Action, quantum computation and the structure of the Yang–Mills Lagrangian

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Abstract We seek to interpret the notions of action and Lagrangian in the quantum domain in terms of a quantum computational paradigm. This effort was inspired by work of Toffoli in the classical case. We claim that in this paradigm an action integral may itself be regarded as a quantum “macroprogram” with the generating Lagrangian density playing the role of a quantum “microprogram” describing an underlying quantum process, in rough analogy with the way a λ -expression (in simply typed λ -calculus) may be unfolded to reveal an underlying natural deduction. We carry out such an unfolding for the Yang–Mills Lagrangian (including a gauge fixing term) to reveal a simple underlying quantum microprogram which in this case effects the variation of a fermion tuple consequent upon a quantized, generically deformed, gauge rotation of the tuple.

Keywords Action · Quantum computation · Variational principles · Yang–Mills Lagrangian

1 Introduction

The meaning or interpretation of the Lagrangian of a physical system has remained rather mysterious from its inception. Feynman showed how action integrals enter most fundamentally into the quantum behavior of a system (and in a certain limit also into its classical behavior) through his formulation of the path integral representation for amplitudes, but acknowledged that this placement of the Lagrangian at the heart of quantum behavior begs the question of its meaning or interpretation ([6]: see also [7]). Even today the Lagrangians of fundamental physics (the standard model, general relativity, string theory, et al.) are not, apart from symmetry requirements imposed ad hoc, derived from prior principles, but rather are assumed to constitute in themselves the postulational bedrock, either directly or in the form of the associated field equations. The tacit consensus seems to be that these expressions ultimately represent experimental input, given by Nature, and that it is meaningless to question their provenance or to try to parse their meaning; like precious jewels they must be plucked from the matrix of Nature and carefully sequestered. It is perhaps not surprising therefore that the first serious attempt to address the issue of the “meaning” of the action in the classical case was undertaken not by a quantum physicist per se but by a versatile and knowledgeable computer scientist [6]. In this compelling work, the author shifts the classical mechanical paradigm to one of the computation, and interprets the notions of Lagrangian and action in terms of this paradigm, drawing conclusions from

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a large array of cleverly designed special cases. Roughly speaking, the author's conclusions entail an interpretation of the action integral as a measure of the *computational capacity* or the number of different *behaviors* or *computations*, offered by the “substrate” to a proposed macroscopic trajectory.

In this paper, we shall pursue this interpretation into the quantum domain, replacing the classical computational paradigm by its quantum analogue. We find that the interpretation largely holds up when the terms are suitably translated into the quantum language. For example, a “count” considered classically as a sum of parts, becomes a *superposition* of operators, or quantum programs. In these terms, the action integral becomes a quantum “macroprogram” whose (Lagrangian density) integrand represents the classical degeneration of a specific quantum “microprogram” describing an underlying quantum process. The integration superposes (“counts”) these elementary quantum microprograms, which unlike the classical case, results not merely in a scalar, but instead effects an actual quantum process. Thus, instead of a “count” we obtain, in the quantum case, an explicit rendition of the actual microprogram itself, whose superposition via integration, along a path, produces the output of the quantum macroprogram effecting the actual process along that path. Thus the action integral in general, as an operator, describes all possible behaviors or computations available to the system.

The layout of the paper is as follows. In the next section we recall some aspects of the quantum computational paradigm in a logical context. In Sect. 3 we recall some notions from the functional formalism often used in quantum field theory and in the following section we introduce the Feynman path integral and proceed from it to the Schwinger Action Principle. We interpret these principles in quantum computational terms and point out the analogy between the Feynman path integral—the amplitude on the left being considered as the end result of the quantum computation represented by the path integral on the right—and the classical case of a λ -expression being the end result of the natural deduction it enfolds. In Sect. 5 we prepare the computational ground needed to interpret amplitudes in terms of the logic of quantum resources, and in a final section we unfold the classical Yang–Mills Lagrangian to reveal its simple underlying quantum microprogram or operator.

2 Computation and the logic of quantum resources

In earlier work [3, 4] we delineated a certain logical construct and showed how it could provide a logical foundation for a theory of quantum computation. This system, which we called **GQ**, is a fragment of a special case of Girard's Gentzen sequent calculus called Linear Logic. The system **GQ** embodies minimal logical rules for making deductions about abstract quantum “resources.” We will not need the full formal calculus for the purposes of this paper: it may be found in the references given above. Suffice it to say, that the logic may be realized in the category of interest for quantum computing, namely the category $\mathcal{H}_F$ of finite dimensional Hilbert spaces over $\mathbb{C}$, the complex numbers. When so realized, the rules reduce to familiar constructs and operations of multilinear algebra, but these now carry logical attributes. Thus we may regard types $A, B \dots$ informally now to be objects in $\mathcal{H}_F$, and a sequent $A \vdash B$ to be a morphism $A \rightarrow B$ in $\mathcal{H}_F$. Multisets (or sequences) $A_1, A_2, \dots, A_n$ of types are realized as tensor products $A_1 \otimes A_2 \otimes \dots \otimes A_n$, and $\mathbb{C}$, being the unit for $\otimes$, is logically (as well as quantum mechanically!) trivial. Thus we assign the empty sequence of formulae, as in $\vdash A$, to the Hilbert space $\mathbb{C}$. Thus the sequent is interpreted as a map $\mathbb{C} \rightarrow A$, that is, as an element of A . We will have more to say about this topic in Sect. 5 but it is worth pointing out here the sort of benefit that comes from thinking in terms of the logic. Strictly speaking, the term “quantum computation” should be applied to unitary operators acting upon certain Hilbert spaces which exhibit the property of “storage capability.” In **GQ**, a type A is *storage capable* if it obeys the rule of Contraction (note the capital C):

$$\frac{A, A \vdash D}{A \vdash D}. \quad (1)$$

For, this means that two uses of A may be replaced by a single use, signifying that it is capable of being stored. Not every type has this property but one may, following Girard, posit an operator $!$ (“of course”) with the property that $!A$ obeys the rule of Contraction for all A . In $\mathcal{H}_F$, $!A$ is modeled, for good quantum logical reasons explained in [4] (or [3]), by the exterior (or Fermi–Dirac) algebra $E(A)$: namely (with A having dimension n)

$$E(A) = \mathbb{C} \oplus A \oplus \bigwedge^2 A \oplus \dots \quad (2)$$

with $\bigwedge^k$ denoting the k th exterior product. We note that $\dim \bigwedge^k A = \binom{n}{k}$, so the series terminates at $k = n$, $\bigwedge^n A$ being one dimensional and $\dim E(A) = 2^n$. If we provide a basis $\{\xi_1, \dots, \xi_n\}$ for A so that an isomorphism $A \cong \bigoplus^n \mathbb{C}$ is determined, then

$$E(A) \cong E\left(\bigoplus^n \mathbb{C}\right) \cong \bigotimes^n E(\mathbb{C}) \quad (3)$$

where $E(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$ is the *qubit*, recognized now as the minimal non-trivial storage capable type or unit. Thus we recover from the logic, without any recourse to classical prejudices, the standard intuition that the minimal system appropriate for quantum computations is the qubit. It then *follows* that the classical degeneration of this unit is exactly the bit. Moreover, the general form of $!A$ is a tensor product, or register, of qubits.

In this paper, we will interpret the term “quantum computation” more liberally as essentially any quantum process describable by a Lagrangian. Thus we may lose storage capability but may gain, in the manner of [6], new perspectives. Thus, for any quantum process, we shall, for the purposes of this paper, naïvely regard the evolution of a normalized “input” state $|\text{in}\rangle$, at time t_1 , say, to a normalized output state $|\text{out}\rangle$, at time $t_2 (> t_1)$ say, to constitute an act of “quantum computation.” The primary numerical attribute of such an evolution is the so-called *amplitude*:

$$\langle \text{out} | \text{in} \rangle, \quad (4)$$

a complex number whose modulus squared $|\langle \text{out} | \text{in} \rangle|^2$ is interpreted, according to the rules of quantum mechanics, as the probability that the transition $|\text{in}\rangle \rightarrow |\text{out}\rangle$ actually occurs. Thus, in terms of the computational paradigm, this is the probability that the system, upon being prepared in the input state, will execute a computation with output state $|\text{out}\rangle$. Consequently, the amplitude $\langle \text{out} | \text{in} \rangle$ must characterize the computational behavior of the system in some way. Those such amplitudes may be expressed in terms of path integrals which was the great discovery of R. P. Feynman. In Sect. 4, we shall indicate an explicit computational interpretation of these expressions. First we digress to discuss some formalism useful in this context.

3 Functional calculus

It will prove convenient to offer a short primer on some notions and notations from a technique favored by quantum field theorists called *functional calculus*. A *functional* is a function, F say, taking tuples of functions as arguments, with values, for the sake of our arguments, in $\mathbb{C}$. The notation in physics is as in

$$F[\varphi^I(x)] := F[\varphi^1(x), \dots, \varphi^n(x)] \quad (5)$$

where the tuple of functions (or *fields*) φ^I is defined on regions of spacetime, whose generic elements are denoted by x . This notation already conflicts with formal mathematical usage, in which arguments to functions such as φ^I are not considered part of the symbol denoting the function; instead, the domain is specified separately. In the physical context it is useful to consider the x variable as a mere label, or index, albeit a continuous one, to be regarded on a par with the discrete index I . Thus, an argument could be made in favor of the notation $F[\varphi^{I,x}]$, which might be more comfortable for mathematical purists, but which we shall not make here, retaining the notation in Eq. (5). We

shall not attempt to specify too closely the nature of the fields $\varphi^I(x)$ except to say that the domain of F may be extended to contain at least elementary distributions, such as the Dirac δ “functions”.

The derivative of such a functional may be defined by analogy with the Taylor expansion of ordinary multivariable functions. In this case, the summation over the indices includes the summation not only over the discrete index I but also over the continuous index x , such a summation now being taken to be an integration over some regions Σ of spacetime, the extent of the x -index. Then (cf. [8]) the *functional derivatives* $\delta F[\varphi^I(x)]/\delta\varphi^J(x')$ are defined by

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(F[\varphi^I(x) + \epsilon\psi^I(x)] - F[\varphi^I(x)] \right) := \int_{\Sigma} \psi^J(x') \frac{\delta F[\varphi^I(x)]}{\delta\varphi^J(x')} dx' \quad (6)$$

Two examples may prove illuminating. First, let I be a singleton, $\varphi^I = \varphi$, and, since we are allowing distributions into the domain of F , put $\psi^I = \delta_y$, the Dirac distribution supported at some fixed y . Then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(F[\varphi(x) + \epsilon\delta_y(x)] - F[\varphi(x)] \right) = \int_{\Sigma} \delta_y(x') \frac{\delta F[\varphi(x)]}{\delta\varphi(x')} dx' = \frac{\delta F[\varphi(x)]}{\delta\varphi(y)} \quad (7)$$

which is often given as the definition of the right hand side.

Next, choose and fix an index I_0 and put $F[\varphi^I(x)] = \varphi^{I_0}(x)$. Then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(F[\varphi^I(x) + \epsilon\psi^I(x)] - F[\varphi^I(x)] \right) = \psi^{I_0}(x) = \int_{\Sigma} \psi^J(x') \frac{\delta\varphi^{I_0}(x)}{\delta\varphi^J(x')} dx', \quad (8)$$

so

$$\frac{\delta\varphi^{I_0}(x)}{\delta\varphi^J(x')} = \delta_{I_0,J} \delta_{x'}(x). \quad (9)$$

It will prove useful to define the *functional differential* of F as the change in F consequent upon changes $\delta\varphi^I$ of the fields in analogy with the case of ordinary functions, namely, with $\psi^I(x) = \delta\varphi^I(x)\delta_y(x)$ for a fixed y , the change in F at y is

$$\begin{aligned} \delta F[\varphi^I(x)](y) &:= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(F[\varphi^I(x) + \epsilon\delta\varphi^I(x)\delta_y(x)] - F[\varphi^I(x)] \right) \\ &= \int_{\Sigma} \delta\varphi^J(x') \delta_y(x') \frac{\delta F[\varphi^I(x)]}{\delta\varphi^J(x')} dx' \\ &= \delta\varphi^J(y) \frac{\delta F[\varphi^I(x)]}{\delta\varphi^J(y)} \end{aligned} \quad (10)$$

so that we may write in general

$$\delta F[\varphi^I(x)] = \frac{\delta F[\varphi^I(x)]}{\delta\varphi^J} \delta\varphi^J, \quad (11)$$

noting that this δ acts as a derivation upon products of functionals.

4 From Feynman to Schwinger

Now we recall the Feynman path integral representation of an amplitude in this context. Namely, we consider a tuple φ^I of fields defined on spacetime, and two space-like surfaces σ_0, σ_t which are time-like separated in that

order. Denoted by $|\varphi^I(\sigma_0)\rangle, |\varphi^I(\sigma_t)\rangle$, the states representing the values of the tuple φ^I on the respective surfaces, suitably normalized. Then the Feynman representation yields (in units chosen so that $\hbar = c = 1$)

$$\langle \varphi^I(\sigma_t) | \varphi^I(\sigma_0) \rangle = \int e^{i \int_P \mathcal{L}(\varphi^I) d^4x} \mathcal{D}\varphi^I(P) \quad (12)$$

where $\mathcal{L}(\varphi^I)$ denotes the operator version of the Lagrangian density of the system of fields and the inner integral's subscript P is understood as follows: the expression $\mathcal{L}(\varphi^I)$ is to be integrated between the surfaces σ_0 and σ_t as φ^I follows a path P in φ^I -space which interpolates between the φ^I configurations on σ_0 and σ_t . These exponentiated contributions are then summed over all possible such paths against an optimistic measure $\mathcal{D}\varphi^I$ on the space of paths. This is symbolized by the outer integral.

As we pointed out in [5] Eq. (12) has an interpretation in terms of quantum computation within the system **GQ**. This is best seen in analogy with the classical case of computation within the system called minimal natural deduction. This system is structurally equivalent to simply typed λ -calculus (via the Curry–Howard correspondence); the conclusion (output) of a natural deduction (called a *proof* in this context) may be translated into a λ -expression which encapsulates the computational path taken during the deduction. Conversely, such a static λ -expression may be unfolded to reveal an underlying computational path or deduction [4]. (The Curry–Howard correspondence makes it possible to view formulae as types and proofs as programs, an insight which has proved extremely fruitful in applications to computational theory.) Moreover, deductions (or proofs, or programs) are executed by removing CUTs. In our quantum analog of natural deduction, namely **GQ** as realized in the category $\mathcal{H}_F$ of finite dimensional Hilbert spaces, CUTs are removed ultimately by contraction (note the small c), namely by evaluating linear functionals upon vectors (giving rise also to operator composition as noted in more detail in the next section). In $\mathcal{H}_F$ this amounts to take inner products. Thus, in the computational paradigm, an amplitude such as the left hand side of Eq. (12) may be the last step in a quantum deduction, like the final reduction of a λ -expression. If it is, where is the analogue of the “encapsulated” computational path or quantum program? We contend that it is to be found on the other side of Eq. (12), which may be parsed as a limit of superposed quantum “macroprograms,” or CUT eliminations hidden ultimately in the Lagrangian expression itself, and that this is the quantum computational import of the path integral representation. That is to say, the path integral on the right hand side of Eq. (12) is the limit of superpositions of the exponentials of action integrals which we regard as quantum macroprograms. It is these action integrals themselves, regarded as quantum macroprograms—specifically their underlying quantum microprograms hidden in the Lagrangian—we aim to analyze for the case of Yang–Mills in the last section.

Since we aim to analyze the action integrals themselves, it will be of interest to try to isolate them as much as possible from the enormous superposition represented by the path integral on the right hand side of Eq. (12). In this we follow Schwinger in applying a functional differential operator like the δ of Eq. (11), noting that the various expressions are functionals of φ^I , to obtain

$$\delta \langle \varphi^I(\sigma_t) | \varphi^I(\sigma_0) \rangle = i \int e^{i \int_P \mathcal{L} d^4x} \delta \left(\int_P \mathcal{L} d^4x \right) \mathcal{D}\varphi^I(P). \quad (13)$$

The right hand side is a sum over paths, each path P contributing in the usual way to the overall value of the general variation $\delta \langle \varphi^I(\sigma_t) | \varphi^I(\sigma_0) \rangle$ of the amplitude with respect to the fields. Consequently, if we choose to vary the field configurations along one particular path, P_0 say, and no other, we would have

$$\delta \left(\int_P \mathcal{L} d^4x \right) = 0 \text{ for } P \neq P_0 \quad (14)$$

and would obtain

$$\delta \langle \varphi^I(\sigma_t) | \varphi^I(\sigma_0) \rangle = i \int e^{i \int_P \mathcal{L} d^4x} \delta \left(\int_P \mathcal{L} d^4x \right) \delta_{P, P_0} \mathcal{D} \varphi^I(P) = i \left\langle \varphi^I(\sigma_t) \left| \delta \left(\int_{P_0} \mathcal{L} d^4x \right) \right| \varphi^I(\sigma_0) \right\rangle. \quad (15)$$

That is to say, the variation of the amplitude, when the field configurations along a path P_0 are varied, is i times the value obtained by sandwiching the concomitant variation of the action integral along P_0 , between the initial and final states. This is the Schwinger Action Principle.

This principle is ostensibly a differential version of the Feynman path integral, but given the mathematical vagueness of the processes involved, one could—with Schwinger—simply adopt Eq. (15) as an axiom or postulate. Benefits may be expected to accrue from the use of a differential version of the path integral, since, among other things, (mythical) differentiation is generally easier than (mythical) integration, and avoids most of the enormous measure theoretic problems inherent in the path integral formulation. Such benefits were repeatedly emphasized by Schwinger himself who even went so far as to devote his entire Nobel address to this topic [1]. Most surprisingly, with careful handling, the principle can be shown to yield virtually the entire structural paraphernalia of quantum mechanics and quantum field theory, including commutation relations, et al. [8, 10].

To interpret this principle according to the computational paradigm, we may think of the x indices as *addresses* and the values $\varphi^I(x)$ as data or instructions at x . Then a path in φ^I -space defines a *computation*, namely, a (continuous) sequence of data transformations as we move from σ_0 to σ_t . To vary a computation then means to vary the data (or φ^I configurations) along the path. Thus, according to this paradigm, the action integral may be taken to represent a quantum *macroprogram*, namely, a limit of superpositions of quantum *microprograms* given by the Lagrangian density $\mathcal{L}(\varphi^I)$. Then the Schwinger Action Principle may be interpreted as asserting that variation of an I/O amplitude consequent upon a small variation of the program parameters along a chosen computational path is necessarily implemented (up to a factor of unit modulus) by running the concomitant variation of the quantum macroprogram (along the path) on the input state and then taking the amplitude. If such a small variation along a computational path leaves the terminal states unchanged, then the concomitant variation of the quantum macroprogram necessarily vanishes. Such computational paths (given by the usual Euler–Lagrange equations) all produce the same I/O behavior and are therefore computationally interchangeable: this is fungibility in the sense of Toffoli [6].

The Schwinger Action Principle is thus seen to be almost tautological, its force residing entirely in the constraint placed upon the mode of implementation (described above) of a *general* variation of the I/O amplitude.

We note that these conclusions are entirely consistent with Toffoli’s hypothesis that the action should relate to the *computational capacity* of the system at hand. For, it is a tautology to say that slightly varying the I/O behavior (i.e. amplitude) of a quantum computational device along a computational path is tantamount to slightly varying that operator implementing the possible behaviors of the device, as it traverses that computational path, should such an operator exist. Assuming it does exist, call it K : this would be an operator version of the observable “computational capacity.” Then, varying only the initial configuration, the amplitudes $\langle \varphi^I(\sigma_t) | \delta | \varphi^I(\sigma_0) \rangle$ and $\langle \varphi^I(\sigma_t) | \delta K | \varphi^I(\sigma_0) \rangle$ should differ only by a factor of unit modulus, so as to produce the same probabilities, i.e. behaviors. The Action Principle presents the solution $K = \int \mathcal{L} d^4x$, with a factor of i .

Moreover, we have then

$$\langle \varphi^I(\sigma_t) | \delta | \varphi^I(\sigma_0) \rangle = i \left\langle \varphi^I(\sigma_t) \left| \delta \left(\int_{P_0} \mathcal{L} d^4x \right) \right| \varphi^I(\sigma_0) \right\rangle \quad (16)$$

so that we surmise, *cum grano salis*, that

$$\delta \propto \mathcal{L} \quad (17)$$

as operators. Thus, the Lagrangian density might be interpreted as a quantum microprogram effecting some sort of small deformation. In the last section we shall demonstrate that this is indeed the case for the Yang–Mills Lagrangian.

5 Some linear heuristics

The idea is to unfold a Lagrangian density itself in rough informal analogy with the unfolding of a λ -expression (in simply typed λ -calculus) to uncover an underlying natural deduction. As already mentioned, the “formal language” we employ to handle quantum resources amounts to the language of multilinear algebraic operations in the category $\mathcal{H}_F$ of Hilbert spaces of finite dimension (which will suffice for our purposes here). The only novel point to make is that the fundamental operation of executing a CUT amounts to the taking of an amplitude. In **GQ** the CUT axiom takes the form

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C} \quad (18)$$

where the consequent $A \rightarrow C$ is the composition of the maps in the antecedent. However, this operation itself may be expressed in terms of the basic contraction $V \otimes V^* \cong V^* \otimes V \rightarrow \mathbb{C}$, where asterisk denotes the linear dual, which may be realized as the taking of amplitudes. First note that $V^* \otimes W \cong \text{Hom}(V, \mathbb{C}) \otimes W \cong \text{Hom}(V, W)$ via the familiar Currying isomorphism. Thus $\text{End } V \cong V^* \otimes V$ and, when operators on V are realized in $V^* \otimes V$, the contraction becomes the trace of the corresponding operator. Now, for operators $P : A \rightarrow B$, $Q : B \rightarrow C$, namely elements in $A^* \otimes B$, $B^* \otimes C$, respectively, the composition $QP : A \rightarrow C$, an element in $A^* \otimes C$, may be rendered in tensorial form as the image of $P \otimes Q$ under the map

$$1_{A^*} \otimes \text{tr} \otimes 1_C : A^* \otimes B \otimes B^* \otimes C \rightarrow A^* \otimes \mathbb{C} \otimes C \cong A^* \otimes C \quad (19)$$

where tr denotes the trace. The taking of a particular amplitude may be realized as a CUT of the form

$$\frac{\vdash B \quad B \vdash}{\vdash} \quad (20)$$

which is realized as

$$\frac{\mathbb{C} \rightarrow B \quad B \rightarrow \mathbb{C}}{\mathbb{C} \rightarrow \mathbb{C}} \quad (21)$$

the antecedent amounting to a pair (element of B , element of B^*) and the consequent being realized by the application of the latter to the former as an element of $\mathbb{C}$.

Thus, as we proceed in a deduction or computation in this calculus, contractions, or traces, accumulate until terms can be reduced no further (via the axioms) and we are left in general with a concluding sequent (for those deductions that do conclude) realized as a map of Hilbert spaces (possibly in the form of a tensor product of constituent operators (for example, of the form $\cdots A^* \otimes A \cdots$). Any remaining constituent maps may be reduced further by taking traces (i.e. forming amplitudes). Thus, we will often be left with products of traces of products of operators, or traces of tensor products of operators. We consider briefly the quantum theoretical import of such a deduction (followed by the taking of traces). A tensor product of Hilbert spaces represents the states of a combined system. Taking amplitudes where possible between component states within such a tensor product amounts physically to the taking into account of transitions between these component states. Continuing with the deduction, we take into account more and more “internal” transitions until the process cannot be taken further (or perhaps proceeds indefinitely). What is left upon termination (if this occurs) is a “maximally external” descriptor of the combined system, since all possible internal transitions have been taken into account. Roughly speaking, the progression through a computation

of this type is a paradigm of the progression from a detailed internal description of a combined system to a coarser, progressively external descriptor of the combined system.

It is this progression we will seek in the next section to *reverse* for the case of the classical Yang–Mills Lagrangian density, which we shall regard as the maximal external representation of an underlying combined quantum system.

6 Unfolding Yang–Mills

The standard classical pure Yang–Mills Lagrangian density may be taken proportional to the expression

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}(\partial^\nu G_\nu)^2 - \frac{1}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu}) \quad (22)$$

where we have included as the first term the Feynman–’t Hooft gauge fixer, and have absorbed coupling constants. The G_μ are $n \times n$ trace free Hermitian matrices for $n > 1$ so that $iG_\mu \in \mathfrak{su}(n)$, the Lie algebra of $SU(n)$, and

$$G^{\mu\nu} := \partial^{[\mu}G^{\nu]} - i[G^\mu, G^\nu]. \quad (23)$$

(Expressing the G^μ in terms of a trace-normalized basis for $\mathfrak{su}(n)$ will give rise to a form perhaps more familiar to some readers.) Omitting the gauge fixing term would force the path integrals to be taken over gauge equivalent fields and this vast overcount would produce fatal divergences. This would be a more serious problem than the concomitant departure from unitarity of the associated scattering matrix, which requires the addition of a ghost term, omitted here. (We will find elsewhere that such a term can be treated in tandem with the treatment of $\mathcal{L}$ to follow.)

To proceed with the unfolding of $\mathcal{L}$ we recall some identities-cum-definitions. Namely,

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}1_V \quad (24)$$

where the γ_μ denote the Dirac matrices, considered to act upon the four-dimensional bispinor space V , $g_{\mu\nu}$ denotes the spacetime metric, and 1_V denotes the identity map on V , and

$$[\gamma_\mu, \gamma_\nu] := -2i\sigma_{\mu\nu} \quad (25)$$

whence

$$\gamma_\mu\gamma_\nu = g_{\mu\nu}1_V - i\sigma_{\mu\nu}. \quad (26)$$

Also ([9], p. 372)

$$\mathrm{tr}(\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho) = 4(g_{\lambda\mu}g_{\nu\rho} + g_{\lambda\rho}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\rho}) \quad (27)$$

from which it follows that

$$\mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau}) = 4(g_{\mu\rho}g_{\nu\tau} - g_{\mu\tau}g_{\nu\rho}). \quad (28)$$

Thus, to begin the unfolding, we have

$$\begin{aligned}
-\frac{1}{2}\mathrm{tr}(\partial^\nu G_\nu)^2 &= \frac{1}{2}\mathrm{tr}(i\partial^\nu G_\nu)^2 \\
&= \frac{1}{8}\mathrm{tr}((i\partial^\nu G_\nu \otimes 1_V)(i\partial^\mu G_\mu \otimes 1_V))
\end{aligned} \tag{29}$$

and

$$-\frac{1}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu}) = \frac{1}{8}(-2\mathrm{tr}(G^{\mu\nu}G_{\mu\nu})) \tag{30}$$

$$= \frac{1}{8}(\mathrm{tr}(G^{\mu\nu}G_{\nu\mu}) - \mathrm{tr}(G^{\mu\nu}G_{\mu\nu})) \tag{31}$$

$$= \frac{1}{8}(\mathrm{tr}(G^{\mu\nu}G^{\rho\tau}g_{\mu\rho}g_{\nu\tau}) - \mathrm{tr}(G^{\mu\nu}G^{\rho\tau}g_{\mu\tau}g_{\nu\rho})) \tag{32}$$

$$= \frac{1}{8}\mathrm{tr}(G^{\mu\nu}G^{\rho\tau})(g_{\mu\rho}g_{\nu\tau} - g_{\mu\tau}g_{\nu\rho}) \tag{33}$$

$$= \frac{1}{8}\left(\frac{1}{4}\mathrm{tr}(G^{\mu\nu}G^{\rho\tau})\mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau})\right) \quad \text{from Eq. (28)} \tag{34}$$

$$= \frac{1}{8}\left(\frac{1}{4}\mathrm{tr}((G^{\mu\nu} \otimes \sigma_{\mu\nu})(G^{\rho\tau} \otimes \sigma_{\rho\tau}))\right). \tag{35}$$

So, by comparing Eqs. (29) and (35) and inspecting $\mathcal{L}$ (Eq. (22)) we are led to consider the following expression, the trace of whose square matches the terms in $\mathcal{L}$, with k some constant to be chosen later:

$$\Delta := k\left(i\partial^\nu G_\nu \otimes 1_V + \frac{1}{2}G^{\mu\nu} \otimes \sigma_{\mu\nu}\right) \tag{36}$$

$$= k\left(i\partial^\nu G_\nu \otimes 1_V + \frac{1}{2}(\partial^{[\mu}G^{\nu]} - i[G^\mu, G^\nu]) \otimes \sigma_{\mu\nu}\right) \tag{37}$$

$$= k\left(\partial^\nu(iG_\nu) \otimes 1_V + \frac{i}{2}(\partial^{[\nu}iG^{\mu]} + [iG^\mu, iG^\nu]) \otimes \sigma_{\mu\nu}\right) \tag{38}$$

$$= k\left(2\partial^\nu\left(\frac{i}{2}G_\nu\right) \otimes 1_V + \frac{i}{2}\left(2\partial^{[\nu}\left[\frac{i}{2}G^{\mu]}\right] + 4\left[\frac{i}{2}G^\mu, \frac{i}{2}G^\nu\right]\right) \otimes \sigma_{\mu\nu}\right) \tag{39}$$

$$= 2k\left(\partial^\nu\left(\frac{i}{2}G_\nu\right) \otimes 1_V + \frac{i}{2}\left(\partial^{[\nu}\left[\frac{i}{2}G^{\mu]}\right] + 2\left[\frac{i}{2}G^\mu, \frac{i}{2}G^\nu\right]\right) \otimes \sigma_{\mu\nu}\right). \tag{40}$$

So, with $k = \frac{1}{2}$ and, to ameliorate the notation, defining $F_\mu := \frac{i}{2}G_\mu$, which we note lies in $\mathfrak{su}(n)$, we obtain from Eqs. (36) and (40),

$$\Delta = \frac{1}{2}\left(i\partial^\nu G_\nu \otimes 1_V + \frac{1}{2}G^{\mu\nu} \otimes \sigma_{\mu\nu}\right) \tag{41}$$

$$= \partial^\nu F_\nu \otimes 1_V + \frac{i}{2}(\partial^{[\nu}F^{\mu]} + 2[F^\mu, F^\nu]) \otimes \sigma_{\mu\nu}. \tag{42}$$

We continue to unfold the last line, now assuming that we are in Minkowski space, with $g_{\mu\nu}$ constant. Then we have

$$\Delta = \partial^\nu F^\mu \otimes g_{\mu\nu}1_V + i(\partial^\nu F^\mu + 2F^\mu F^\nu) \otimes \sigma_{\mu\nu} \tag{43}$$

$$= \partial^\nu F^\mu \otimes (g_{\mu\nu}1_V + i\sigma_{\mu\nu}) + F^\mu F^\nu \otimes 2i\sigma_{\mu\nu} \tag{44}$$

$$= \partial^\nu F^\mu \otimes (g_{\nu\mu}1_V - i\sigma_{\nu\mu}) + F^\mu F^\nu \otimes [\gamma_\nu, \gamma_\mu] \tag{45}$$

$$= \partial^\nu F^\mu \otimes \gamma_\nu\gamma_\mu + F^\mu F^\nu \otimes [\gamma_\nu, \gamma_\mu]. \tag{46}$$

At this juncture we pause to consider the following expression, where B denotes the space carrying the representation of $\mathfrak{su}(n)$ we are tacitly assuming:

$$\mathcal{L}_Q := \text{tr}(e^\Delta - 1_{B \otimes V}) \quad (47)$$

$$= \text{tr} \left(\Delta + \frac{1}{2!} \Delta^2 + \dots \right) \quad (48)$$

$$= \text{tr} \left(\frac{1}{2!} \Delta^2 + \dots \right) \quad \text{since } \text{tr} \Delta = 0, \quad (49)$$

$$= \text{tr} \left(\frac{1}{2!} \left[\frac{1}{2} \left(i \partial^\nu G_\nu \otimes 1_V + \frac{1}{2} G^{\mu\nu} \otimes \sigma_{\mu\nu} \right) \right]^2 + \dots \right) \quad \text{from Eq. (41)} \quad (50)$$

$$= \frac{1}{2!4} \text{tr} \left(-(\partial^\nu G_\nu)^2 \otimes 1_V + \text{trace free term} + \frac{1}{4} (G^{\mu\nu} G^{\rho\tau} \otimes \sigma_{\mu\nu} \sigma_{\rho\tau}) + \dots \right) \quad (51)$$

$$= \frac{1}{2!4} \left(-\text{tr}(\partial^\nu G_\nu)^2 \cdot 4 + \frac{1}{4} \text{tr}(G^{\mu\nu} G^{\rho\tau}) \text{tr}(\sigma_{\mu\nu} \sigma_{\rho\tau}) + \dots \right) \quad (52)$$

$$= \frac{1}{2!4} \left(-\text{tr}(\partial^\nu G_\nu)^2 \cdot 4 - 2 \text{tr}(G^{\mu\nu} G_{\mu\nu}) + \dots \right) \quad \text{from Eq. (34)} \quad (53)$$

$$= -\frac{1}{2} \text{tr}(\partial^\nu G_\nu)^2 - \frac{1}{4} \text{tr}(G^{\mu\nu} G_{\mu\nu}) + \dots \quad (54)$$

a series whose lowest order term matches $\mathcal{L}$ exactly. What could have happened to the higher order terms? The form shown in the last equation has been obtained by unpicking traces and essentially unfolding or reversing a computation which putatively gives rise to the classical, maximally external expression $\mathcal{L}$. Thus, as we pointed out in the previous section, it presumably reveals more of the hitherto hidden internal or microscopic structure, perhaps contained in the higher order terms. Therefore, let us express the x_μ coordinates in terms of some very small unit of length, τ say, of the order of the Planck length, for instance, without committing to any assumptions concerning its status as a fundamental unit of length. This then entails replacing x_μ by $\tau^{-1} x_\mu$, ∂^μ by $\tau \partial^\mu$, G^μ by τG^μ and the measure $d^4 x$ by $\tau^{-4} d^4 x$. Consequently, the expression $\mathcal{L}_Q$ if used in a path integral relative to the new measure would acquire the factor τ^{-4} relative to the original measure so the new density would assume the form

$$\tau^{-4} \mathcal{L}_Q = \tau^{-4} \left(-\frac{1}{2} \text{tr}(\tau^2 \partial^\nu G_\nu)^2 - \frac{1}{4} \text{tr}(\tau^4 G^{\mu\nu} G_{\mu\nu}) + O(\tau^6) \right) \quad (55)$$

$$= -\frac{1}{2} \text{tr}(\partial^\nu G_\nu)^2 - \frac{1}{4} \text{tr}(G^{\mu\nu} G_{\mu\nu}) + O(\tau^2) \quad \text{as } \tau \rightarrow 0. \quad (56)$$

Thus, for field intensities much smaller than the enormous τ^{-2} the higher order terms in the last expression would be invisible, leaving the original form as it appears classically.

Our unfolding of the Yang–Mills Lagrangian has brought us to the presumably underlying quantum microprogram expressed by the operator $e^\Delta - 1$ (omitting, as in what follows, the subscripts on 1). Our task now is to interpret what it does. This in turn devolves upon unfolding the operator Δ which we shall attempt using its expression in Eq. (46). The first term may be written as,

$$\partial^\nu F^\mu \otimes \gamma_\nu \gamma_\mu = (\partial^\nu F^\mu \otimes \gamma_\nu)(1 \otimes \gamma_\mu) \quad (57)$$

while the second may be written as,

$$F^\mu F^\nu \otimes [\gamma_\nu, \gamma_\mu] = (F^\mu \otimes 1)(F^\nu \otimes \gamma_\nu \gamma_\mu - F^\nu \otimes \gamma_\mu \gamma_\nu) \quad (58)$$

$$= (F^\mu \otimes 1)((F^\nu \otimes \gamma_\nu)(1 \otimes \gamma_\mu) - (1 \otimes \gamma_\mu)(F^\nu \otimes \gamma_\nu)). \quad (59)$$

Recall first that the Dirac maps γ_μ acting on the space V of bispinors, may be regarded quite literally as quantized versions of the differentials dx_μ . This is best seen by expressing Minkowski vectors in terms of (spinor) $\otimes$ (conjugate spinor) pairs, applying a derivation to these pairs (in the appropriate tensor algebra) and then finding appropriate Lorentz covariant representations of the two resulting terms: details may be found in [2] or [3]. Now consider the setting of a general gauge theory. Its main ingredient is a space of fermion n -tuples (also called multiplets) accommodating a representation of the gauge group, assumed to be a Lie group, with the representation being liftable to the Lie algebra of the gauge group. (We will generally drop the n reference and just write tuple). The action of the gauge group is supposed to supply a physically ignorable unitary phase to the space of tuples. The proper mathematical way—please see the next paragraph—to describe such a space of tuples is to separate the n -dimensional representation space—call it B —of the gauge group (or Lie algebra $\mathfrak{g}$) from the space of bispinors V and form their tensor product $B \otimes V \cong \bigoplus^n V$. This is the space of fermion n -tuples, with $\mathfrak{g}$ acting on B . In our case $\mathfrak{g}$ is $\mathfrak{su}(n)$, the Lie algebra of $SU(n)$.

Let us return to the unfolded forms of Eqs. (57) and (59) and note that these expressions are operators exactly on the space $B \otimes V$ whose choice is seen therefore to be not mere mathematical whimsy, but actually emerges from the unfolding process. Moreover, we can now interpret the Dirac forms

$$F^\mu \otimes \gamma_\mu := \not{F} \quad (60)$$

that have spontaneously arisen there. As we noted, γ_μ is the quantized form of dx_μ , so $\not{F}$ is the quantized form of $F^\mu dx_\mu$. Now notice that for an element $w \in B$

$$\delta w := e^{F^\mu dx_\mu} w - w \quad (61)$$

$$= F^\mu(w) dx_\mu \quad (62)$$

which is the variation in w consequent upon this small generic gauge rotation. Consequently, $\not{F}$ is a quantized, Dirac slashed version of the variation in $w \otimes s$ consequent upon this small generic gauge rotation, and we may write, for a generating element $w \otimes s \in B \otimes V$

$$\delta(w \otimes s) := F^\mu(w) \otimes \gamma_\mu(s) \quad (63)$$

$$= \not{F}(w \otimes s) \quad (64)$$

Now we can return to unfolding Eq. (59) which we apply to a generating element $w \otimes s$:

$$(F^\mu F^\nu \otimes [\gamma_\nu, \gamma_\mu])(w \otimes s) = (F^\mu \otimes 1)[(F^\nu \otimes \gamma_\nu)(w \otimes \gamma_\mu(s)) - (1 \otimes \gamma_\mu)(F^\nu \otimes \gamma_\nu)(w \otimes s)] \quad (65)$$

$$= (F^\mu \otimes 1)[\not{F}(w \otimes \gamma_\mu(s)) - (1 \otimes \gamma_\mu)\not{F}(w \otimes s)] \quad (66)$$

$$= (F^\mu \otimes 1)[\delta(w \otimes \gamma_\mu(s)) - (1 \otimes \gamma_\mu)\delta(w \otimes s)] \quad (67)$$

$$= (F^\mu \otimes 1)[\delta((1 \otimes \gamma_\mu)(w \otimes s)) - (1 \otimes \gamma_\mu)\delta(w \otimes s)] \quad (68)$$

$$= (F^\mu \otimes 1)(\delta(1 \otimes \gamma_\mu))(w \otimes s). \quad (69)$$

Note also that the ordinary Dirac operator $\not{F}\partial$ is the quantized version of the usual operator d for which we shall use the δ notation for consistency. Then Eq. (57) may be rewritten

$$(\partial^\nu F^\mu \otimes \gamma_\nu)(1 \otimes \gamma_\mu) = (\delta(F^\mu \otimes 1))(1 \otimes \gamma_\mu) \quad (70)$$

so that, as the sum of the maps in Eqs. (69) and (70),

$$\Delta = (\delta(F^\mu \otimes 1))(1 \otimes \gamma_\mu) + (F^\mu \otimes 1)(\delta(1 \otimes \gamma_\mu)) \quad (71)$$

$$= \delta((F^\mu \otimes 1)(1 \otimes \gamma_\mu)) \quad (72)$$

$$= \delta(F^\mu \otimes \gamma_\mu) \quad (73)$$

$$= \delta \mathcal{F} \quad (74)$$

$$= \delta \left(\frac{i}{2} \mathcal{G} \right) \quad (75)$$

This is our sought for expression for Δ . In the form $e^{\delta(\frac{i}{2}\mathcal{G})}$ it represents a gauge rotation whose infinitesimal generator is the generic deformation of a Dirac quantized gauge potential. Then, the quantum microprogram underlying the classical Yang–Mills Lagrangian, namely $e^{\delta(\frac{i}{2}\mathcal{G})} - 1$, acts on a generating tuple to give

$$\left(e^{\delta(\frac{i}{2}\mathcal{G})} - 1 \right) (w \otimes s) = e^{\delta(\frac{i}{2}\mathcal{G})} (w \otimes s) - (w \otimes s). \quad (76)$$

That is to say, it represents the variation induced upon a fermion tuple consequent upon the gauge rotation whose infinitesimal generator is the generic deformation of a Dirac quantized gauge potential. There are therefore two variations involved, the inner one $\delta(\frac{i}{2}\mathcal{G})$ and the outer one of the last equation, essentially a variation of a variation. Presumably this accounts ultimately for the appearance of curvature forms in the macroworld, geometrical considerations notwithstanding.

7 Conclusions

We have tried to propel Toffoli's insights concerning the computational nature of the classical notions of action and Lagrangian into the quantum domain and have found that his interpretation of them in terms of computational capacity holds up well, and even more strongly, when translated into the appropriate quantum language. In particular, the Lagrangian density itself may be interpreted as a quantum microprogram whose action integrals then represent quantum macroprograms as the limit of superpositions of quantum microprograms. This then opens the possibility of unfolding a classical Lagrangian density (presumed to be the irreducible reduction of a quantum deduction) to reveal an underlying quantum microprogram driving the quantum process, in rough (and so far informal) analogy with the unfolding of a simply typed λ -expression to reveal an underlying natural deduction. We have carried this out for the Yang–Mills Lagrangian, including the Feynman–'t Hooft gauge fixing term, and found its origin to lie in the simple quantum program that effects the variation in a tuple consequent upon a quantized generically deformed gauge rotation of the tuple.

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