On the existence of coalition-proof Bertrand equilibrium

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Received: 13 March 2013 / Accepted: 21 March 2013 / Published online: 11 April 2013
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Abstract In Baye and Morgan (Econ Theory 19:271–282, 2002) and Hoernig (Econ Theory 31:573–585, 2007), sufficient conditions for the existence of Bertrand equilibrium with arbitrary sharing rules were provided. Here, sufficient conditions for the existence of a coalition-proof Bertrand equilibrium with an arbitrary tie-breaking rule are provided. A classical Bertrand game in which sellers have convex costs is analyzed and sufficient conditions for the existence of coalition-proof Bertrand equilibrium which admit discontinuities in tied payoffs and a general class of tie-breaking rules are stated. Finally, an example is provided where one of the conditions is violated and there is non-existence of coalition-proof Bertrand equilibrium. This work generalizes an existence result of Chowdhury and Sengupta (Econ Theory 24:307–324, 2004).

Keywords Equilibrium existence · Discontinuous payoffs · Coalition-proofness · Bertrand games

JEL Classification D43 · C72

1 Introduction

In recent years there has been a substantial amount of research which has established what conditions market primitives, cost and demand functions, must satisfy

R. R. Routledge would like to thank numerous colleagues for advice which improved an earlier draft of the paper, three referees who helped to clarify some of the ideas, and the editor, Nicholas Yannelis, for helpful comments.

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to guarantee the existence of equilibrium in classical Bertrand games where sellers produce a single homogeneous good.¹ The most recent research has considered what conditions the sharing rule at price ties must satisfy to ensure existence of equilibrium. Given that the preferences of consumers are unlikely to coincide with the specific sharing rules considered in the literature, and that non-convex preferences may result in discontinuities in tied payoffs, the existence results can be applied in a wide variety of settings. However, as Bertrand games have infinite action spaces and discontinuous payoffs establishing the conditions, the tied payoffs must satisfy is a non-trivial problem.² This paper furthers this line of investigation. Rather than establishing the existence of Bertrand equilibrium, the aim is to provide conditions which guarantee the existence of coalition-proof Bertrand equilibrium with an arbitrary tie-breaking rule. The notion of coalition-proofness was introduced by [Bernheim et al. \(1987\)](#) and is appropriate in situations where the players in a game can engage in pre-play communication but cannot write binding contracts. Therefore, any coalitional deviation must be self-enforcing.

This work is related to a number of papers on price-setting games. [Hoernig \(2007\)](#) provided an extensive analysis of which properties arbitrary sharing rules must satisfy to guarantee the existence of pure and mixed strategy Bertrand equilibrium. It was proved that if non-tied payoffs are continuous, the sum of players' payoffs is upper semicontinuous and the sharing rule is coalition monotone and weakly tie-decreasing then there exists a mixed strategy Bertrand equilibrium. Coalition monotone means that the sum of players' payoffs increases as more players join a price tie and is often satisfied in games with convex costs. Weak tie-decreasing means that non-tied payoffs are at least as great as tied payoffs whenever the latter are positive. [Baye and Morgan \(2002\)](#) considered the "winner-takes-all" sharing rule where one seller is chosen randomly from the set of sellers tying at the minimum price to serve all the demand forthcoming. They permitted discontinuities in both cost and demand functions and established sufficient conditions for the existence of an equilibrium in which all sellers earn zero expected profit. [Bagh \(2010\)](#) considered a general model in which sellers had convex costs and the sharing rule was deterministic but arbitrary and the market demand could be discontinuous. It was established that a pure strategy Bertrand equilibrium exists when sellers have symmetric costs and the profit functions are left lower semicontinuous. A sufficient condition for the existence of equilibrium with asymmetric sellers was also provided. Finally, [Chowdhury and Sengupta \(2004\)](#) established the existence of coalition-proof Bertrand equilibrium when the tie-breaking rule is capacity sharing. If the sellers are symmetric the Bertrand game admits a unique coalition-proof Bertrand equilibrium. Moreover, they demonstrated that as the set of sellers becomes large the coalition-proof equilibrium set converges on the competitive equilibrium, provided all sellers are active in the limit.

¹ A seminal paper was [Dastidar \(1995\)](#) which proved that in a market with symmetric sellers, strictly convex costs and equal sharing at price ties, a Bertrand equilibrium always exists and is non-unique.

² This problem of establishing conditions which guarantee the existence of equilibrium in discontinuous games has been addressed recently by [Carmona \(2011\)](#) which introduced the notion of weak better-reply security, and [Reny \(2011\)](#) which provided conditions to establish whether a discontinuous game admits a strategic approximation.

The existence proof regarding coalition-proof equilibrium which is provided here considers a market with sellers with convex costs and a sharing rule which is deterministic. In this context, the result of [Bagh \(2010\)](#) is invoked to guarantee that the Bertrand equilibrium set is non-empty. As coalition-proofness is a refinement of the Nash equilibrium the existence of a Bertrand equilibrium is a necessary condition for the existence of a coalition-proof Bertrand equilibrium. It is then shown that the set of prices at which no strict subset of the sellers has a profitable deviation is non-empty and compact. Finally, it is shown that each seller's maximizer on the set exists, and as a result, one can then use a similar line of reasoning as Proposition 1 in [Chowdhury and Sengupta \(2004\)](#) to establish the existence of coalition-proof equilibrium.

Two examples are presented which illustrate the result and the importance of the assumptions that are used. The first example is a market with symmetric sellers and a sharing rule which is different from equal sharing. Nevertheless, one can apply the result to show that the market possesses a coalition-proof Bertrand equilibrium. The second example violates a continuity condition and fails to possess a coalition-proof Bertrand equilibrium despite the game possessing a Bertrand equilibrium. As far as the author is aware, this is the first example of the non-existence of coalition-proof Bertrand equilibrium to be presented in the literature. The paper concludes with a number of remarks regarding possible generalizations of the existence result and a remark regarding equilibrium refinements similar to coalition-proofness.

2 The Bertrand game

There is a set of price-makers given by $N = \{1, \dots, n\}$ with $n \geq 2$. The sellers produce a single perfectly homogeneous good and sellers play a classical Bertrand game in which each seller independently and simultaneously chooses a price. The sellers commit to supplying all the demand forthcoming from buyers. The market demand is $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and the cost function of each seller is $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The following standard assumptions are imposed on the market primitives.

Assumption 1 There exists a positive finite real number, $\bar{p}$, such that $D(p) = 0$ for every $p \geq \bar{p}$ and $D(\cdot)$ is negative monotone³ on $[0, \bar{p}]$. Every seller's cost function is continuous, positive monotone, strictly convex and satisfies $C_i(0) = 0$.

The possibility that some coalition of sellers ties at the minimum price will be considered. When ties occur there must be a sharing rule which describes how the market demand is split between the sellers tying at the minimum price. Let $\Omega = \{M : M \in 2^N \setminus \{\emptyset\}, |M| \geq 2\}$ so Ω is the set of all coalitions with cardinality at least as great as two. For any coalition $A \in \Omega$, let $L(A)$ denotes the $|A| - 1$ -dimension unit simplex and $\text{int}L(A)$ denotes the interior of the unit simplex. A sharing rule for coalition is $A \in \Omega$ is a mapping $f_A : \mathbb{R}_+ \rightarrow \text{int}L(A)$ with the domain being the price at which sellers tie. Therefore, the sharing rule gives the share of the market demand which a member of a coalition tied at the minimum price receives.⁴ The complete

³ A function, $f : \mathbb{R} \rightarrow \mathbb{R}$, is positive (negative) monotone if $f(x') > (<) f(x)$ for every $x' > x$.

⁴ Although this type of sharing rule is quite general, it is not completely arbitrary as it excludes the "winner-takes-all" rule and extreme cases where some sellers' quantity is zero under capacity sharing. Nevertheless,

set of sharing rules is $\{f_A\}_{A \in \Omega}$. Let i 's share of the market demand when it ties with $A \setminus \{i\}$ be denoted by $f_{i,A}$. Although the sharing rule may depend upon the price at which sellers tie at, this is suppressed to simplify the notation.

Assumption 2 $f_{i,N}D(p)$ is negative monotone on $[0, \bar{p}]$ for every $i \in N$.

The specification of the Bertrand game can be completed by summarizing the payoffs. Denote the strategy spaces of the sellers by $S = [0, \bar{p}]$ and let $\pi_i(p) = pD(p) - C_i(D(p))$, $\hat{\pi}_i(p, A) = pf_{i,A}D(p) - C_i(f_{i,A}D(p))$ and $\tilde{\pi}_i(p, A) = \pi_i(p) - \hat{\pi}_i(p, A)$. The function $\pi_i(p)$ is the monopoly profit of seller i , $\hat{\pi}_i(p, A)$ is the shared profit which i obtains when it ties with $A \setminus \{i\}$ at the minimum price and $\tilde{\pi}_i(p, A)$ is the difference between the values of these functions.⁵ The following assumptions are imposed on the tied and monopoly profit functions.

Assumption 3 For each $i \in N$ there exist prices $p'_i, p''_i \in (0, \bar{p})$ such that $\pi_i(p'_i) > 0$ and $\hat{\pi}_i(p''_i, N) > 0$.

Assumption 4 For every $i \in N$ the equations $\pi_i(p) = 0$ and $\hat{\pi}_i(p, N) = 0$ possess solutions in $(0, \bar{p})$.

Assumption 5 The profit functions $\hat{\pi}_i(p, N)$ are upper semicontinuous in price.⁶

Assumption 3 is a standard condition that there exist prices at which the monopoly and shared profits of the sellers are positive. The other two assumptions permit us to consider cases where there are discontinuities in tied payoffs. If continuity were imposed on tied payoffs, then Assumption 4 would follow from Assumption 3 and the intermediate value theorem. However, this is not done here so that the model permits the possibility that there are discontinuities in tied payoffs which may arise if the market demand is discontinuous. For more on this, please refer to Remark 1 below. Given a vector of prices, $\mathbf{p} = (p_1, \dots, p_n)$, the the payoffs to seller i can be summarized as:

$$u_i(\mathbf{p}) = \begin{cases} \pi_i(p_i), & \text{if } p_i < p_j \text{ for all } j \neq i; \\ \hat{\pi}_i(p_i, A) & \text{if } i \text{ ties with } A \setminus \{i\} \text{ at min. price;} \\ 0 & \text{if } p_i > p_j \text{ for some } j \neq i. \end{cases} \quad (1)$$

Footnote 4 continued

there are many cases where it might be expected that sellers' quantities may be close to equal sharing or capacity sharing. As present, the literature has nothing to say on these small deviations from the existing sharing rules.

⁵ To be mathematically precise about these functions, they are defined on the following domains $\pi_i : S \rightarrow \mathbb{R}$, $\hat{\pi}_i : S \times \Omega \rightarrow \mathbb{R}$ and $\tilde{\pi}_i : S \times \Omega \rightarrow \mathbb{R}$.

⁶ It is standard to impose continuity conditions upon the profit functions. However, one might wonder whether there are conditions upon the primitives which guarantee this condition. If the sharing rule and market demand are continuous then the tied profits are continuous. If the market demand and/or sharing rule are discontinuous then the following general condition can be given which guarantees the tied profit functions are upper semicontinuous. Let $z_i(p)$ denotes the competitive supply of seller i at price p : $z_i(p) = \arg \max_{Q \in \mathbb{R}_+} pQ - C_i(Q)$. If p^* is a discontinuity point of the tied profit of coalition A , then the tied profits are upper semicontinuous at p^* if $|\bar{f}_{i,A}(p^*)\bar{D}(p^*) - z_i(p^*)| > |f_{i,A}(p^*)D(p^*) - z_i(p^*)|$ for every $i \in A$, where $\bar{f}_{i,A}$ and $\bar{D}(p^*)$ denote the share and demand that converge upon at p^* .

Denote the simultaneous-move Bertrand game, with payoffs defined by Eq. 1, by $\mathcal{G} = \langle N, (S, u_i)_{i \in N} \rangle$. In the rest of the paper, the possibility that a coalition of sellers can change their prices to achieve a higher payoff will be considered. This is now defined formally.

Definition 1 A coalition of sellers, $A \subset N$, has a profitable deviation at price vector $\mathbf{p}$ if there exist prices $\mathbf{p}' = \{p'_i\}_{i \in A}$ such that $u_i(\mathbf{p}', \mathbf{p}(N \setminus A)) > u_i(\mathbf{p})$ for every $i \in A$.

In words, a coalition has a profitable deviation at a price vector if they can change their prices collectively and achieve a higher payoff provided the sellers not in the coalition do not change their prices. Given this definition of an improvement it is possible to define a pure strategy Bertrand equilibrium.

Definition 2 A price vector $\mathbf{p}$ is a pure strategy Bertrand equilibrium if no coalition of sellers $A \subset N$, $|A| = 1$, has a profitable deviation at $\mathbf{p}$.

Denote the pure strategy Bertrand equilibrium set of the game $\mathcal{G}$ by $\mathcal{E}(\mathcal{G})$. Now although a coalition may have a profitable deviation from a price vector, not all profitable deviations are self-enforcing. This is because once a coalition deviates some subset of the deviating sellers may have a new profitable deviation. The analysis is restricted to consider profitable deviations which do not have this unattractive property. The notion of a self-enforcing profitable deviation is defined inductively. Formally, a coalition $A \subset N$, $|A| = 1$, has a *self-enforcing* profitable deviation if A has a profitable deviation. Now suppose that self-enforcing profitable deviations for a coalition with cardinality less than or equal to l have been defined. A coalition A with $|A| = l + 1$ has a self-enforcing profitable deviation if A has a profitable deviation, which will be denoted by $\mathbf{p}'$, and there is no coalition $B \subset A$, $B \neq A$, which has a self-enforcing profitable deviation upon $\mathbf{p}'$. Given this concept of a self-enforcing profitable deviation, a coalition-proof Bertrand equilibrium can be defined.

Definition 3 A price vector $\mathbf{p}$ is a coalition-proof Bertrand equilibrium if there is no coalition which has a self-enforcing profitable deviation upon $\mathbf{p}$.

The coalition-proof Bertrand equilibrium set of the game $\mathcal{G}$ will be denoted by $\mathcal{E}^C(\mathcal{G})$.

Definition 4 For any $A \in \Omega$ and $i \in A$ let $\hat{p}_i(A) = \inf\{p : p \in [0, \bar{p}) \text{ and } \hat{\pi}_i(p, A) \geq 0\}$.

Definition 5 For any $A \in \Omega$ and $i \in A$ let $\tilde{p}_i(A) = \sup\{p : p \in [0, \bar{p}) \text{ and } \tilde{\pi}_i(p, A) \leq 0\}$.

The prices in Definitions 4 and 5 provide bounds on the Bertrand equilibrium set for any coalition as $\hat{p}_i(A)$ is the minimum price which could be a Bertrand equilibrium in which all sellers charge the same price for coalition A .⁷ If coalition A was to post

⁷ The discontinuities in demand/sharing rule may result in tied profits being negative at $\hat{p}_i(A)$. The use of $\inf/\sup$ ensures that the point is well-defined. This is not a problem in the model here as we have assumed the existence of the prices in Assumption 4.

a price lower than $\hat{p}_i(A)$, then seller i would have a profitable deviation by posting price $\bar{p}$. The price $\tilde{p}_i(A)$ is an upper bound on the Bertrand equilibrium set in which coalition A posts the same price. If the coalition posted a price higher than $\tilde{p}_i(A)$, then seller i would have a profitable deviation by undercutting this price and obtaining its monopoly profit. Let $I(N) = \cap_{i \in N} [\hat{p}_i(N), \tilde{p}_i(N)]$.⁸ The two final assumptions are stated below.

Assumption 6 $\max\{\hat{p}_i(N) : i \in N\} \leq \min\{\hat{p}_j(A) : j \in A, A \in \Omega \setminus \{N\}\}$.

Assumption 7 $I(N) \neq \emptyset$.

Assumption 6 guarantees that the minimum symmetric Bertrand equilibrium price for the coalition N is at least as small as for any other coalition of sellers. It is easily verified that in a market with symmetric sellers and the equal sharing rule this condition is satisfied and in many other cases as well (see Example 1 below). What is required for Assumptions 6 and 7 to be satisfied is that the asymmetries between sellers, caused either by costs or the sharing rule, are not “too great”.

With equal sharing and very asymmetric costs the assumptions may not be satisfied. For example, suppose $N = \{1, 2, 3\}$, $C_1(Q) = 3Q^2$ and $C_2(Q) = C_3(Q) = Q^2$. The market demand is given by $D(p) = \max\{0, 12 - 6p\}$. There is equal sharing at price ties. Then $\max\{\hat{p}_i(N) : i \in N\} = 12/7$. However, given $A = \{2, 3\}$ we have $\hat{p}_2(A) = \hat{p}_3(A) = 3/2 < 12/7$. Therefore, Assumption 6 is violated. Alternatively, one could consider asymmetries caused by the sharing rule. Suppose the costs of the sellers are identical and given by $C(Q) = Q^2$. However, the sharing rule is different from equal sharing. When all three sellers tie at the same price then seller one obtains $2/3$ of the market demand, sellers two and three each obtain $1/6$ of the market demand. When any two sellers tie at the same price they each obtain $1/2$ of the market demand. Then $\max\{\hat{p}_i(N) : i \in N\} = 8/5$. However, for any A , with $|A| = 2$, $\hat{p}_i(A) = \hat{p}_j(A) = 3/2 < 8/5$.

Assumption 7 guarantees that the set of Bertrand equilibrium is non-empty. As $\mathcal{E}^C(\mathcal{G}) \subset \mathcal{E}(\mathcal{G})$, the non-emptiness of the Bertrand equilibrium set is a necessary condition for the existence of a coalition-proof equilibrium. In the rest of the paper, let (p, N) denotes the case where all the sellers quote price p . Define the sets $\Gamma(p) = \{A \subset N, A \neq N : A \text{ has a profitable deviation upon } (p, N)\}$ and $\Delta = \{p \in [0, \bar{p}] : \Gamma(p) = \emptyset\}$. The set Δ is those prices such that when all sellers post the price in the market no strict subset of the sellers has a profitable deviation. Note that $\Gamma(p) \subset 2^N$ and $\Delta \subset I(N)$. Let $p^* = \sup \Delta$. With this set-up, the existence result regarding the coalition-proof Bertrand equilibrium set can be given.

Theorem 1 *If A1–A7 are satisfied then $\mathcal{E}^C(\mathcal{G}) \neq \emptyset$.*

Proof First, the Bertrand game, $\mathcal{G}$, defined here satisfies all the conditions of Theorem 3 in Bagh (2010) so we can invoke that result to establish that for any $p \in I(N)$ all sellers quoting p is a pure strategy Bertrand equilibrium. That is, if $p \in I(N)$ then (p, N) is a pure strategy Bertrand equilibrium.

⁸ One can refer to Bagh (2010) to verify $\hat{p}_i(N) < \tilde{p}_i(N)$.

We now demonstrate that Δ is non-empty and compact. Consider the price $p' = \min I(N)$. We know that (p', N) is a pure strategy Bertrand equilibrium. If a coalition $A \subset N$, $A \neq N$, has a profitable deviation then they must post a common price $p'' < p'$. The profit which each seller earns at this price is $\hat{\pi}_i(p'', A)$. However, from Assumption 6, it must be that $\hat{\pi}_i(p'', A) < 0$ for some $i \in A$ which contradicts A having a profitable deviation. Therefore, $\Delta \neq \emptyset$.

The boundedness of Δ follows from $\Delta \subset [0, \bar{p}]$. Now suppose that Δ is *not* a compact set. Then, there must be a sequence of prices $\{p_n\}_{n \in \mathbb{N}}$ such that $p_n \in \Delta$ for every $n \in \mathbb{N}$ and $p^L \notin \Delta$ with $p^L = \lim_{n \rightarrow \infty} p_n$. The profit which each seller obtains at p^L is $\hat{\pi}_i(p^L, N)$. It must be that $\hat{\pi}_i(p^L, N) \geq 0$ for every $i \in N$ otherwise the upper semicontinuity of $\hat{\pi}_i(p, N)$ would mean that there is a price sufficiently far in the sequence $\{p_n\}_{n \in \mathbb{N}}$ which yields negative tied profit for some seller and contradicts $p_n \in \Delta$ for every $n \in \mathbb{N}$. However, if a strict subset of sellers has a profitable deviation from (p^L, N) the deviation must involve posting a common price strictly less than p^L . But the upper semicontinuity of $\hat{\pi}_i(p, N)$ would mean that there is a price sufficiently far in the sequence $\{p_n\}_{n \in \mathbb{N}}$ upon which the coalition has a profitable deviation and contradicts $p_n \in \Delta$ for every $n \in \mathbb{N}$. Therefore, it must be that $p^L \in \Delta$ and Δ is compact.

Note that it must be that $p' = \min \Delta$. This is because Assumption 6 implies that some seller's profit is negative for prices in the set $[0, p')$, regardless of which coalition they tie with, and they could always profitably deviate to $\bar{p}$. Now recall that $p^* = \max \Delta = \sup \Delta$ and define the set $P = \{p \in [p', p^*] : \hat{\pi}_i(p, N) \geq \hat{\pi}_i(p^*, N) \text{ for every } i \in N\}$. As the tied profit functions $\hat{\pi}_i(p, N)$ are upper semicontinuous, P is a compact set. Further, let $p_i^m = \arg \max_{p \in P} \hat{\pi}_i(p, N)$. As P is compact and $\hat{\pi}_i(p, N)$ is upper semicontinuous the maximizer, p_i^m , exists as upper semicontinuous functions achieve their maximum on a compact set.⁹

We now show that a coalition-proof Bertrand equilibrium exists. Consider (p_i^m, N) . Suppose the grand coalition has a self-enforcing profitable deviation. They must charge a common price, p^N . This common price cannot be in P as seller i cannot be made better-off by charging a price in P . Moreover, the common price cannot be less than p' as Assumption 6 implies that some seller's profit would be negative. Therefore, the only possible profitable deviation is for the sellers to charge a common price $p^N > p^*$. However, if the sellers post a common price $p^N > p^*$ there must be a coalition, $A \subset N$, $A \neq N$, which has a profitable deviation from (p^N, N) . To enact this deviation, coalition A must charge a common price, p^A , such that $p^A < p^N$. If this deviation is self-enforcing for coalition A then this contradicts N having a self-enforcing deviation by charging p^N . If the deviation to p^A by coalition A is not self-enforcing, then there must be a coalition $B \subset A$, $B \neq A$, which has a self-enforcing profitable deviation by posting a common price, p^B such that $p^B < p^A < p^N$. This self-enforcing deviation for coalition B then contradicts the deviation by the grand coalition to p^N being self-enforcing.

Therefore, if a coalition has a self-enforcing deviation from (p_i^m, N) it must be a coalition $A \subset N$, $A \neq N$. To enact this self-enforcing deviation coalition, A must

⁹ See Kolmogorov and Fomin (1999, p.66, Theorem 3a).

charge a common price $p^A < p_i^m \leq p^*$. Furthermore, it must be that $\pi_j(p^A, A) > \pi_j(p_i^m, A) \geq \pi_j(p^*, N)$ for every $j \in A$. However, this contradicts $p^* \in \Delta$. It can be concluded that there is no coalition which has a self-enforcing deviation upon (p_i^m, N) . $\square$

Remark 1 It was noted earlier that continuity was not imposed on the tied profit functions. This meant that it had to be assumed that there exist prices which make the monopoly and tied profit functions equal to zero (Assumption 3). However, this assumption could be dispensed with if one were to impose left lower semicontinuity¹⁰ upon the tied profit functions as was noted in [Bagh \(2010\)](#) and [Baye and Morgan \(2002\)](#). The reason this was not done is because left lower semicontinuity does not guarantee that the tied profit functions achieve their maximum on a compact set which was used in the proof of Theorem 1. It is not clear whether one can relax upper semicontinuity as an example is presented below which shows that when tied profit functions are not upper semicontinuous a coalition-proof Bertrand equilibrium may fail to exist.

Remark 2 The proof of Theorem 1 used Assumption 6 which stated that the lowest bound of symmetric pure strategy Bertrand equilibrium set is at least as small as the lower bound upon the symmetric equilibrium set for any other coalition. One might reasonably ask when this condition is satisfied. It has been noted that the bounds on pure strategy Bertrand equilibrium tend to increase as the set of sellers in the market is reduced ([Vives 1999](#), p.122). Therefore, this condition will be satisfied in a market with symmetric sellers providing that the sharing rule is not too different from equal sharing. However, the capacity sharing rule used in [Chowdhury and Sengupta \(2004\)](#) may not always satisfy this condition as if one adds a seller which has very high costs to a coalition with low costs, the high-cost seller may not be able to profitably tie at the minimum price. The other commonly cited sharing rule, that of winner-takes-all sharing introduced in [Baye and Morgan \(2002\)](#), will satisfy the assumption but is not covered by Theorem 1 as the sharing rule is random and assigns all the demand to one seller.¹¹ However, there are many cases where the split of the market demand may be different from capacity sharing, so one cannot apply Proposition 1 in [Chowdhury and Sengupta \(2004\)](#), yet the conditions in Theorem 1 are satisfied, such as Example 1 below.

Remark 3 The result in Theorem 1 establishes existence but does not indicate anything about uniqueness of the equilibrium. However, if one assumes that the tied profit functions, $\hat{\pi}_i(p, N)$, are positive monotone on the set Δ then the symmetric coalition-proof equilibrium set is unique. This is because if the sellers were to post any common price $p < p^*$, then the grand coalition would have a self-enforcing profitable deviation by posting price p^* .

¹⁰ A function, $f(x)$, is left lower semicontinuous at x' if $\liminf_{x \uparrow x'} f(x) \geq f(x')$. Loosely speaking, the function “jumps down” as approached from the left.

¹¹ This sharing rule does not fit into the present model as it is a profit sharing rather than a quantity sharing rule.

3 Examples

In this section two examples are presented. The first example is a market with symmetric sellers and a tie-breaking rule which is different from equal sharing. In this example the conditions of Theorem 1 are satisfied and there exists a coalition-proof Bertrand equilibrium. In the second example, the assumptions of Theorem 1 are not satisfied and it is shown that there is non-existence of coalition-proof Bertrand equilibrium.

Example 1 Consider a market in which there are three sellers, $N = \{1, 2, 3\}$, and sellers have the same cost function $C(Q) = Q^2$. The demand for the good is piecewise-affine and given by $D(p) = \max\{0, 10 - p\}$. The way in which the market demand is shared at price ties is given as follows. When all three sellers tie at the same price seller one obtains a demand of $\frac{5}{12}D(p)$ and sellers two and three both obtain a demand of $\frac{7}{24}D(p)$. When sellers one and two tie at the same price, seller one obtains a demand of $\frac{5}{12}D(p)$ and seller two obtains $\frac{7}{12}D(p)$. When sellers two and three tie at the same price they split the market demand equally so each seller obtain a demand of $\frac{1}{2}D(p)$. Finally, when sellers one and three tie at the same price seller one obtains a demand of $\frac{5}{12}D(p)$ and seller three obtains $\frac{7}{12}D(p)$. Routine calculations reveal that the market satisfies Assumptions 1–7 and it can be concluded that there exists a coalition-proof Bertrand equilibrium despite the tie-breaking rule being different from equal sharing. Indeed, all sellers charging the common price $p = 4\frac{18}{43}$ is a coalition-proof Bertrand equilibrium.

Example 2 Again consider a market in which there are three sellers, $N = \{1, 2, 3\}$, and sellers have the same cost function $C(Q) = Q^2$. The demand for the good is piecewise-affine and given by:

$$D(p) = \begin{cases} 20 - p & \text{if } 0 \leq p \leq 4\frac{6}{11}; \\ 10 - p & \text{if } 4\frac{6}{11} < p < 10; \\ 0 & \text{if } p \geq 10. \end{cases} \quad (2)$$

The market demand is shared equally at price ties. Given these market primitives and sharing rule, the pure strategy Bertrand equilibrium set has the sellers posting the same price in the interval $p \in (4\frac{6}{11}, 5\frac{5}{7}]$. However, there is no coalition-proof Bertrand equilibrium as two sellers always have a self-enforcing profitable deviation by posting a common price $p' = p - \epsilon > 4\frac{6}{11}$. It is worth noting that in this case A5 is violated as the tied profit functions fail to be upper semicontinuous at $p = 4\frac{6}{11}$. In Fig. 1 below the graphs of the monopoly profit function (dashed) and the joint tied profit function (solid) are presented.

4 Conclusion

This paper has established sufficient conditions for the existence of a coalition-proof Bertrand equilibrium with an arbitrary tie-breaking rule and potential discontinuities in tied payoffs. Given that consumers' preferences are unlikely to generate the specific

Fig. 1 The profit functions in Example 2

sharing rules which have been considered previously, and that non-convex preferences may result in discontinuities, the result can be applied to a wide range of payoffs. Moreover, an example has been exhibited where the Bertrand equilibrium set is non-empty yet a coalition-proof equilibrium fails to exist. The examples indicate that the existence of coalition-proof Bertrand equilibrium is intimately related to the types of discontinuity that occur in the tied profit functions and it remains unclear whether the result presented here can be generalized further to permit different discontinuities. It is worth noting that the proof of Theorem 1 demonstrates that not only does a coalition-proof Bertrand equilibrium exist but that it is symmetric: all sellers post the same price in the market. This is the same as the existence result of Chowdhury and Sengupta (2004). However, it may well be possible to relax the assumptions of the model further to establish results regarding coalition-proof Bertrand equilibrium in which a strict subset of the sellers ties at the minimum price.

Although our interest was in coalition-proof Bertrand equilibrium, it should be noted that the logic of coalition-proofness has been criticized by several authors. Much of this criticism centres around the assumption, implicit in the definition of coalition-proofness, that deviations by coalitions are done secretly. As a result, there has been interest in refinements of the Nash equilibrium by coalitions which are done openly, rather than secretly. Xue (2000) introduced the notion of negotiation-proof Nash equilibrium in which an agent proposing a deviation takes into account the set of counter deviations by all possible coalitions. This notion avoids the inductive definition and secretiveness which is implicit in coalition-proof deviations. In Example 2, it was shown that the Bertrand equilibrium set is non-empty but a coalition-proof Bertrand equilibrium fails to exist. However, does there exist a negotiation-proof Bertrand equilibrium? More generally, do Bertrand games possess negotiation-proof Nash equilibria? These are interesting open questions.

Finally, a number of recent papers have analyzed price-setting games which admit discontinuities in the cost function.¹² Both continuity and convexity were imposed on the cost function here, but it may be possible that existence results can be provided for coalition-proof Bertrand equilibrium which admits discontinuities in the cost and demand functions.

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¹² See, for example, [Saporiti and Coloma \(2010\)](#) which establishes an interesting connection between subadditivity of the cost function and the existence of Bertrand equilibrium. Moreover, [Dastidar \(2011\)](#) has demonstrated that in market with symmetric sellers, subadditive costs and equal sharing at price ties, there is no Bertrand equilibrium in either pure or mixed strategies.