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# Quantum Ornstein–Uhlenbeck semigroups

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**Abstract** Based on nuclear infinite-dimensional algebra of entire functions with a certain exponential growth condition with two variables, we define a class of operators which gives in particular three semigroups acting on continuous linear operators, called the quantum Ornstein–Uhlenbeck (O–U) semigroup, the left quantum O–U semigroup and the right quantum O–U semigroup. Then, we prove that the solution of the Cauchy problem associated with the quantum number operator, the left quantum number operator and the right quantum number operator, respectively, can be expressed in terms of such semigroups. Moreover, probabilistic representations of these solutions are given. Eventually, using a new notion of positive white noise operators, we show that the aforementioned semigroups are Markovian.

**Keywords** Space of entire function · Quantum O–U semigroup · Quantum number operator · Cauchy problem · Positive operators · Markovain semigroups

**Mathematics Subject Classification** 46F25 · 46G20 · 46A32 · 60H15 · 60H40 · 81S25

## 1 Introduction

Piech [25] introduced the number operator N (Beltrami Laplacian) as infinite-dimensional analog of a finite-dimensional Laplacian. This infinite-dimensional Laplacian has been extensively studied in [18,20] and the references cited therein. In particular, Kuo [18] formulated the number operator as continuous linear operator acting on the space of test white noise functionals. As applications, Kuo [17] studied the heat equation associated with the number operator N; this solution is related to the Ornstein–Uhlenbeck (O–U) semigroup. Based on the white noise theory, Kuo formulated the O–U semigroup as continuous linear operator acting on the space of test white noise functionals; see [18] and references cited therein. In [7], based on nuclear algebra of entire functions, some results are extended about operator–parameter transforms involving the O–U semigroup.

In this paper, based on nuclear algebra of entire functions with two variables, three semigroups appear naturally: the quantum, the left quantum and the right quantum O–U semigroups, respectively. We extend some results

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about these semigroups and their infinitesimal generators called quantum, left quantum and right quantum number operators, respectively. Moreover, we prove that the solution of the Cauchy problems associated with these operators can be expressed in terms of the O–U semigroups. Such semigroups are shown to be Markovian.

The paper is organized as follows. In Sect. 2, we briefly recall well-known results on nuclear algebras of entire holomorphic functions. In Sect. 3, we extend some regularity properties about quantum number operator $\widetilde{\mathcal{N}}$, left quantum number operator $\widetilde{\mathcal{N}_1}$, right quantum number operator $\widetilde{\mathcal{N}_2}$ and quantum O–U semigroups. In Sect. 4, we construct semigroups with infinitesimal generator $-\widetilde{\mathcal{N}}$, $-\widetilde{\mathcal{N}_1}$ and $-\widetilde{\mathcal{N}_2}$, respectively. Then, we deduce the solution of the associated Cauchy problems where its probabilistic representations are given. In Sect. 5, using an adequate definition of positive operators, we prove that these quantum O–U semigroups are Markovian.

## 2 Preliminaries

First, we review the basic concepts, notations and some results which will be needed in the present paper. The development of these and similar results can be found in Refs. [7,11,15,20,21,24].

In mathematics, a nuclear space is a locally convex topological vector space such that for any seminorm p we can find a larger seminorm q, so that the natural map from $V_q$ to $V_p$ is nuclear. Such spaces preserve many of the good properties of finite-dimensional vector spaces. As main examples of nuclear spaces we recall the Schwartz space of smooth functions for which the derivatives of all orders are rapidly decreasing and the space of entire holomorphic functions on the complex plane with $\theta-$exponential growth. Using a separable Hilbert space and a positive self-adjoint operator with Hilbert–Schmidt inverse, we can construct a real nuclear space. For $i = 1, 2$, let $H_i$ be a real separable (infinite-dimensional) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|_0$. Let $A_i \geq 1$ be a positive self-adjoint operator in $H_i$ with Hilbert–Schmidt inverse. Then there exist a sequence of positive numbers $1 < \lambda_{i,1} \leq \lambda_{i,2} \leq \cdots$ and a complete orthonormal basis of $H_i$, $\{e_{i,n}\}_{n=1}^{\infty} \subseteq \mathrm{Dom}(A_i)$, such that

$$A_i e_{i,n} = \lambda_{i,n} e_{i,n}\,, \quad \sum_{n=1}^{\infty} \lambda_{i,n}^{-2} = \left\| A_i^{-1} \right\|_{HS}^2 < \infty.$$

For every $p \in \mathbb{R}$, we define:

$$|\xi|_p^2 := \sum_{n=1}^{\infty} \langle \xi, e_{i,n}\rangle^2 \lambda_{i,n}^{2p} = \left|A_i^p \xi\right|_0^2\,, \quad \xi \in H_i.$$

The fact that, for $\lambda > 1$, the map $p \mapsto \lambda^p$ is increasing implies that:

(i) for $p \geq 0$, the space $(X_i)_p$, of all $\xi \in H_i$ with $|\xi|_p < \infty$, is a Hilbert space with norm $|\cdot|_p$ and, if $p \leq q$, then $(X_i)_q \subseteq (X_i)_p$;
(ii) denoting by $(X_i)_{-p}$, the $|\cdot|_{-p}$-completion of $H_i$ ($p \geq 0$), if $0 \leq p \leq q$, then $(X_i)_{-p} \subseteq (X_i)_{-q}$.

This construction gives a decreasing chain of Hilbert spaces $\{(X_i)_p\}_{p\in\mathbb{R}}$ with natural continuous inclusions $i_{q,p} : (X_i)_q \hookrightarrow (X_i)_p$ ($p \leq q$). Defining the countably Hilbert nuclear space (see, e.g., [12]):

$$X_i := \mathrm{projlim}_{p\to\infty} (X_i)_p \cong \bigcap_{p\geq 0} (X_i)_p,$$

the strong dual space $X_i'$ of $X_i$ is:

$$X_i' := \mathrm{indlim}_{p\to\infty} (X_i)_{-p} \cong \bigcup_{p\geq 0} (X_i)_{-p}$$

and the triple

$$X_i \subset H_i \equiv H_i' \subset X_i' \tag{1}$$

is called a real standard triple [20]. For $i = 1, 2$, let $N_i$ be the complexification of the real nuclear space $X_i$. For $p \in \mathbb{N}$, we denote by $(N_i)_p$ the complexification of $(X_i)_p$ and by $(N_i)_{-p}$, respectively, $N_i'$ the strong dual space of $(N_i)_p$ and $N_i$. Then, we obtain

$$N_i = \text{proj} \lim_{p\to\infty} (N_i)_p \quad \textit{and} \quad N_i' = \text{ind} \lim_{p\to\infty} (N_i)_{-p}. \tag{2}$$

The spaces $N_i$ and $N_i'$ are, respectively, equipped with the projective and inductive limit topology. For all $p \in \mathbb{N}$, we denote by $|.|_{-p}$ the norm on $(N_i)_{-p}$ and by $\langle ., . \rangle$ the $\mathbb{C}-$bilinear form on $N_i' \times N_i$. In the following, $\mathscr{H}$ denote by the direct Hilbertian sum of $(N_1)_0$ and $(N_2)_0$, i.e., $\mathscr{H} = (N_1)_0 \oplus (N_2)_0$. For $n \in \mathbb{N}$, we denote by $N_i^{\widehat{\otimes} n}$ the n-fold symmetric tensor product on $N_i$ equipped with the $\pi-$topology and by $(N_i)_p^{\widehat{\otimes} n}$ the n-fold symmetric Hilbertian tensor product on $(N_i)_p$. We will preserve the notation $|.|_p$ and $|.|_{-p}$ for the norms on $(N_i)_p^{\widehat{\otimes} n}$ and $(N_i)_{-p}^{\widehat{\otimes} n}$, respectively.

Let $\theta$ be a Young function, i.e., it is a continuous, convex and increasing function defined on $\mathbb{R}^+$ and satisfies the two conditions: $\theta(0) = 0$ and $\lim_{r\to\infty} \dfrac{\theta(r)}{r} = \infty$. Obviously, the conjugate function $\theta^*$ of $\theta$ defined by

$$\forall x \geq 0, \quad \theta^*(x) := \sup_{t\geq 0}(tx - \theta(t)),$$

is also a Young function. For every $n \in \mathbb{N}$, let

$$(\theta)_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}. \tag{3}$$

Throughout the paper, we fix a pair of Young function $(\theta_1, \theta_2)$. From now on, we assume that the Young functions $\theta_i$ satisfy

$$\lim_{r\to\infty} \frac{\theta_i(r)}{r^2} < \infty. \tag{4}$$

Note that, if a Young function $\theta$ satisfies condition (4), there exist constant numbers $\alpha$ and $\gamma$ such that

$$(\theta)_n \leq \alpha \left(\frac{2e\gamma}{n}\right)^{n/2} \tag{5}$$

and, for $r > 0$ such that $r\gamma < 1$,

$$\sum_{n=0}^{\infty} r^n n! (\theta)_{2n} < \infty. \tag{6}$$

For a complex Banach space $(\mathscr{C}, \|\cdot\|)$, let $\mathscr{H}(\mathscr{C})$ denotes the space of all entire functions on $\mathscr{C}$, i.e., of all continuous $\mathbb{C}$-valued functions on $\mathscr{C}$ whose restrictions to all affine lines of $\mathscr{C}$ are entire on $\mathbb{C}$. For each $m > 0$, we denote by $\text{Exp}(\mathscr{C}, \theta, m)$ the space of all entire functions on $\mathscr{C}$ with $\theta-$exponential growth of finite type m, i.e.,

$$\text{Exp}(\mathscr{C}, \theta, m) = \left\{ f \in \mathscr{H}(\mathscr{C}); \quad \|f\|_{\theta,m} := \sup_{z\in\mathscr{C}} |f(z)| e^{-\theta(m\|z\|)} < \infty \right\}.$$

The projective system $\{\text{Exp}((N_i)_{-p}, \theta, m);\ p \in \mathbb{N},\ m > 0\}$ gives the space

$$\mathscr{F}_\theta(N_i') := \text{proj} \lim_{p\to\infty; m\downarrow 0} \text{Exp}((N_i)_{-p}, \theta, m). \tag{7}$$

It is noteworthy that, for each $\xi \in N_i$, the exponential function

$e_\xi(z) := e^{\langle z,\xi\rangle}, \quad z \in N_i'$,

belongs to $\mathscr{F}_\theta(N_i')$ and the set of such test functions spans a dense subspace of $\mathscr{F}_\theta(N_i')$.

For all positive numbers $m_1, m_2 > 0$ and all integers $(p_1, p_2) \in \mathbb{N}\times\mathbb{N}$, we define the space of all entire functions on $(N_1)_{-p_1} \oplus (N_2)_{-p_2}$ with $(\theta_1, \theta_2)-$exponential growth by

$$\begin{aligned}&\mathrm{Exp}((N_1)_{-p_1} \oplus (N_2)_{-p_2}, (\theta_1, \theta_2), (m_1, m_2))\\ &\quad= \{f \in \mathscr{H}((N_1)_{-p_1} \oplus (N_2)_{-p_2});\ \|f\|_{(\theta_1,\theta_2);(p_1,p_2);(m_1,m_2)} < \infty\}\end{aligned}$$

where $\mathscr{H}((N_1)_{-p_1} \oplus (N_2)_{-p_2})$ is the space of all entire functions on $(N_1)_{-p_1} \oplus (N_2)_{-p_2}$ and

$$\|f\|_{(\theta_1,\theta_2);(p_1,p_2);(m_1,m_2)} = \sup\{|f(z_1, z_2)|e^{-\theta_1(m_1|z_1|_{-p_1})-\theta_2(m_2|z_2|_{-p_2})}\}$$

for $(z_1, z_2) \in (N_1)_{-p_1} \oplus (N_2)_{-p_2}$. So, the space of all entire functions on $(N_1)_{-p_1} \oplus (N_2)_{-p_2}$ with $(\theta_1, \theta_2)-$exponential growth of minimal type is naturally defined by

$$\mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2') = \underset{p_1,p_2\to\infty,m_1,m_2\downarrow 0}{\mathrm{proj}\lim} Exp((N_1)_{-p_1} \oplus (N_2)_{-p_2}, (\theta_1, \theta_2), (m_1, m_2)). \tag{8}$$

By definition, $\varphi \in \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$ admits the Taylor expansions:

$$\varphi(x, y) = \sum_{n,m=0}^{\infty} \langle x^{\otimes n} \otimes y^{\otimes m}, \varphi_{n,m}\rangle, \quad (x, y) \in N_1' \times N_2' \tag{9}$$

where for all $n, m \in \mathbb{N}$, we have $\varphi_{n,m} \in N_1^{\widehat{\otimes} n} \otimes N_2^{\widehat{\otimes} m}$ and we used the common symbol $\langle ., .\rangle$ for the canonical $\mathbb{C}-$bilinear form on $(N_1^{\otimes n} \times N_2^{\otimes m})' \times N_1^{\otimes n} \times N_2^{\otimes m}$. So, we identify in the next all test function $\varphi \in \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$ by their coefficients of its Taylors series expansion at the origin $(\varphi_{n,m})_{n,m\in\mathbb{N}}$. As important example of elements in $\mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$, we define the exponential function as follows. For a fixed $(\xi, \eta) \in N_1 \times N_2$,

$$e_{(\xi,\eta)}(a, b) = (e_\xi \otimes e_\eta)(a, b) = \exp\{\langle a, \xi\rangle + \langle b, \eta\rangle\}, \quad (a, b) \in N_1' \times N_2'.$$

Let $\varphi \sim (\varphi_{n,m})_{n\geq 0}$ in $\mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$. Then, from [15] for any $p_1, p_2 \geq 0$ and $m_1, m_2 > 0$, there exist $q_1 > p_1$ and $q_2 > p_2$ such that

$$\begin{aligned}|\varphi_{n,m}|_{p_1,p_2} &\leq e^{n+m}(\theta_1)_n(\theta_2)_m m_1^n m_2^m \|i_{q_1,p_1}\|_{HS}^n \|i_{q_2,p_2}\|_{HS}^m\\ &\quad\times\|\varphi\|_{(\theta_1,\theta_2);(q_1,q_2);(m_1,m_2)}.\end{aligned} \tag{10}$$

Denoted by $\mathscr{F}^*_{\theta_1,\theta_2}(N_1' \oplus N_2')$ the topological dual of $\mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$ called the space of distribution on $N_1' \oplus N_2'$. In the particular case where $N_2 = \{0\}$, we obtain the following identification

$$\mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus \{0\}) = \mathscr{F}_{\theta_1}(N_1')$$

and therefore

$$\mathscr{F}^*_{\theta_1,\theta_2}(N_1' \oplus \{0\}) = \mathscr{F}^*_{\theta_1}(N_1').$$

So, the space $\mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$ can be considered as a generalization of the space $\mathscr{F}_{\theta_1}(N_1')$ studied in [11].

## 3 Quantum O–U semigroup and quantum number operator

### 3.1 Quantum O–U semigroup

Let $\varphi(y_1, y_2) = \sum_{n,m=0}^{\infty}\langle y_1^{\otimes n} \otimes y_2^{\otimes m}, \varphi_{n,m}\rangle \in \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$. For $s, t \geq 0$, let $a_t = \sqrt{1 - \exp(-2t)}$ and $b_t = \exp(-t)$. Then, we define $O_{s,t}$ by

$$O_{s,t}\varphi(y_1,y_2)=\int_{X_1'\times X_2'}\varphi(a_sx_1+b_sy_1,a_tx_2+b_ty_2)d\mu_1(x_1)d\mu_2(x_2),$$

where $\mu_j$ is the standard Gaussian measure on $X_j'$ $(for j=1,2)$ uniquely specified by its characteristic function

$$e^{-\frac{1}{2}|\xi|_0^2}=\int_{X_j'}e^{i\langle x,\xi\rangle}\mu_j(dx),\quad \xi\in X_j.$$

**Proposition 1** *Let $s,t\geq 0$. Then, the operator $O_{s,t}$ is continuous linear from $\mathscr{F}_{\theta_1,\theta_2}(N_1'\oplus N_2')$ into itself.*

*Proof* Let $\varphi\in\mathscr{F}_{\theta_1,\theta_2}(N_1'\oplus N_2')$. For any $p_1,p_2\geq 0$ and $m_1,m_2>0$, there exist $p_1',p_2'\geq 0$ and $m_1',m_2'>0$ such that $\left|O_{s,t}\varphi(y_1,y_2)\right|$

$$\begin{aligned}&\leq\int_{X_1'\times X_2'}|\varphi(a_sx_1+b_sy_1,a_tx_2+b_ty_2)|\,d\mu_1(x_1)d\mu_2(x_2)\\&\leq\|\varphi\|_{(\theta_1,\theta_2);(p_1',p_2');(m_1',m_2')}\int_{X_1'}\exp\left\{\theta_1\left(\frac{1}{2}m_1|a_sx_1+b_sy_1|_{-p_1}\right)\right\}d\mu_1(x_1)\\&\quad\times\int_{X_2'}\exp\left\{\theta_2\left(\frac{1}{2}m_2|a_tx_2+b_ty_2|_{-p_2}\right)\right\}d\mu_2(x_2).\end{aligned}$$

Since, for $i=1,2$, $\theta_i$ are convex, we have

$$\theta_i\left(\frac{1}{2}m_i\,|a_sx_i+b_sy_i|_{-p_i}\right)\leq\frac{1}{2}\theta_i(m_i\,|a_s|\,|x_i|_{-p_i})+\frac{1}{2}\theta_i(m_i\,|b_s|\,|y_i|_{-p_i}).$$

Therefore, we obtain $\left|O_{s,t}\varphi(y_1,y_2)\right|$

$$\begin{aligned}&\leq\|\varphi\|_{(\theta_1,\theta_2);(p_1',p_2');(m_1',m_2')}\exp\{\theta_1(m_1\,|b_s|\,|y_1|_{-p_1})+\theta_2(m_2\,|b_t|\,|y_2|_{-p_2})\}\\&\quad\times\int_{(X_1)_{-p_1}}\exp\{\theta_1(m_1\,|a_s|\,|x_1|_{-p_1})\}d\mu_1(x_1)\int_{(X_2)_{-p_2}}\exp\{\theta_2(m_2\,|a_t|\,|x_2|_{-p_2})\}d\mu_2(x_2).\end{aligned}$$

Recall that, for $p_i>1$ and $i=1,2$, $(H_i,(X_i)_{-p_i})$ is an abstract Wiener space. Then, under the condition $\lim_{r\to\infty}\dfrac{\theta_i(r)}{r^2}<\infty$, the measure $\mu_i$ satisfies the Fernique theorem, i.e., there exist some $\alpha_i>0$ such that

$$\int_{(X_i)_{-p_i}}\exp\{\alpha_i|x_i|_{-p_i}^2\}d\mu_i(x_i)<\infty.\tag{11}$$

Hence, in view of (11), we obtain

$$\begin{aligned}&\left|O_{s,t}\varphi(y_1,y_2)\right|\exp\{-\theta_1(m_1\,|b_s|\,|y_1|_{-p_1})-\theta_2(m_2\,|b_t|\,|y_2|_{-p_2})\}\\&\quad\leq I_{p_1,p_2}^{m_1,m_2}\|\varphi\|_{(\theta_1,\theta_2);(p_1',p_2');(m_1',m_2')},\end{aligned}$$

where the constant $I_{p_1,p_2}^{m_1,m_2}$ is given by

$$\begin{aligned}I_{p_1,p_2}^{m_1,m_2}=&\int_{(X_1)_{-p_1}}\exp\{\theta_1(m_1\,|a_s|\,|x_1|_{-p_1})\}d\mu_1(x_1)\\&\times\int_{(X_2)_{-p_2}}\exp\{\theta_2(m_2\,|a_t|\,|x_2|_{-p_2})\}d\mu_2(x_2).\end{aligned}$$

This follows that

$$\left\| O_{s,t}\varphi \right\|_{(\theta_1,\theta_2);(p_1,p_2);(m_1,m_2)} \leq I^{m_1,m_2}_{p_1,p_2} \left\|\varphi\right\|_{(\theta_1,\theta_2);(p'_1,p'_2);(m'_1,m'_2)} .$$

This completes the proof. □

Later on, we need the following Lemma for Taylor expansion.

**Lemma 1** *For $s,t \geq 0$ and $n,m \in \mathbb{N}$, we have* $\int_{X'_1\times X'_2}(a_s x_1 + b_s y_1)^{\otimes n} \otimes (a_t x_2 + b_t y_2)^{\otimes m} d\mu_1(x_1)d\mu_2(x_2)$

$$= \sum_{k=0}^{[n/2]}\sum_{l=0}^{[m/2]} \frac{n!m!a_s^{2k}a_t^{2l}b_s^{n-2k}b_t^{m-2l}}{(n-2k)!(m-2l)!2^{l+k}k!l!}(\tau_1^{\otimes k}\widehat{\otimes}y_1^{\otimes n-2k}) \otimes (\tau_2^{\otimes l}\widehat{\otimes}y_2^{\otimes m-2l}),$$

*where $\tau_i$ is the usual trace on $N_i$ for i=1,2.*

*Proof* Using the following equality,

$$(ax+by)^{\otimes n} = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(ax)^{\otimes k}\widehat{\otimes}(by)^{\otimes n-k},$$

then, for $\xi_1 \in N_1$ and $\xi_2 \in N_2$, we easily obtain

$$\left\langle \int_{X'_1\times X'_2}(a_s x_1 + b_s y_1)^{\otimes n} \otimes (a_t x_2 + b_t y_2)^{\otimes m} d\mu(x_1)d\mu(x_2), \xi_1^{\otimes n}\otimes\xi_2^{\otimes m}\right\rangle$$
$$= \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}a_s^k b_s^{n-k}\left\langle y_1^{\otimes n-k},\xi_1^{\otimes n-k}\right\rangle \int_{X'_1}\left\langle x_1^{\otimes k},\xi_1^{\otimes k}\right\rangle d\mu_1(x_1)$$
$$\times\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}a_t^l b_t^{m-l}\left\langle y_2^{\otimes m-l},\xi_2^{\otimes m-l}\right\rangle \int_{X'_2}\left\langle x_2^{\otimes l},\xi_2^{\otimes l}\right\rangle d\mu_2(x_2).$$

We recall the following identity for the Gaussian white noise measure; see [20],

$$\int_{X'_i}\langle x_i^{\otimes k},\xi_i^{\otimes k}\rangle d\mu_i(x_i) = \begin{cases} \frac{(2j)!}{2^j j!}\,|\xi_i|_0^2 & if \quad k=2j \\ 0 & if \quad k=2j+1 \end{cases},$$

from which we deduce that

$$\left\langle \int_{X'_1\times X'_2}(a_s x_1 + b_s y_1)^{\otimes n} \otimes (a_t x_2 + b_t y_2)^{\otimes m} d\mu(x_1)d\mu(x_2), \xi_1^{\otimes n}\otimes\xi_2^{\otimes m}\right\rangle$$
$$= \sum_{k=0}^{[n/2]}\frac{n!a_s^{2k}b_s^{n-2k}\left\langle y_1^{\otimes n-2k},\xi_1^{\otimes n-2k}\right\rangle}{(2k)!(n-2k)!}\frac{(2k)!|\xi_1|^{2k}}{2^k k!}$$
$$\times\sum_{l=0}^{[m/2]}\frac{m!a_t^{2l}b_t^{m-2l}\left\langle y_2^{\otimes m-2l},\xi_2^{\otimes m-2l}\right\rangle}{(2l)!(m-2l)!}\frac{(2l)!|\xi_2|^{2l}}{2^l l!}$$
$$= \sum_{k=0}^{[n/2]}\sum_{l=0}^{[m/2]}\frac{n!m!a_s^{2k}a_t^{2l}b_s^{n-2k}b_t^{m-2l}}{(n-2k)!(m-2l)!2^{l+k}k!l!}$$
$$\times\langle(\tau_1^{\otimes k}\widehat{\otimes}y_1^{\otimes n-2k}) \otimes (\tau_2^{\otimes l}\widehat{\otimes}y_2^{\otimes m-2l}),\xi_1^{\otimes n}\otimes\xi_2^{\otimes m}\rangle.$$

The above equalities hold for all $\xi_1^{\otimes n}$ and $\xi_2^{\otimes m}$ with $\xi_1 \in N_1$ and $\xi_2 \in N_2$; thus, the statement follows by the polarization identity (see [18,20]). □

Now, we can use Lemma (1) to represent $O_{s,t}$ by Taylor expansion.

**Proposition 2** *Let $s, t \geq 0$, then for any $\varphi \in \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$ given by $\varphi(y_1, y_2) = \sum_{n,m=0}^{\infty} \langle y_1^{\otimes n} \otimes y_2^{\otimes m}, \varphi_{n,m}\rangle$, we have*

$$(O_{s,t}\varphi)(y_1, y_2) = \sum_{n,m=0}^{\infty} \langle y_1^{\otimes n} \otimes y_2^{\otimes m}, g_{n,m}\rangle,$$

*where $g_{n,m}$ is given by*

$$g_{n,m} = \frac{b_s^n b_t^m}{n!m!} \sum_{k,l=0}^{\infty} \frac{(n+2k)!(m+2l)!}{2^{l+k}k!l!} a_s^{2k} a_t^{2l} (\tau_1^{\otimes k} \otimes \tau_2^{\otimes l}) \widehat{\otimes}_{2k,2l} \varphi_{n+2k,m+2l}$$

*and, for $\xi_1 \in N_1$, $\xi_2 \in N_2$,*

$$(\tau_1^{\otimes k} \otimes \tau_2^{\otimes l}) \widehat{\otimes}_{2k,2l} (\xi_1^{\otimes n+2k} \otimes \xi_2^{\otimes m+2l}) = \langle \xi_1, \xi_1\rangle^k \langle \xi_2, \xi_2\rangle^l (\xi_1^{\otimes n} \otimes \xi_2^{\otimes m}).$$

*Proof* Consider $\varphi_{(\nu_1,\nu_2)}(z_1, z_2) = \sum_{n,m=0}^{\nu_1,\nu_2} \langle z_1^{\otimes n} \otimes z_2^{\otimes m}, \varphi_{n,m}\rangle$ as an approximating sequence of $\varphi \in \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$. Then, for any $p_i \in \mathbb{N}$, $i = 1, 2$ and $m_i > 0$, there exist $M \geq 0$ such that

$$\left|\varphi_{(\nu_1,\nu_2)}(z_1, z_2)\right| \leq M e^{\theta_1(m_1|z_1|_{-p_1}) + \theta_2(m_2|z_2|_{-p_2})}.$$

Hence, in view of (11), we can apply the Lebesgue dominated convergence theorem to get

$$O_{s,t}\varphi(y_1, y_2) = \sum_{n,m=0}^{\infty} \int_{X_1' \times X_2'} \left\langle (a_s x_1 + b_s y_1)^{\otimes n} \otimes (a_t x_2 + b_t y_2)^{\otimes m}, \varphi_{n,m}\right\rangle d\mu_1(x_1) d\mu_2(x_2).$$

Then, by Lemma (1),

$$O_{s,t}\varphi(y_1, y_2) = \sum_{n,m=0}^{\infty} \sum_{k=0}^{[n/2]} \sum_{l=0}^{[m/2]} \frac{n!m! a_s^{2k} a_t^{2l} b_s^{n-2k} b_t^{m-2l}}{(n-2k)!(m-2l)!2^{l+k}k!l!} \times \left\langle (\tau_1^{\otimes k} \widehat{\otimes} y_1^{\otimes n-2k}) \otimes (\tau_2^{\otimes l} \widehat{\otimes} y_2^{\otimes m-2l}), \varphi_{n,m}\right\rangle.$$

By changing the order of summation (which can be justified easily), we get

$$O_{s,t}\varphi(y_1, y_2) = \sum_{k,l=0;}^{\infty} \sum_{n=2k;}^{\infty} \sum_{m=2l}^{\infty} \frac{n!m! a_s^{2k} a_t^{2l} b_s^{n-2k} b_t^{m-2l}}{(n-2k)!(m-2l)!2^{l+k}k!l!} \times \left\langle y_1^{\otimes n-2k} \otimes y_2^{\otimes m-2l}, (\tau_1^{\otimes k} \otimes \tau_2^{\otimes l}) \widehat{\otimes}_{2k,2l} \varphi_{n,m}\right\rangle.$$

Therefore, we sum over $n - 2k = j$ for $j \geq 0$ and $m - 2l = i$ for $i \geq 0$ to get

$$\begin{aligned} O_{s,t}\varphi(y_1, y_2) &= \sum_{k,l,j,i=0}^{\infty} \frac{(j+2k)!(i+2l)! a_s^{2k} a_t^{2l} b_s^j b_t^i}{j!i!2^{l+k}k!l!} \left\langle y_1^{\otimes j} \otimes y_2^{\otimes i}, (\tau_1^{\otimes k} \otimes \tau_2^{\otimes l}) \widehat{\otimes}_{2k,2l} \varphi_{j+2k,i+2l}\right\rangle \\ &= \sum_{j,i=0}^{\infty} \left\langle y_1^{\otimes j} \otimes y_2^{\otimes i}, \sum_{k,l=0}^{\infty} \frac{(j+2k)!(i+2l)! a_s^{2k} a_t^{2l} b_s^j b_t^i}{j!i!2^{l+k}k!l!} (\tau_1^{\otimes k} \otimes \tau_2^{\otimes l}) \widehat{\otimes}_{2k,2l} \varphi_{j+2k,i+2l}\right\rangle. \end{aligned}$$

This proves the desired statement. □

Denoting by $\mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ to be the space of continuous linear operators from a nuclear space $\mathfrak{X}$ to another nuclear space $\mathfrak{Y}$. From the nuclearity of the spaces $\mathscr{F}_{\theta_i}(N_i')$, we have by Kernel Theorem the following isomorphisms:

$$\mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1'), \mathscr{F}_{\theta_2}(N_2')) \simeq \mathscr{F}_{\theta_1}(N_1') \otimes \mathscr{F}_{\theta_2}(N_2') \simeq \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2'). \tag{12}$$

So, for every $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1'), \mathscr{F}_{\theta_2}(N_2'))$, the associated kernel $\Phi_\Xi \in \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$ is defined by

$$\langle\langle \Xi\varphi, \psi \rangle\rangle = \langle\langle \Phi_\Xi, \varphi \otimes \psi \rangle\rangle, \quad \forall \varphi \in \mathscr{F}^*_{\theta_1}(N_1'),\ \forall \psi \in \mathscr{F}^*_{\theta_2}(N_2'). \tag{13}$$

Using the topological isomorphism:

$$\mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1'), \mathscr{F}_{\theta_2}(N_2')) \ni \Xi \longmapsto \mathscr{K}\,\Xi = \Phi_\Xi \in \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2'), \tag{14}$$

we can define the quantum O–U semigroup as follows. For the operator $O_{s,t}$ defined in this section, we write $\widetilde{O_{s,t}} = \mathscr{K}^{-1} O_{s,t} \mathscr{K} \in \mathscr{L}\left(\mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1'), \mathscr{F}_{\theta_2}(N_2'))\right)$.

The operator $\widetilde{O_{t,t}}$, denoted by $\widetilde{O_t}$ for simplicity, is called the **quantum O–U semigroup**. The operator $\widetilde{O_{s,0}}$ is called the **left quantum O–U semigroup** and the operator $\widetilde{O_{0,t}}$ is called the **right quantum O–U semigroup**.

Recall that the classical O–U semigroup studied in [17,18] is defined by

$$q_t\varphi(y) = \int_{X_i'} \varphi(a_t x + b_t y) d\mu(x), \quad y \in N_i',\ \varphi \in \mathscr{F}_\theta(N_i'). \tag{15}$$

Then, we have the following

**Proposition 3** *Let $s, t \geq 0$, then we have*

$O_{s,t} = q_s \otimes q_t,$

*where $q_t$ is the classical O–U semigroup.*

*Proof* We can easily check that

$$q_t e_{\xi_i} = \exp\left\{\frac{a_t^2}{2}|\xi_i|_0^2\right\} e_{b_t\xi_i}, \quad \text{for} \quad i = 1, 2$$

and

$$O_{s,t} e_{(\xi_1,\xi_2)} = \exp\left\{\frac{a_s^2}{2}|\xi_1|_0^2 + \frac{a_t^2}{2}|\xi_2|_0^2\right\} e_{(b_s\xi_1, b_t\xi_2)}. \tag{16}$$

Then, since $\{e_{(\xi_1,\xi_2)}, \xi_1 \in N_1, \xi_2 \in N_2\}$ spans a dense subspace of $\mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$, we have the result. □

**Theorem 1** *Let $s, t \geq 0$, then we have*

$\widetilde{O_{s,t}}(\Xi) = q_s \Xi q_t^*, \quad \Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1'), \mathscr{F}_{\theta_2}(N_2')),$

*where $q_t^*$ is the adjoint operator of $q_t$.*

*Proof* Let $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1'), \mathscr{F}_{\theta_2}(N_2'))$, $\phi \in \mathscr{F}^*_{\theta_1}(N_1')$ and $\varphi \in \mathscr{F}^*_{\theta_2}(N_2')$. Then, by Proposition 3, we have

$$\begin{aligned}\langle\langle \widetilde{O_{s,t}}(\Xi)\phi, \varphi \rangle\rangle &= \langle\langle O_{s,t}(\mathscr{K}\,\Xi), \varphi \otimes \phi \rangle\rangle \\ &= \langle\langle \mathscr{K}\,\Xi, (q_s^*\varphi) \otimes (q_t^*\phi) \rangle\rangle \\ &= \langle\langle \Xi q_t^*\phi, q_s^*\varphi \rangle\rangle \\ &= \langle\langle q_s \Xi q_t^*\phi, \varphi \rangle\rangle,\end{aligned}$$

which gives the result. □

### 3.2 Quantum number operator

Let $\varphi(x,y)=\sum_{n,m=0}^{\infty}\langle x^{\otimes n}\otimes y^{\otimes m},\varphi_{n,m}\rangle$ in $\mathscr{F}_{\theta_1,\theta_2}(N_1'\oplus N_2')$, then we define the three following operators by:

$$\mathscr{N}\varphi(x,y):=\sum_{n,m=0;(n,m)\neq(0,0)}^{\infty}(n+m)\langle x^{\otimes n}\otimes y^{\otimes m},\varphi_{n,m}\rangle. \tag{17}$$

$$\mathscr{N}_1\varphi(x,y):=\sum_{n=1,m=0}^{\infty}n\langle x^{\otimes n}\otimes y^{\otimes m},\varphi_{n,m}\rangle, \tag{18}$$

$$\mathscr{N}_2\varphi(x,y):=\sum_{n=0,m=1}^{\infty}m\langle x^{\otimes n}\otimes y^{\otimes m},\varphi_{n,m}\rangle. \tag{19}$$

**Proposition 4** *$\mathscr{N}$, $\mathscr{N}_1$ and $\mathscr{N}_2$ are linear continuous operators from $\mathscr{F}_{\theta_1,\theta_2}(N_1'\oplus N_2')$ into itself.*

*Proof* Let $p_1,p_2\geq 0$. From (17), we deduce that

$$|\mathscr{N}\varphi(x,y)|\leq\sum_{n,m=0;(n,m)\neq(0,0)}^{\infty}(n+m)|x|_{-p_1}^{n}|y|_{-p_2}^{m}|\varphi_{n,m}|_{p_1,p_2}.$$

Therefore, using the fact that $(n+m)\leq 2^{n+m}$ and the inequality (10), for $q_1>p_1$, $q_2>p_2$ and $m_1,m_2>0$, we have

$$|\mathscr{N}\varphi(x,y)|\leq\|\varphi\|_{(\theta_1,\theta_2);(q_1,q_2);(m_1,m_2)}$$
$$\times\sum_{n,m=0}^{\infty}\{2m_1e\|i_{q_1,p_1}\|_{HS}\}^n|x|_{-p_1}^{n}(\theta_1)_n\{2m_2e\|i_{q_2,p_2}\|_{HS}\}^m|y|_{-p_2}^{m}(\theta_2)_m.$$

Then, using (3), for $m_1',m_2'>0$, $m_1,m_2>0$, $q_1>p_1$ and $q_2>p_2$ such that

$$\max\left\{2\frac{m_1}{m_1'}e\|i_{q_1,p_1}\|_{HS},2\frac{m_2}{m_2'}e\|i_{q_2,p_2}\|_{HS}\right\}<1,$$

we get

$$\|\mathscr{N}\varphi\|_{(\theta_1,\theta_2);(p_1,p_2);(m_1',m_2')}\leq\|\varphi\|_{(\theta_1,\theta_2);(q_1,q_2);(m_1,m_2)}c_{p_1,p_2,q_1,q_2}$$

where

$$c_{p_1,p_2,q_1,q_2}=\left\{1-(2\frac{m_1}{m_1'}e\|i_{q_1,p_1}\|_{HS})\right\}^{-1}\left\{1-(2\frac{m_2}{m_2'}e\|i_{q_2,p_2}\|_{HS})\right\}^{-1}.$$

Hence, we prove the continuity of $\mathscr{N}$. Similarly, we complete the proof. □

Recall that the standard number operator on $\mathscr{F}_{\theta_i}(N_i')$ is given by

$$N\varphi(x)=\sum_{n=1}^{\infty}\langle x^{\otimes n},n\varphi_n\rangle, \tag{20}$$

where $\varphi(x)=\sum_{n=0}^{\infty}\langle x^{\otimes n},\varphi_n\rangle\in\mathscr{F}_{\theta_i}(N_i')$.

*Remark 1* From (17) and (20), we can easily see that $\mathscr{N}_1$, $\mathscr{N}_2$ and $\mathscr{N}$ have the following decompositions

$$\mathscr{N}_1 = N \otimes I, \quad \mathscr{N}_2 = I \otimes N, \quad \mathscr{N} = N \otimes I + I \otimes N,$$

respectively.

**Definition 1** We define the following operator on $\mathscr{L}(\mathscr{F}^*_{\theta_1}(N'_1), \mathscr{F}_{\theta_2}(N'_2))$ by

$$\widetilde{\mathscr{N}_1} := \mathscr{K}^{-1}(\mathscr{N}_1)\mathscr{K}, \quad \widetilde{\mathscr{N}_2} := \mathscr{K}^{-1}(\mathscr{N}_2)\mathscr{K}, \quad \widetilde{\mathscr{N}} := \mathscr{K}^{-1}\mathscr{N}\mathscr{K} = \widetilde{\mathscr{N}_1} + \widetilde{\mathscr{N}_2}.$$

The operator $\widetilde{\mathscr{N}_1}$ is called **left quantum number operator**, $\widetilde{\mathscr{N}_2}$ is called **right quantum number operator** and $\widetilde{\mathscr{N}}$ is called **quantum number operator**.

**Proposition 5** *For any* $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N'_1), \mathscr{F}_{\theta_2}(N'_2))$, *we have*

$$\widetilde{\mathscr{N}_1}\Xi = N\Xi, \quad \widetilde{\mathscr{N}_2}\Xi = \Xi N, \quad \widetilde{\mathscr{N}}\Xi = N\Xi + \Xi N.$$

*Proof* Let $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N'_1), \mathscr{F}_{\theta_2}(N'_2))$. Then, for any $\psi \in \mathscr{F}^*_{\theta_1}(N'_1)$ and $\varphi \in \mathscr{F}^*_{\theta_2}(N'_2)$, we have

$$\begin{aligned}
\langle\langle \widetilde{\mathscr{N}_1}\Xi\psi, \varphi\rangle\rangle &= \langle\langle \mathscr{K}^{-1}\mathscr{N}_1 K\Xi\psi, \varphi\rangle\rangle \\
&= \langle\langle \mathscr{N}_1\mathscr{K}\Xi, \varphi \otimes \psi\rangle\rangle \\
&= \langle\langle \mathscr{K}\Xi, (N\varphi) \otimes \psi\rangle\rangle \\
&= \langle\langle \Xi\psi, N\varphi\rangle\rangle \\
&= \langle\langle N\Xi\psi, \varphi\rangle\rangle,
\end{aligned}$$

which follows that, for any $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N'_1), \mathscr{F}_{\theta_2}(N'_2))$,

$$\widetilde{\mathscr{N}_1}\Xi = N\Xi.$$

Similarly, we get $\langle\langle \widetilde{\mathscr{N}_2}\Xi\psi, \varphi\rangle\rangle = \langle\langle \Xi N\psi, \varphi\rangle\rangle$ to obtain $\widetilde{\mathscr{N}_2}\Xi = \Xi N$. Finally, we get

$$\widetilde{\mathscr{N}}\Xi = \widetilde{\mathscr{N}_1}\Xi + \widetilde{\mathscr{N}_2}\Xi = N\Xi + \Xi N.$$

This completes the proof. □

Note that Definition 1 holds true on $\mathscr{L}(\mathscr{F}_{\theta_1}(N'_1), \mathscr{F}^*_{\theta_2}(N'_2))$.

## 4 Cauchy problem associated with quantum number operator

First, we will construct a semigroup $\{\widetilde{Q}_t,\ t \geq 0\}$, $\{\widetilde{Q}_{s,0},\ s \geq 0\}$ and $\{\widetilde{Q}_{0,t},\ t \geq 0\}$ on $\mathscr{L}(\mathscr{F}^*_{\theta_1}(N'_1), \mathscr{F}_{\theta_2}(N'_2))$ with infinitesimal generator $-\widetilde{\mathscr{N}}$, $-\widetilde{\mathscr{N}_1}$ and $-\widetilde{\mathscr{N}_2}$, respectively. It reminds constructing a semigroup $\{Q_t,\ t \geq 0\}$, $\{Q_{s,0},\ s \geq 0\}$ and $\{Q_{0,t},\ t \geq 0\}$ on $\mathscr{F}_{\theta_1,\theta_2}(N'_1 \oplus N'_2)$ with infinitesimal generator $-\mathscr{N}$, $-\mathscr{N}_1$ and $-\mathscr{N}_2$, respectively. Observe that symbolically $Q_{s,t} = e^{-s\mathscr{N}_1 - t\mathscr{N}_2}$. Thus, we can define the operator $Q_{s,t}$ as follows. For $\varphi \sim (\varphi_{n,m})$, we define

$$Q_{s,t}\varphi(x, y) := \sum_{n,m=0}^{\infty} \left\langle x^{\otimes n} \otimes y^{\otimes m}, e^{-sn-tm}\varphi_{n,m}\right\rangle, \tag{21}$$

and let $Q_{t,t}$ denoted by $Q_t$.

**Lemma 2** *For any* $s, t \geq 0$, *the linear operator* $Q_{s,t}$ *is continuous from* $\mathscr{F}_{\theta_1,\theta_2}(N'_1 \oplus N'_2)$ *into itself.*

*Proof* Let $\varphi \sim (\varphi_{n,m})$. For any $p_1,p_2 \geq 0$, we have

$$\left|Q_{s,t}\varphi(x,y)\right| \leq \sum_{n,m=0}^{\infty} e^{-sn-tm}|x|_{-p_1}^{n}|y|_{-p_2}^{m}|\varphi_{n,m}|_{p_1,p_2}$$
$$\leq \sum_{n,m=0}^{\infty} |x|_{-p_1}^{n}|y|_{-p_2}^{m}|\varphi_{n,m}|_{p_1,p_2}.$$

Therefore, using the inequality (10), for $q_1 > p_1$, $q_2 > p_2$ and $m_1, m_2 > 0$, we get

$$\left|Q_{s,t}\varphi(x,y)\right| \leq \|\varphi\|_{(\theta_1,\theta_2);(q_1,q_2);(m_1,m_2)} \times \sum_{n,m=0}^{\infty} \{m_1 e\|i_{q_1,p_1}\|_{HS}\}^n |x|_{-p_1}^{n}(\theta_1)_n \{m_2 e\|i_{q_2,p_2}\|_{HS}\}^m |y|_{-p_2}^{m}(\theta_2)_m. \tag{22}$$

Then, using (3), for $m_1',m_2' > 0$, $m_1,m_2 > 0$, $q_1 > p_1$ and $q_2 > p_2$ such that

$$\max\left\{\frac{m_1}{m_1'}e\|i_{q_1,p_1}\|_{HS}, \frac{m_2}{m_2'}e\|i_{q_2,p_2}\|_{HS}\right\} < 1,$$

we get

$$\|Q_{s,t}\varphi\|_{(\theta_1,\theta_2);(p_1,p_2);(m_1',m_2')} \leq \|\varphi\|_{(\theta_1,\theta_2);(q_1,q_2);(m_1,m_2)} K_{p_1,p_2,q_1,q_2}, \tag{23}$$

where $K_{p_1,p_2,q_1,q_2}$ is given by

$$K_{p_1,p_2,q_1,q_2} = \left\{1 - \left(\frac{m_1}{m_1'}e\|i_{q_1,p_1}\|_{HS}\right)\right\}^{-1}\left\{1 - \left(\frac{m_2}{m_2'}e\|i_{q_2,p_2}\|_{HS}\right)\right\}^{-1}.$$

This proves the desired statement. □

*Remark 2* Using (21), Lemma 2, Proposition 2 and a similar classical argument used in [18], we can show that $Q_{s,t} = O_{s,t}$. Moreover, we see that

$$\widetilde{Q}_{s,t} := \mathscr{K}^{-1}Q_{s,t}\mathscr{K} = \widetilde{O}_{s,t} \in \mathscr{L}(\mathscr{L}(\mathscr{F}_{\theta_1}^{*}(N_1'), \mathscr{F}_{\theta_2}(N_2')));$$

in particular, $\widetilde{Q}_t = \widetilde{O}_t$, $\widetilde{Q}_{s,0} = \widetilde{O}_{s,0}$ and $\widetilde{Q}_{0,t} = \widetilde{O}_{0,t}$.

**Theorem 2** *The families $\{\widetilde{Q}_t, t \geq 0\}$, $\{\widetilde{Q}_{s,0}, s \geq 0\}$ and $\{\widetilde{Q}_{0,t}, t \geq 0\}$ are strongly continuous semigroup of continuous linear operators from $\mathscr{L}(\mathscr{F}_{\theta_1}^{*}(N_1'), \mathscr{F}_{\theta_2}(N_2'))$ into itself with the infinitesimal generator $-\widetilde{\mathscr{N}}$, $-\widetilde{\mathscr{N}_1}$ and $-\widetilde{\mathscr{N}_2}$, respectively. Moreover, the quantum Cauchy problems*

$$\begin{cases} \frac{d\Pi_t}{dt} = -\widetilde{\mathscr{N}}\Pi_t \\ \Pi_0 = \Xi \in \mathscr{L}(\mathscr{F}_{\theta_1}^{*}(N_1'), \mathscr{F}_{\theta_2}(N_2')) \end{cases} \tag{24}$$

$$\begin{cases} \frac{d\Lambda_s}{ds} = -\widetilde{\mathscr{N}_1}\Lambda_s \\ \Lambda_0 = \Xi \in \mathscr{L}(\mathscr{F}_{\theta_1}^{*}(N_1'), \mathscr{F}_{\theta_2}(N_2')) \end{cases} \tag{25}$$

$$\begin{cases} \frac{d\Upsilon_t}{dt} = -\widetilde{\mathscr{N}_2}\Upsilon_t \\ \Upsilon_0 = \Xi \in \mathscr{L}(\mathscr{F}_{\theta_1}^{*}(N_1'), \mathscr{F}_{\theta_2}(N_2')) \end{cases} \tag{26}$$

*have a unique solutions given respectively by*

$$\Pi_t = \widetilde{Q}_t\Xi, \quad \Lambda_s = \widetilde{Q}_{s,0}\Xi \quad \textit{and} \quad \Upsilon_t = \widetilde{Q}_{0,t}\Xi. \tag{27}$$

*Proof* We start by proving that the family $\{Q_t, t \geq 0\}$ is a strongly continuous semigroup of continuous linear operators from $\mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2')$ into itself with the infinitesimal generator $-\mathscr{N}$ and the function $U(t, x_1, x_2) = Q_t\varphi(x_1, x_2)$ satisfies

$$\begin{cases} \frac{\partial U(t,x_1,x_2)}{\partial t} = -\mathscr{N} U(t, x_1, x_2), \\ \lim_{t\to 0^+} U(t, x_1, x_2) = \varphi \quad in \quad \mathscr{F}_{\theta_1,\theta_2}(N_1' \oplus N_2'). \end{cases}$$

To this end, it is obvious that $Q_t Q_s = Q_{t+s}$ for any $t, s \geq 0$. Thus, we should show the strong continuity of $\{Q_t, t \geq 0\}$. Suppose $t \leq 1$, then we can use the inequality $|e^x - 1| \leq |x|e^{|x|}$, $x \in \mathbb{R}$, to obtain

$$\begin{aligned} |Q_t\varphi(x, y) - \varphi(x, y)| &\leq \sum_{n,m=0}^{\infty} (e^{-t(n+m)} - 1)|x|_{-p_1}^n |y|_{-p_2}^m |\varphi_{n,m}|_{p_1,p_2} \\ &\leq t \sum_{n,m=0}^{\infty} e^{(n+m)}|x|_{-p_1}^n |y|_{-p_2}^m |\varphi_{n,m}|_{p_1,p_2}. \end{aligned}$$

Then, similarly to the proof of Lemma 2, for any $q_1 > p_1, q_2 > p_2$ and $m_1, m_2, m_1', m_2' > 0$ such that

$$\max\left\{\frac{m_1}{m_1'}e^2\|i_{q_1,p_1}\|_{HS}, \frac{m_2}{m_2'}e^2\|i_{q_2,p_2}\|_{HS}\right\} < 1,$$

we get

$$\begin{aligned} &\|Q_t\varphi - \varphi\|_{(\theta_1,\theta_2);(p_1,p_2);(m_1',m_2')} \\ &\quad \leq t\|\varphi\|_{(\theta_1,\theta_2);(q_1,q_2);(m_1,m_2)} \left\{\left(1 - \left(\frac{m_1}{m_1'}e^2\|i_{q_1,p_1}\|_{HS}\right)\right)\left(1 - \left(\frac{m_2}{m_2'}e^2\|i_{q_2,p_2}\|_{HS}\right)\right)\right\}^{-1}. \end{aligned}$$

This implies the strong continuity of $\{Q_t, t \geq 0\}$. To check that $-\mathscr{N}$ is the infinitesimal generator of $\{Q_t, t \geq 0\}$, let

$$\left(\frac{Q_t\varphi - \varphi}{t} + \mathscr{N}\varphi\right) \sim (Q_{n,m}),$$

where $Q_{n,m}$ is given by

$$Q_{n,m} = \left\{\frac{e^{-t(n+m)} + t(n+m) - 1}{t}\right\}\varphi_{n,m},$$

which follows that, for $p_1$,$p_2 \geq 0$,

$$\left|Q_{n,m}\right|_{p_1,p_2} \leq \left|\frac{e^{-t(n+m)} - 1 + t(n+m)}{t}\right|\left|\varphi_{n,m}\right|_{p_1,p_2}.$$

Using the obvious inequality $|e^x - 1 - x| \leq x^2e^{|x|}$ for all $x \in \mathbb{R}$, we get

$$\left|Q_{n,m}\right|_{p_1,p_2} \leq |t|(n+m)^2e^{|t|(n+m)}|\varphi_{n,m}|_{p_1,p_2}.$$

By using (10) and the inequality $(n+m)^2 \leq 2^{2n+2m}$, we get, for $q_1 > p_1, q_2 > p_2$ and $m_1, m_2 > 0$,

$$\begin{aligned} \left|Q_{n,m}\right|_{p_1,p_2} &\leq t\|\varphi\|_{(\theta_1,\theta_2);(q_1,q_2);(m_1,m_2)} \\ &\quad \times(4m_1e\|i_{q_1,p_1}\|_{HS}e^t)^n(4m_2e\|i_{q_2,p_2}\|_{HS}e^t)^m(\theta_1)_n(\theta_2)_m. \end{aligned}$$

Suppose $t \leq 1$. Hence, by (3), for $m_1',m_2' > 0$, $m_1,m_2 > 0$, $q_1 > p_1$ and $q_2 > p_2$ such that

$$\max\left\{4\frac{m_1}{m_1'}e^2\|i_{q_1,p_1}\|_{HS}, 4\frac{m_2}{m_2'}e^2\|i_{q_2,p_2}\|_{HS}\right\} < 1,$$

we get

$$\left\|\frac{Q_t\varphi-\varphi}{t}+\mathscr{N}\varphi\right\|_{(\theta_1,\theta_2);(p_1,p_2);(m_1',m_2')} \leq tc_3\|\varphi\|_{(\theta_1,\theta_2);(q_1,q_2);(m_1,m_2)}$$

where $c_3$ is given by

$$c_3 = \left\{1-\left(4\frac{m_1}{m_1'}e^2\|i_{q_1,p_1}\|_{HS}\right)\right\}^{-1}\left\{1-\left(4\frac{m_2}{m_2'}e^2\|i_{q_2,p_2}\|_{HS}\right)\right\}^{-1}.$$

Then, we obtain

$$\lim_{t\to 0^+}\left\|\frac{Q_t\varphi-\varphi}{t}+\mathscr{N}\varphi\right\|_{(\theta_1,\theta_2);(p_1,p_2);(m_1',m_2')} = 0. \tag{28}$$

This means that

$$t^{-1}(Q_t\varphi-\varphi) \longrightarrow -\mathscr{N}\varphi \quad in \quad \mathscr{F}_{\theta_1,\theta_2}(N_1'\oplus N_2'),$$

i.e., $-\mathscr{N}$ is the infinitesimal generator of $\{Q_t, t\geq 0\}$. Moreover, we can write

$$\frac{Q_{t+s}\varphi-Q_t\varphi}{s} = \frac{Q_s(Q_t\varphi)-(Q_t\varphi)}{s}.$$

Since $Q_t\varphi \in \mathscr{F}_{\theta_1,\theta_2}(N_1'\oplus N_2')$, we can apply (28) to see that the equation

$$\frac{\partial U(t,x_1,x_2)}{\partial t} = -\mathscr{N}U(t,x_1,x_2)$$

is satisfied by $U(t,x_1,x_2) = Q_t\varphi(x_1,x_2)$. Then, using the topological isomorphism $\mathscr{K}$, we complete the proof of the first assertion. Similarly, we complete the proof. □

Now, we consider two $N_1'$ and $N_2'$-valued stochastic integral equations:

$$U_t = x + \sqrt{2}\int_0^t dW_s - \int_0^t U_s ds$$
$$V_t = y + \sqrt{2}\int_0^t dY_s - \int_0^t V_s ds,$$

where $W_t$ and $Y_s$ are standard $N_1'$-valued and $N_2'$-valued Wiener process, respectively, starting at 0.

**Theorem 3** *The solutions of the Cauchy problems* (24), (25) *and* (26) *have the following probabilistic representations:*

$$\mathscr{K}(\Pi_t)(x,y) = \mathbb{E}(f_1(U_t)/U_0=x)\mathbb{E}(f_2(V_t)/V_0=y)$$
$$\mathscr{K}(\Lambda_s)(x,y) = \mathbb{E}(g_2(y)g_1(U_s)/U_0=x)$$
$$\mathscr{K}(\Upsilon_t)(x,y) = \mathbb{E}(h_1(x)h_2(V_t)/V_0=y)$$

*where* $\mathscr{K}(\Pi_0) = f_1\otimes f_2$, $\mathscr{K}(\Lambda_0) = g_1\otimes g_2$, $\mathscr{K}(\Upsilon_0) = h_1\otimes h_2$, $f_1, g_1, h_1 \in \mathscr{F}_{\theta_1}(N_1')$ *and* $f_2, g_2, h_2 \in \mathscr{F}_{\theta_2}(N_2')$.

*Proof* Applying the kernel map $\mathscr{K}$ to the solution (27) of the Cauchy problem (24), we get

$$\begin{aligned}\mathscr{K}(\Pi_t)(x,y) &= Q_t(\mathscr{K}(\Pi_0))(x,y)\\ &= Q_t(f_1\otimes f_2)(x,y)\end{aligned}$$

for $\mathscr{K}(\Pi_0) = f_1 \otimes f_2$, $f_1 \in \mathscr{F}_{\theta_1}(N_1^{'})$ and $f_2 \in \mathscr{F}_{\theta_2}(N_2^{'})$. Then, using Remark 2 and Proposition 3, we obtain

$$\mathscr{K}(\Pi_t)(x,y) = q_t(f_1)(x)q_t(f_2)(y).$$

On the other hand, it is well known from [18] that

$$q_t(f_1)(x) = \mathbb{E}(f_1(U_t)/U_0 = x), \tag{29}$$
$$q_t(f_2)(y) = \mathbb{E}(f_2(V_t)/V_0 = y), \tag{30}$$

for $f_1 \in \mathscr{F}_{\theta_1}(N_1^{'})$ and $f_2 \in \mathscr{F}_{\theta_2}(N_2^{'})$. Similarly, we have

$$\mathscr{K}(\Lambda_s)(x,y) = Q_{s,0}(\mathscr{K}(\Lambda_0))(x,y) = Q_{s,0}(g_1\otimes g_2)(x,y)$$
$$\mathscr{K}(\Upsilon_t)(x,y) = Q_{0,t}(\mathscr{K}(\Upsilon_0))(x,y) = Q_{0,t}(h_1\otimes h_2)(x,y).$$

Then, from Proposition 3, we get

$$\mathscr{K}(\Lambda_s)(x,y) = q_s(g_1)(x)q_0(g_2)(y) = q_s(g_1)(x)g_2(y)$$
$$\mathscr{K}(\Upsilon_t)(x,y) = q_0(h_1)(x)q_t(h_2)(y) = h_1(x)q_t(h_2)(y);$$

hence, from (29) and (30), we obtain

$$\mathscr{K}(\Lambda_s)(x,y) = \mathbb{E}(g_1(U_s)/U_0 = x)g_2(y)$$
$$\mathscr{K}(\Upsilon_t)(x,y) = h_1(x)\mathbb{E}(h_2(V_t)/V_0 = y),$$

which completes the proof. □

## 5 Markovianity of the quantum O–U semigroups

Recall from [22] that $\mathscr{F}_{\theta_1,\theta_2}(N_1^{'}\oplus N_2^{'})$ is a nuclear algebra with the involution* defined by

$$\varphi^*(z,w) := \overline{\varphi(\overline{z},\overline{w})},\quad z\in N_1^{'},\, w\in N_2^{'}$$

for all $\varphi \in \mathscr{F}_{\theta_1,\theta_2}(N_1^{'}\oplus N_2^{'})$. Using the isomorphism $\mathscr{K}$, we can define the involution (denoted by the same symbol*) on $\mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$ as follows:

$$\Xi^* := \mathscr{K}^{-1}((\mathscr{K}(\Xi))^*),\quad \forall \Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))\cdot$$

Since $\mathscr{F}_{\theta_1,\theta_2}(N_1^{'}\oplus N_2^{'})$ is closed under multiplication, there exists a unique element $\varphi \in \mathscr{F}_{\theta_1,\theta_2}(N_1^{'}\oplus N_2^{'})$, such that

$$\varphi = \mathscr{K}(\Xi_1)\mathscr{K}(\Xi_2).$$

Then by the topology isomorphism $\mathscr{K}$, there exists $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$ such that

$$\mathscr{K}(\Xi) = \mathscr{K}(\Xi_1)\mathscr{K}(\Xi_2), \tag{31}$$

which is equivalent to

$$\Xi = \mathscr{K}^{-1}(\mathscr{K}(\Xi_1)\mathscr{K}(\Xi_2))\cdot \tag{32}$$

Denoted by $\Xi$ to be the product $\boxtimes$ between $\Xi_1$ and $\Xi_2$,

$$\Xi = \Xi_1 \boxtimes \Xi 2\cdot$$

Note that from (31) we see that the product $\boxtimes$ is commutative. Now, define the following cones

$$B := \{\Xi^* \boxtimes \Xi; \quad \Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))\}\cdot$$

Elements in B are said to be B-positive operators. Let $S, T \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$; we say that $S \le T$, if $T - S \in B$. Denoted by $I_0 = \mathscr{K}^{-1}\left(1_{\mathscr{F}_{\theta_1,\theta_2}(N_1^{'} \oplus N_2^{'})}\right)$.

**Definition 2** A map P : $\mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'})) \to \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$ is said to be

(i) positive if $P(B) \subseteq B$
(ii) Markovian, if it is positive and $P(\Xi) \le I_0$ whenever $\Xi = \Xi^*$ and $\Xi \le I_0$.

A one-parameter semigroup $\{P_t, t \ge 0\}$ on $\mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$ is said to be positive (resp. Markovian) provided $P_t$ is positive (resp. Markovian) for all $t \ge 0$.

**Theorem 4** *The quantum O–U semigroup* $\{\widetilde{Q_t}, t \ge 0\}$*, the right quantum O–U semigroup* $\{\widetilde{Q_{0,t}}, t \ge 0\}$ *and the left quantum O–U semigroup* $\{\widetilde{Q_{s,0}}, s \ge 0\}$ *are Markovian.*

*Proof* Let $\Xi \in B$, then there exists $S \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$ such that

$$\Xi = S^* \boxtimes S\cdot$$

Then, for all $t \ge 0$, $z \in N_1^{'}$ and $w \in N_2^{'}$ , we have

$$\begin{aligned}
\mathscr{K}\widetilde{Q_t}(\Xi)(z, w) &= Q_t(\mathscr{K}(\mathscr{K}^{-1}(\mathscr{K}(S^*)\mathscr{K}(S))))(z, w) \\
&= Q_t((\mathscr{K}(S^*))\mathscr{K}(S))(z, w) \\
&= (\mathscr{K}(S^*)\mathscr{K}(S))(e^{-t}z, e^{-t}w) \\
&= Q_t(\mathscr{K}(S^*))(z, w)Q_t(\mathscr{K}(S))(z, w).
\end{aligned}$$

Using (32), we get

$$\begin{aligned}
\widetilde{Q_t}(\Xi) &= \mathscr{K}^{-1}(Q_t(\mathscr{K}(S^*))Q_t(\mathscr{K}(S))) \\
&= \mathscr{K}^{-1}(\mathscr{K}(\widetilde{Q_t}(S^*))\mathscr{K}(\widetilde{Q}_t(S))) \\
&= \widetilde{Q_t}(S^*) \boxtimes \widetilde{Q_t}(S).
\end{aligned}$$

On the other hand, we have

$$\mathscr{K}(\widetilde{Q_t}(S^*)) = Q_t(\mathscr{K}(S^*))\cdot$$

But we know that

$$S^* = \mathscr{K}^{-1}((\mathscr{K}(S))^*)\cdot$$

Then, we get

$$\begin{aligned}
\mathscr{K}(\widetilde{Q_t}(S^*))(z, w) &= Q_t((\mathscr{K}(S))^*)(z, w) \\
&= (\mathscr{K}(S))^*(e^{-t}z, e^{-t}w) \\
&= \overline{\mathscr{K}(S)(e^{-t}\overline{z}, e^{-t}\overline{w})} \\
&= \overline{(Q_t\mathscr{K}(S))(\overline{z}, \overline{w})} \\
&= (Q_t\mathscr{K}(s))^*(z, w).
\end{aligned}$$

From this we obtain

$$\begin{aligned}\widetilde{Q_t}(S^*) &= \mathscr{K}^{-1}((Q_t\mathscr{K}(S))^*)\\ &= \mathscr{K}^{-1}((\mathscr{K}\widetilde{Q_t}(S))^*)\\ &= (\widetilde{Q_t}(S))^*. \end{aligned} \tag{33}$$

Hence, we get

$$\widetilde{Q_t}(\Xi) = (\widetilde{Q_t}(S))^* \boxtimes Q_t(S)\cdot$$

This proves that $\widetilde{Q_t}(\Xi) \in B$ for all $t \geq 0$, which implies that $\{\widetilde{Q}_t; t \geq 0\}$ is positive. To complete the proof, let $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$ such that $\Xi \leq I_0$ and $\Xi = \Xi^*$. This gives $I_0 - \Xi \in B$, which means that there exists $T \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$, such that

$$I_0 - \Xi = T^* \boxtimes T\cdot$$

This implies that

$$(1 - (\mathscr{K}(\Xi)) = ((\mathscr{K}(T))^*(\mathscr{K}(T). \tag{34}$$

On the other hand, we have

$$\begin{aligned}\mathscr{K}(I_0 - \widetilde{Q_t}(\Xi))(z, w) &= 1 - Q_t(\mathscr{K}(\Xi))(z, w)\\ &= 1 - \mathscr{K}(\Xi)(e^{-t}z, e^{-t}w)\\ &= (1 - \mathscr{K}(\Xi))(e^{-t}z, e^{-t}w).\end{aligned}$$

Then, using (34), we get

$$\begin{aligned}I_0 - \widetilde{Q_t}(\Xi) &= \mathscr{K}^{-1}(Q_t((\mathscr{K}(T))^*)Q_t(\mathscr{K}(T)))\\ &= \mathscr{K}^{-1}(\{\mathscr{K}\widetilde{Q_t}\mathscr{K}^{-1}(\mathscr{K}(T))^*\}\{\mathscr{K}\widetilde{Q_t}(T)\})\\ &= \mathscr{K}^{-1}(\mathscr{K}(\widetilde{Q_t}(T^*))\mathscr{K}(\widetilde{Q_t}(T)))\\ &= \mathscr{K}^{-1}(\mathscr{K}(\widetilde{Q_t}(T^*) \boxtimes \widetilde{Q}_t(T)))\\ &= \widetilde{Q_t}(T^*) \boxtimes \widetilde{Q_t}(T).\end{aligned}$$

Then, using (33), we obtain

$$I - \widetilde{Q_t}(\Xi) = (\widetilde{Q_t}(T))^* \boxtimes \widetilde{Q_t}(T)\cdot$$

This means that

$$I_0 - \widetilde{Q_t}(\Xi) \in B,$$

which is equivalent to say that

$$\widetilde{Q_t}(\Xi) \leq I_0, \quad \forall t \geq 0\cdot$$

This completes the proof of the Markovianity of the quantum O–U semigroup $\{\widetilde{Q_t}, t \geq 0\}$. Similarly, we show the Markovianity of the others semigroups. □

*Remark 3* Let $\Xi_1, \Xi_2 \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$, define the following scalar product:

$$(((\Xi_1, \Xi_2))) := \int_{X_1^{'} \times X_2^{'}} \mathscr{K}(\Xi_1)(x, y)\overline{\mathscr{K}(\Xi_2)(x, y)}d\mu_1(x)d\mu_2(y)\cdot$$

Using Theorem 4, $\{\widetilde{Q_t}; \quad t \geq 0\}$ is a positive semigroup. Let $\Xi_1, \Xi_2 \in B$, such that $\Xi_1, \Xi_2 \neq 0$. Then, there exist $S, T \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(N_1^{'}), \mathscr{F}_{\theta_2}(N_2^{'}))$, such that

$$\Xi_1 = S^* \boxtimes S, \quad \Xi_2 = T^* \boxtimes T.$$

From this one can get

$(((\Xi_1, \widetilde{Q_t}(\Xi_2)))) \geq 0.$

But it is important to show that: for all $\Xi_1, \Xi_2 \in B, \quad \Xi_1, \Xi_2 \neq 0$, there exists $t > 0$ such that

$(((\Xi_1, \widetilde{Q_t}(\Xi_2)))) > 0,$

i.e., $\{\widetilde{Q_t},\ t \geq 0\}$ is ergodic, which gives scope for new work.

## References


1. Accardi, L., Ouerdiane, H., Smolyanov, O.G.: Lévy Laplacian acting on operators. Rus. J. Math. Phys. **10**(4), 359–380 (2003)
2. Accardi, L., Smolyanov, O.G.: On Laplacians and traces. Conf. del Sem. di Mate. dell'Univ. di Bari **250**, 1–28 (1993)
3. Accardi, L., Smolyanov, O.G.: Transformations of Gaussian measures generated by the Lévy Laplacian and generalized traces. Dokl. Akad. Nauk SSSR **350**, 5–8 (1996)
4. Barhoumi, A., Kuo, H.-H., Ouerdiane, H.: Generalized Gross heat equation with noises. Soochow J. Math. **32**(1), 113–125 (2006)
5. Barhoumi, A., Ouerdiane, H., Rguigui, H.: QWN-Euler operator and associated cauchy problem. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **15**(1), 1250004 (2012)
6. Barhoumi, A., Ouerdiane, H., Rguigui, H.: Stochastic heat equation on algebra of generalized functions. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **15**(4), 1250026 (2012)
7. Barhoumi, A., Ouerdiane, H., Rguigui, H.: Generalized Euler heat equation. Quantum Probab. White Noise Anal. **25**, 99–116 (2010)
8. Cipriani, F.: Dirichlet forms on noncommutative spaces, quantum potential theory. Lecturs notes in mathematics. Springer, Berlin (2008)
9. Chung, D.M., Ji, U.C.: Transform on white noise functionals with their application to Cauchy problems. Nagoya Math. J. **147**, 1–23 (1997)
10. Chung, D.M., Ji, U.C.: Transformation groups on white noise functionals and their application. Appl. Math. Optim. **37**(2), 205–223 (1998)
11. Gannoun, R., Hachaichi, R., Ouerdiane, H., Rezgui, A.: Un théorème de dualité entre espace de fonctions holomorphes à croissance exponentielle. J. Func. Anal. **171**(1), 1–14 (2002)
12. Gel'fand, I.M., Vilenkin, N.Y.: Generalized unctions, vol. 4, Academic press New York (1964).
13. Gross, L.: Potential theory on Hilbert space. J. Funct. Anal. **1**, 123–181 (1967)
14. Ji, U.C.: Quantum extensions of Fourier–Gauss and Fourier–Mehler transforms. J. Korean Math. Soc. **45**(6), 1785–1801 (2008)
15. Ji, U.C., Obata, N., Ouerdiane, H.: Analytic charachterization of generalized Fock space operator as two-variable entire function with growth condition. World Sci. **5**(3), 395–407 (2002)
16. Khalgui, M.S., Torasso, P.: Poisson-Plancherel formula for a real semialgebraic group. 1. Fourier-transform of orbital integrals. J. Func. Anal. **116**(2), 359–440 (1993)
17. Kuo, H.-H.: Potential theory associated with O-U process. J. Funct. Anal. **21**, 63–75 (1976)
18. Kuo, H.-H.: Withe Noise distribution theory. CRC Press, (1996)
19. Lee, Y.J.: Integral transform of analytic function on abstract Wiener space. J. Funct. Anal. **47**(2), 153–164 (1982)
20. Obata, N.: White noise calculus and Fock space. Lecture Notes in Math. 1577 (1994)
21. Obata, N.: Quantum white noise calculus based on nuclear algebras of entire function. Trends in Infinite Dimensional Analysis and Quantum Probability (Kyoto 2001), RIMS No. 1278, pp. 130–157
22. Ouerdiane, H.: Infinite dimensional entire functions and application to stochastic differential equations. Not. S. Afr. Math. Soc. **35**(1), 23–45 (2004)
23. Ouerdiane, H., Rezgui, A.: Representation integrale de fonctionnelles analytiques positives. Can. Math. Soc. Conf. Proc. **28**, 283–290 (2000)
24. Ouerdiane, H., Rguigui, H.: QWN-conservation operator and associated differential equation. Commun. Stoch. Anal. **6**(3), 437–450 (2012)
25. Piech, M.A.: Parabolic equations associated with the number operator. Trans. Am. Math. Soc. **194**, 213–222 (1974)