the anomaly of the weak value cannot survive for any θ and θ' in the case of $s \leq 2$. However, when the values of θ and θ' are taken in the blue regions of Fig. 6c, we can observe the anomaly of the weak value in the case of $s = 3$. The regions of θ and θ' that satisfy the inequalities (59), (61) and (62) become smaller as the temperature of the environment is higher or the qubit-environment coupling is stronger.

6 Summary and conclusion

In this paper, we have investigated the weak values of the qubit under the influence of the environment which has the dephasing coupling with the qubit. The weak measurement is performed on the irreversibly evolved qubit from the pre-selected state to the post-selected one. We have also taken into account of the initial correlation between the qubit and the environment, which is created in the preparation of the initial qubit state (the pre-selection). We have investigated how the post-selection and measurement times affect the decoherence of the weak value and how long the anomaly of the weak value can survive in the dephasing process.

Fig. 6 The parameter region that the weak value $\sigma_{x,w}$ is greater than unity. **a**, **b** represent the parameter regions where the anomaly of the weak value can be observe in the limit $t_f \rightarrow \infty$ respectively for the measurement time $t_m = 0$ and $t_m = t_f$. In the *red* regions *i* and *ii*, we can observe the anomaly of the weak value in both cases of $s \leq 2$ and $s = 3$. In the *blue* regions *iii* and *iv*, we can observe the anomaly of the weak value only in the case of $s = 3$. The **c** shows the parameter region, where we can observe the anomaly of the weak value in the limit $t_f \rightarrow \infty$ with $s = 3$ for any measurement time t_m ($0 < t_m < t_f$). In all the figures, we set $\Omega\beta = 5$, $\lambda = 0.3$, $\phi = \phi' = 0$ and $\Omega = \omega_0$. In this case, we have $C(5, 0.3) \approx 0.718$ in Eqs. (59), (61) and (62)

The property of the weak value is greatly different for the sub-Ohmic, Ohmic and super-Ohmic environments. We have derived the condition that the anomaly of the weak value can be observed even in the limit $t_f \rightarrow \infty$. The post-selection time, the measurement time, the coupling strength, the temperature and the spectral density greatly affect the decoherence of the weak values. The parameter region which makes it possible to observe the anomaly of the weak value decreases when the environmental temperature is higher and the system-reservoir coupling is stronger. We have found that the magnitude of the coherence of the pre- and post-selected state determines the preferable measurement time in the dephasing process. Moreover, we have considered the measurement-induced initial correlation between the qubit and the environment. It has been found from the result of the numerical calculation that the effect of the initial correlation always suppresses the anomaly of the weak value. This result is consistent with the previous one that the effect of the initial correlation enhances the decoherence [36].

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