Analysis of vibration suppression of master structure in nonlinear systems using nonlinear delayed absorber

Yueli Chen · Kwok-wai Chung · Jian Xu · Yixia Sun

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Abstract In this paper, a nonlinear delayed absorber is proposed by a delayed feedback loop and is utilized to absorb or suppress the vibration of a two-degree-of-freedom nonlinear system when the primary resonance and the 1:1 internal resonance occur simultaneously. To explain analytically mechanism of performance of the absorber, an integral equation method is provided to obtain the second order approximation and the amplitude equations. As a result, the feedback gain and time delay which make the amplitude of the main system equal to zero can be derived analytically. Optimal working conditions of the system are extracted from the time delay-response curves, the force-response curves and the frequency-response curves. In the illustrative example, the appropriate choice of feedback gains and time delays can reduce the amplitude of the main system by more than 98 % in comparison with the amplitude with no time delay. All analytical predictions are in excellent agreement with the numerical simulations.

Keywords Delayed differential equation · Nonlinear vibration · Nonlinear delayed absorber · Nonlinear state feedback · Integral equation method

1 Introduction

From the viewpoint of engineering, the active control of mechanical and structural vibrations is superior to the passive control in many aspects. However, the presence of unavoidable time delays in controllers and actuators may induce complex dynamic behavior such as undesirable bifurcations, large-amplitude vibrations, quasi-periodic motions and chaotic behavior etc. It limits the performance of the active control.

Over the past few decades, the controlled systems using time delay have received much attention. Among the previous works, the stability which has arisen from delay has attracted special interest. For example, the Hopf bifurcation of an autonomous difference-differential equation was investigated in [1]. In [2], the case of multi-delayed differential equations was extended from the former work [1] by Belair and Campbell. Stépán and Haller [3] investigated the quasi-periodic oscillations of a delayed robot system and demonstrated that a codimension two Hopf bifurcation occurred for an infinite set of parameter values. Reddy et al. [4] demonstrated that time delay can induce amplitude death of the coupled limit cycle oscillators. The dynamics of two coupled van der Pol oscillators with delay coupling was investigated by Wirkus and Rand [5]. They focused on the bifurcation accompanying the change in number and stability of solutions. In [6], we studied a van der Pol-Duffing oscillator with time delayed position feedback and found two routes to chaos, namely period-doubling cascade and torus breaking.

Time delayed feedback control has been employed to suppress the dangerous vibrations of nonlinear systems. Hu et al. [7] investigated the primary resonance and the 1/3 sub-harmonic resonance of a Duffing oscillator under linear state feedback control with a time delay. They demonstrated that the appropriate choice of the feedback gains and the time

Y. Chen · J. Xu · Y. Sun
School of Aerospace Engineering and Applied Mechanics,
Tongji University, Shanghai 200092, People's Republic of China
e-mail: chenyeuli1986@126.com

J. Xu
e-mail: xujian@tongji.edu.cn

K. Chung (✉)
Department of Mathematics, City University of Hong Kong,
Kowloon, Hong Kong, People's Republic of China
e-mail: makchung@cityu.edu.hk

delay can enhance the performance of vibration suppression. Maccari [8] studied the parametric resonance of the van der Pol oscillator with a time delay state feedback and found that proper choices for the feedback gains and time delay can exclude the possibility of quasi-periodic motion and reduce the amplitude peak of the parametric resonance. Siewe et al. [9] investigated the principal parametric resonance of a Rayleigh–Duffing oscillator with time-delayed feedback position and linear velocity terms. It was shown that the proper choice of time delay can broaden the stable region of the non-trivial steady-state solutions and enhance the control performance. El-Gohary and El-Ganaini [10] considered the vibration suppression of a dynamical system to multi-parametric excitations via time-delay absorber. They demonstrated that time delay absorber is effective like ordinary one within a specified range of time delay, which is an important factor in selecting the absorber. Leung et al. [11] studied the steady state bifurcation of a periodically excited system under three kinds of delayed feedback controls. They indicated that different feedback gains not only change the bifurcating curves but also can eliminate the saddle-node bifurcation under resonant response. Zhao and Xu [12] investigated the effects of linear delayed feedback control on a nonlinear vibration absorber system. It was shown that the feedback gains and the time delays not only suppress the vibration but also induce complex behaviors, such as quasi-periodic motions and chaotic motions.

In this paper, we investigate a two-degree-of-freedom system with external excitation. Here, a called nonlinear delayed absorber is proposed to suppress the vibration of the main system. The integral equation method, which was introduced by Schmidt and Tondl [14] for the case without the time delay, is extended to obtain the second order approximation and the amplitude equations. For some parametrically excited and periodically forced systems, Chen and Xu [15] demonstrated that the accuracy of the analytical solutions obtained from the integral equation method is much higher than that from the method of multiple scales. The feedback gain and time delay which make the amplitude of the main system equal to zero can be derived analytically. Optimal working conditions of the system are extracted from the time delay-response curves, the force-response curves and the frequency-response curves. The appropriate choice of the feedback gain and time delay can suppress the vibration of the main system drastically. Numerical simulations are presented to validate the analytical predictions. It is found that they are in good agreement.

2 Mathematical modeling

Although the effects of time delayed feedback control on the vibration suppression of various nonlinear systems have been widely studied, most papers are concerned with the linear case [7–10, 12, 13, 16]. However, the works on nonlinear time

Fig. 1 The model describing the nonlinear state feedback control system

delayed feedback control are relatively rare and they focused on the single degree of freedom [11, 17]. It is well known that the appropriate choice of feedback gains and time delays can enhance the performance of vibration reduction. Motivated by this advantage, we investigate the vibration suppression by a called nonlinear delayed absorber as shown in Fig. 1.

In Fig. 1, the system consists of a slave mass m_1 and the main system m_2 . The parameters c_1 and c_2 represent the damping coefficients of the controller and the main system, respectively. The spring f_1 is assumed to be linear with the stiffness k_1 while the spring f_2 is considered to be nonlinear with force–displacement ($f_2 - x$) relationship given as $f_2 = k_2x + k_3x^3$. The main system is excited by the periodic excitation $f_3(t) = f \sin(\omega t)$. The nonlinear delayed absorber is composed of a nonlinear delayed feedback loop or control $gx_1^3(t - \tau)$ and the slave mass m_1 . Such control can be realized by a servo motor and a charge amplifier. The servo motor is activated by the feedback signals derived from measurement of the slave system. The signals are coded to be any desired signals (including nonlinear signals) and actively to drive actions of a controlled beams in an actively delayed format. The controlled beam is connected to the main system m_2 at one end and to the slave mass m_1 at another end. The charge amplifier is used to enlarge the feedback signals forward to the servo motor and then the gain can be obtained. Thus, the nonlinear delayed absorber is coupled into the system under consideration. The governing equations of the structure shown in Fig. 1 are given by

$$\begin{aligned} m_1 \ddot{x}_1 + k_1(x_1 - x_2) + c_1(\dot{x}_1 - \dot{x}_2) + gx_1^3(t - \tau) &= 0, \\ m_2 \ddot{x}_2 + k_2x_2 + c_2\dot{x}_2 + k_1(x_2 - x_1) + c_1(\dot{x}_2 - \dot{x}_1) &+ k_3x_2^3 - gx_1^3(t - \tau) = f \sin(\omega t). \end{aligned} \quad (1)$$

By introducing the following dimensionless time and new parameters

$$\begin{aligned} t^* &= \frac{\omega}{\Omega} t, \quad \tau^* = \frac{\omega}{\Omega} \tau, \quad y_1 = \frac{x_1}{x_c}, \quad y_2 = \frac{x_2}{x_c}, \quad \omega_1^2 = \frac{k_1 \Omega^2}{m_1 \omega^2}, \\ \omega_2^2 &= \frac{k_2 \Omega^2}{m_2 \omega^2}, \quad \zeta_1 = \frac{c_1 \Omega}{m_1 \omega}, \quad \zeta_2 = \frac{c_2 \Omega}{m_2 \omega}, \quad \alpha_1 = \frac{k_3 \Omega^2 x_c^2}{m_2 \omega^2}, \\ \gamma &= \frac{m_1}{m_2}, \quad g^* = \frac{g \Omega^2 x_c^2}{m_1 \omega^2}, \quad F = \frac{f \Omega^2}{m_2 \omega^2 x_c}, \end{aligned} \quad (2)$$

Equation (1) can be written as

$$\begin{aligned} \ddot{y}_1 + \omega_1^2(y_1 - y_2) + \zeta_1(\dot{y}_1 - \dot{y}_2) + g^*y_1^3(t - \tau) &= 0, \\ \ddot{y}_2 + \omega_2^2y_2 + \zeta_2\dot{y}_2 + \gamma\omega_1^2(y_2 - y_1) + \gamma\zeta_1(\dot{y}_2 - \dot{y}_1) &+ \alpha_1y_2^3 - \gamma g^*y_1^3(t - \tau) = F \sin(\Omega t). \end{aligned} \quad (3)$$

Here, for simplicity, we drop the “*” of t^* and τ^* .

3 The integral equation method

Vibrations of various kinds with time delay can be governed by delay differential equations of the form

$$\begin{aligned} \ddot{x}_i + \lambda_i x_i &= \phi_i(x_i, \dot{x}_i, x_{i\tau}, \dot{x}_{i\tau}, x_j, \dot{x}_j, x_{j\tau}, \dot{x}_{j\tau}; \varepsilon_p, t) \\ &= \phi_i[t], i, j = 1, 2, \dots \end{aligned} \quad (4)$$

where $x_{i\tau} = x_i(t - \tau)$ and ε_p is a parameter.

Theorem 1 Every periodic solution of the differential equation (4) is a solution of the integro-differential equation (see [14])

$$x_i(t) = \int_0^{2\pi} G_i[t, \sigma] \phi_i[\sigma] d\sigma + \delta_{\lambda_i}^{n_i^2} (r_i \cos n_i t + s_i \sin n_i t) \quad (5)$$

where $G_i[t, \sigma] = \frac{1}{\pi} (\frac{1}{2\lambda_i} + \sum_{j=1}^{\infty} \frac{\cos j(t-\sigma)}{\vartheta \lambda_i - j^2})$ is the corresponding Generalized Green's function.

And if $\lambda_i = n_i^2$ (n_i being an integer) (the resonance case), the parameters r_i, s_i are to be determined by the periodicity equations (see [14])

$$\begin{aligned} r_i &= \frac{1}{\pi} \int_0^{2\pi} x_i(t) \cos n_i t dt \\ s_i &= \frac{1}{\pi} \int_0^{2\pi} x_i(t) \sin n_i t dt \end{aligned} \quad (6)$$

which are equivalent to

$$\int_0^{2\pi} \phi_i[t] \cos n_i t dt = \int_0^{2\pi} \phi_i[t] \sin n_i t dt = 0 \quad (7)$$

The solutions of (5) can be found by successive approximations of $x_{ik}(t)$, $k = 1, 2, 3, \dots$, which are given by

$$\begin{aligned} x_{ik}(t) &= \int_0^{2\pi} G_i[t, \sigma] \phi_{i,k-1}[\sigma] d\sigma \\ &+ \delta_{\lambda_i}^{n_i^2} (r_i \cos nt + s_i \sin nt), \end{aligned} \quad (8)$$

where

$$\begin{aligned} \phi_{i0}[t] &= \phi_i[0, 0, 0, 0; \varepsilon_p, t] \\ \phi_{ik}[t] &= \phi_i[x_{ik}(t), \dot{x}_{ik}(t), x_{ik}(t - \tau), \dot{x}_{ik}(t - \tau), \\ &x_{jk}(t), \dot{x}_{jk}(t), x_{jk}(t - \tau), \dot{x}_{jk}(t - \tau); \varepsilon_p, t] \\ &k = 1, 2, 3, \dots \end{aligned} \quad (9)$$

The integral equation method introduced by Schmidt and Tondl [14] has some advantages although it is less known than the other perturbation methods. In our previous work [15], we demonstrated that the accuracy of analytical solutions obtained from the integral equation method is much higher than that from the method of multiple scales in most cases. In [14], the authors showed that the use of small parameters could lead to solutions of every degree of accuracy and the mechanism of finding the successive approximations was simple and clear.

4 Analytical solutions

To analyze the primary resonance and 1:1 internal resonance for which $\Omega \approx \omega_2$ and $\omega_1 \approx \omega_2$, two detuning parameters σ_1, σ_2 are introduced as follows

$$\omega_1 = \omega_2 + \sigma_1, \quad \Omega = \omega_2 + \sigma_2. \quad (10)$$

We introduce a dimensionless time again by

$$t_1 = \Omega t. \quad (11)$$

The original Eq. (3) can be transformed into the following form:

$$\begin{aligned} y_1'' + y_1 &= \frac{1}{\Omega^2} [-(2(\sigma_1 - \sigma_2)\Omega + (\sigma_1 - \sigma_2)^2)y_1 \\ &+ \omega_1^2 y_2 - \zeta_1 \Omega (y_1' - y_2') - g^* y_1^3(t - \tau)], \\ y_2'' + y_2 &= \frac{1}{\Omega^2} [-(2\Omega\sigma_2 + \sigma_2^2)y_2 - \zeta_2 \Omega y_2' \\ &- \gamma\omega_1^2(y_2 - y_1) - \gamma\zeta_1 \Omega (y_2' - y_1') - \alpha_1 y_2^3 \\ &+ \gamma g^* y_1^3(t - \tau) + F \sin t], \end{aligned} \quad (12)$$

where ' denotes $\frac{d}{dt_1}$. For simplicity, we also replace t_1 by t in (12).

For this system, $\lambda_i = 1$ and the Generalized Green's functions become

$$G_i[t, \sigma] = \frac{1}{\pi} \left(\frac{1}{2} - \cos(t - \sigma) + \sum_{j=2}^{\infty} \frac{\cos(j(t - \sigma))}{1 - j^2} \right), \quad i = 1, 2.$$

Since $\phi_{10}[\sigma] = 0, \phi_{20}[\sigma] = F \sin \sigma$, we have from Eq. (8)

$$y_{11} = r_1 \cos t + s_1 \sin t, \quad (13)$$

$$y_{21} = r_2 \cos t + s_2 \sin t. \quad (14)$$

Substituting Eqs. (13) and (14) into Eq. (8), we obtain the second approximation as

$$\begin{aligned}
y_{12} &= \int_0^{2\pi} \frac{1}{\pi} \left(\frac{1}{2} - \cos(t - \sigma) + \sum_{j=2}^{\infty} \frac{\cos(j(t - \sigma))}{1 - j^2} \right) \frac{1}{\Omega^2} (\\
&\quad - (2\Omega(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_2)^2) y_{11}(\sigma) + \omega_1^2 y_{21}(\sigma) \\
&\quad + \zeta_1 \Omega (y'_{21} - y'_{11}) - g^* y_{11}(\sigma - \tau)^3) d\sigma + r_1 \cos t \\
&\quad + s_1 \sin t \\
&= \left(s_1 + \frac{\sigma_1^2 s_1}{\Omega^2} - \frac{2s_1 \sigma_1 \sigma_2}{\Omega^2} + \frac{s_1 \sigma_2^2}{\Omega^2} - \frac{r_1 \zeta_1}{\Omega} + \frac{r_2 \zeta_1}{\Omega} \right. \\
&\quad + \frac{2s_1 \sigma_1}{\Omega} - \frac{2s_1 \sigma_2}{\Omega} - \frac{s_2 \omega_1^2}{\Omega^2} + \frac{3g^* r_1^2 s_1 \cos \tau}{4\Omega^2} + \frac{3g^* s_1^3 \cos \tau}{4\Omega^2} \\
&\quad + \frac{3g^* r_1^3 \sin \tau}{4\Omega^2} + \frac{3g^* r_1 s_1^2 \sin \tau}{4\Omega^2} \left. \right) \sin t + \left(r_1 + \frac{r_1 \sigma_1^2}{\Omega^2} \right. \\
&\quad - \frac{2r_1 \sigma_1 \sigma_2}{\Omega^2} + \frac{r_1 \sigma_2^2}{\Omega^2} + \frac{s_1 \zeta_1}{\Omega} - \frac{s_2 \zeta_1}{\Omega} + \frac{2r_1 \sigma_1}{\Omega} - \frac{2r_1 \sigma_2}{\Omega} \\
&\quad - \frac{r_2 \omega_1^2}{\Omega^2} + \frac{3g^* r_1^3 \cos \tau}{4\Omega^2} + \frac{3g^* r_1 s_1^2 \cos \tau}{4\Omega^2} - \frac{3g^* r_1 s_1 \sin \tau}{4\Omega^2} \\
&\quad - \frac{3g^* s_1^3 \sin \tau}{4\Omega^2} \left. \right) \cos t + \left(\frac{3g^* r_1^2 s_1 \cos(3\tau)}{32\Omega^2} - \frac{g^* s_1^3 \cos(3\tau)}{32\Omega^2} \right. \\
&\quad - \frac{3g^* r_1 s_1^2 \sin(3\tau)}{32\Omega^2} + \frac{g^* r_1^3 \sin(3\tau)}{32\Omega^2} \left. \right) \sin(3t) + \left(\frac{g^* r_1^3 \cos(3\tau)}{32\Omega^2} \right. \\
&\quad - \frac{3g^* r_1 s_1^2 \cos(3\tau)}{32\Omega^2} - \frac{3g^* r_1 s_1 \sin(3\tau)}{32\Omega^2} \\
&\quad + \left. \frac{g^* s_1^3 \sin(3\tau)}{32\Omega^2} \right) \cos(3t),
\end{aligned} \quad (15)$$

$$\begin{aligned}
y_{22} &= \int_0^{2\pi} \frac{1}{\pi} \left(\frac{1}{2} - \cos(t - \sigma) + \sum_{j=2}^{\infty} \frac{\cos(j(t - \sigma))}{1 - j^2} \right) \frac{1}{\Omega^2} ((2\Omega\sigma_2 \\
&\quad - \sigma_2^2) y_{21} - \zeta_2 \Omega y'_{21} - \gamma \omega_1^2 (y_{21} - y_{11}) \\
&\quad + \gamma \zeta_1 \Omega (y'_{11} - y'_{21}) - \alpha_1 y_{21}^2 + \gamma g^* y_{11}(\sigma - \tau)^3 \\
&\quad + F \sin(\sigma)) d\sigma + r_2 \cos t \\
&\quad + s_2 \sin t \\
&= \left(s_2 - \frac{F}{\Omega^2} + \frac{3r_2^2 s_2 \alpha_1}{4\Omega^2} + \frac{3s_2^3 \alpha_1}{4\Omega^2} + \frac{s_2 \sigma_2^2}{\Omega^2} + \frac{\gamma r_1 \zeta_1}{\Omega} - \frac{\gamma r_2 \zeta_1}{\Omega} \right. \\
&\quad - \frac{r_2 \zeta_2}{\Omega} - \frac{2s_2 \sigma_2}{\Omega} - \frac{\gamma s_1 \omega_1^2}{\Omega^2} - \frac{3\gamma g^* r_1^2 s_1 \cos \tau}{4\Omega^2} - \frac{3g^* s_1^3 \gamma \cos \tau}{4\Omega^2} \\
&\quad + \frac{s_2 \gamma \omega_1^2}{\Omega^2} - \frac{3g^* r_1 s_1^2 \gamma \sin \tau}{4\Omega^2} - \frac{3g^* r_1^3 \gamma \sin \tau}{4\Omega^2} \left. \right) \sin t + \left(r_2 \right. \\
&\quad + \frac{3r_2^3 \alpha_1}{4\Omega^2} + \frac{3r_2 s_2^2 \alpha_1}{4\Omega^2} + \frac{r_2 \sigma_2^2}{\Omega^2} - \frac{s_1 \gamma \zeta_1}{\Omega} + \frac{s_2 \gamma \zeta_1}{\Omega} + \frac{s_2 \zeta_2}{\Omega} \\
&\quad - \frac{2r_2 \sigma_2}{\Omega} - \frac{r_1 \gamma \omega_1^2}{\Omega^2} + \frac{r_2 \gamma \omega_1^2}{\Omega^2} - \frac{3g^* r_1 s_1^2 \gamma \cos \tau}{4\Omega^2} - \frac{3g^* r_1^3 \gamma \cos \tau}{4\Omega^2} \\
&\quad + \frac{3g^* r_1^2 s_1 \gamma \sin \tau}{4\Omega^2} + \frac{3g^* s_1^3 \gamma \sin \tau}{4\Omega^2} \left. \right) \cos t + \left(\frac{3r_2^2 s_2 \alpha_1}{32\Omega^2} - \frac{s_2^3 \alpha_1}{32\Omega^2} \right. \\
&\quad - \frac{3g^* r_1^2 s_1 \gamma \cos(3\tau)}{32\Omega^2} + \frac{g^* s_1^3 \gamma \cos(3\tau)}{32\Omega^2} - \frac{g^* r_1^3 \gamma \sin(3\tau)}{32\Omega^2} \\
&\quad + \frac{3g^* r_1 s_1^2 \gamma \sin(3\tau)}{32\Omega^2} \left. \right) \sin(3t) + \left(\frac{r_2^3 \alpha_1}{32\Omega^2} - \frac{3r_2 s_2^2 \alpha_1}{32\Omega^2} \right. \\
&\quad - \frac{g^* r_1^3 \gamma \cos(3\tau)}{32\Omega^2} + \frac{3g^* r_1 s_1^2 \gamma \cos(3\tau)}{32\Omega^2} + \frac{3g^* r_2^2 s_1 \gamma \sin(3\tau)}{32\Omega^2} \\
&\quad - \left. \frac{g^* s_1^3 \gamma \sin(3\tau)}{32\Omega^2} \right) \cos(3t).
\end{aligned} \quad (16)$$

It follows from the first harmonics of Eqs. (15) and (16) that

$$\begin{aligned}
&\frac{r_1 \sigma_1^2}{\Omega^2} - \frac{2r_1 \sigma_1 \sigma_2}{\Omega^2} + \frac{r_1 \sigma_2^2}{\Omega^2} + \frac{s_1 \zeta_1}{\Omega} - \frac{s_2 \zeta_1}{\Omega} + \frac{2r_1 \sigma_1}{\Omega} \\
&- \frac{2r_1 \sigma_2}{\Omega} + \frac{3g^* r_1^3 \cos(\Omega \tau)}{4\Omega^2} - \frac{r_2 \omega_1^2}{\Omega^2} + \frac{3g^* r_1 s_1^2 \cos(\Omega \tau)}{4\Omega^2} \\
&- \frac{3g^* r_1^2 s_1 \sin(\Omega \tau)}{4\Omega^2} - \frac{3g^* s_1^3 \sin(\Omega \tau)}{4\Omega^2} = 0,
\end{aligned} \quad (17)$$

$$\begin{aligned}
&\frac{s_1 \sigma_1^2}{\Omega^2} - \frac{2s_1 \sigma_1 \sigma_2}{\Omega^2} + \frac{s_1 \sigma_2^2}{\Omega^2} - \frac{r_1 \zeta_1}{\Omega} + \frac{r_2 \zeta_1}{\Omega} + \frac{2s_1 \sigma_1}{\Omega} \\
&- \frac{2s_1 \sigma_2}{\Omega} + \frac{3g^* r_1^2 s_1 \cos(\Omega \tau)}{4\Omega^2} - \frac{s_2 \omega_1^2}{\Omega^2} + \frac{3g^* s_1^3 \cos(\Omega \tau)}{4\Omega^2} \\
&+ \frac{3g^* r_1^3 \sin(\Omega \tau)}{4\Omega^2} + \frac{3g^* r_1 s_1^2 \sin(\Omega \tau)}{4\Omega^2} = 0,
\end{aligned} \quad (18)$$

$$\begin{aligned}
&\frac{3r_2^3 \alpha_1}{4\Omega^2} + \frac{3r_2 s_2^2 \alpha_1}{4\Omega^2} + \frac{r_2 \sigma_2^2}{\Omega^2} - \frac{s_1 \gamma \zeta_1}{\Omega} + \frac{s_2 \gamma \zeta_1}{\Omega} + \frac{s_2 \zeta_2}{\Omega} \\
&- \frac{2r_2 \sigma_2}{\Omega} - \frac{3g^* r_1^3 \gamma \cos(\Omega \tau)}{4\Omega^2} + \frac{r_2 \gamma \omega_1^2}{\Omega^2} - \frac{3g^* r_1 s_1^2 \gamma \cos(\Omega \tau)}{4\Omega^2} \\
&- \frac{r_1 \gamma \omega_1^2}{\Omega^2} + \frac{3g^* r_1^2 s_1 \gamma \sin(\Omega \tau)}{4\Omega^2} + \frac{3g^* s_1^3 \gamma \sin(\Omega \tau)}{4\Omega^2} = 0,
\end{aligned} \quad (19)$$

$$\begin{aligned}
&- \frac{F}{\Omega^2} + \frac{3r_2^2 s_2 \alpha_1}{4\Omega^2} + \frac{3s_2^3 \alpha_1}{4\Omega^2} + \frac{s_2 \sigma_2^2}{\Omega^2} + \frac{r_1 \gamma \zeta_1}{\Omega} - \frac{r_2 \gamma \zeta_1}{\Omega} \\
&- \frac{r_2 \zeta_2}{\Omega} - \frac{2s_2 \sigma_2}{\Omega} - \frac{s_1 \gamma \omega_1^2}{\Omega^2} + \frac{s_2 \gamma \omega_1^2}{\Omega^2} - \frac{3g^* r_1^2 s_1 \gamma \cos(\Omega \tau)}{4\Omega^2} \\
&- \frac{3g^* r_1^3 \gamma \cos(\Omega \tau)}{4\Omega^2} - \frac{3g^* r_1 s_1^2 \gamma \sin(\Omega \tau)}{4\Omega^2} - \frac{3g^* r_1^3 \gamma \sin(\Omega \tau)}{4\Omega^2} = 0.
\end{aligned} \quad (20)$$

For convenience to find the appropriate feedback gain g^* and time delay τ which make the amplitude $A_2 = \sqrt{r_2^2 + s_2^2}$ of the main system reach a minimum value, we introduce the following abbreviations

$$\begin{aligned}
A_1^2 &= r_1^2 + s_1^2, \quad A_2^2 = r_2^2 + s_2^2, \quad g_1 = g^* A_1^2, \\
\theta &= \Omega \tau, \quad \sigma = \sigma_1 - \sigma_2, \quad a_1 = 2\sigma \Omega + \sigma^2 + \frac{3}{4} g_1 \cos \theta, \\
a_2 &= \Omega \zeta_1 - \frac{3}{4} g_1 \sin \theta, \quad a_3 = -\omega_1^2, \quad a_4 = -\Omega \zeta_1, \\
b_1 &= \gamma(\sigma^2 + 2\Omega \sigma - \omega_1^2), \quad b_3 = \frac{3}{4} \alpha_1 A_2^2 + \sigma_2^2 - 2\Omega \sigma_2, \\
b_4 &= \Omega \zeta_2.
\end{aligned} \quad (21)$$

The amplitude equations (17)–(20) can be simplified as

$$a_1 r_1 + a_2 s_1 + a_3 r_2 + a_4 s_2 = 0, \quad (22)$$

$$-a_2 r_1 + a_1 s_1 - a_4 r_2 + a_3 s_2 = 0, \quad (23)$$

$$b_1 r_1 + b_3 r_2 + b_4 s_2 = 0, \quad (24)$$

$$b_1 s_1 - b_4 r_2 + b_3 s_2 = F. \quad (25)$$

Solving Eqs. (22)–(25), we obtain two equations in the amplitudes A_1 and A_2 as

$$f_1(g_1, \theta, A_1, A_2) := (a_1^2 + a_2^2) A_1^2 - (a_3^2 + a_4^2) A_2^2 = 0, \quad (26)$$

$$\begin{aligned}
f_2(g_1, \theta, A_1, A_2) &:= (a_3^2 + a_4^2) F^2 - ((a_3^2 + a_4^2) b_1^2 \\
&\quad - 2a_4 b_1 (a_2 b_3 + a_1 b_4) + 2a_3 b_1 (a_2 b_4 - a_1 b_3) \\
&\quad + (a_1^2 + a_2^2) (b_3^2 + b_4^2)) A_1^2 = 0.
\end{aligned} \quad (27)$$

Theorem 2 The amplitude of the main system A_2 vanishes when the feedback gain, the time delay and the amplitude of the controller are given by

$$\begin{cases} g^* = \frac{4b_1^2}{3F^2} \sqrt{(2\sigma\Omega + \sigma^2)^2 + (\Omega\zeta_1)^2} \\ \tau = \frac{1}{\Omega} \left[\arctan \left(-\frac{\Omega\zeta_1}{2\sigma\Omega + \sigma^2} \right) + 2k\pi \right], k = 0, 1, 2, \dots \end{cases} \quad (28a)$$

or

$$\begin{cases} g^* = -\frac{4b_1^2}{3F^2} \sqrt{(2\sigma\Omega + \sigma^2)^2 + (\Omega\zeta_1)^2} \\ \tau = \frac{1}{\Omega} \left[\arctan \left(-\frac{\Omega\zeta_1}{2\sigma\Omega + \sigma^2} \right) + (2k+1)\pi \right], k = 0, 1, 2, \dots \end{cases} \quad (28b)$$

and

$$A_1 = \frac{F}{|b_1|}. \quad (28c)$$

Proof Since A_1 and F_1 are nonzero, we observe from Eqs. (22) to (25) that $A_2 = 0$ (i.e. $r_2 = s_2 = 0$) is equivalent to $r_1 = a_1 = a_2 = 0$.

From the abbreviation (21), we have

$$\begin{cases} 2\sigma\Omega + \sigma^2 + \frac{3}{4}g_1 \cos \theta = 0, \\ \Omega\zeta_1 - \frac{3}{4}g_1 \sin \theta = 0. \end{cases} \quad (29)$$

Since $a_1 = a_2 = 0$, we have from Eq. (27)

$$A_1^2 = \frac{F^2}{b_1^2}.$$

Then, Eqs. (28a) and (28b) follow from Eq. (29), the above equation and the fact that $g^* = \frac{g_1}{A_1^2}$. $\square$

In the following sections, we investigate the optimal conditions of vibration reduction. Here, g^* and τ are considered as variable parameters and other parameters are fixed at $\omega_1 = 1, \omega_2 = 1, \Omega = 1, \gamma = 0.1, \zeta_1 = 0.1, \zeta_2 = 0.1, F = 0.1, \alpha_1 = -0.2$, unless otherwise specified.

Figure 2 shows how g^* and τ change typically with respect to Ω when the amplitude of the main system vanishes. For $k \in \{0, 1\}$, we obtain from Eqs. (28a) and (28b) four optimal solutions for vibration reduction and the amplitude of the main system can be reduced to almost zero:

$$(g^*, \tau, A_2) \in \{(0.133, 1.57, 0), (0.133, 7.85, 0), (-0.133, 4.71, 0), (-0.133, 11, 0)\}.$$

5 Stability analysis of Eq. (3)

To investigate the stability of the vibration described by Eq. (3), we use the linear variational equation of Eq. (12)

Fig. 2 The relation between g^* , τ and Ω when the amplitude of the main system vanishes. (Red line for the positive feedback control while blue line for the negative feedback control). (Color figure online)

$$\begin{aligned} z_1'' + z_1 &= \frac{1}{\Omega^2} [-(2(\sigma_1 - \sigma_2)\Omega + (\sigma_1 - \sigma_2)^2)z_1 + \omega_1^2 z_2 \\ &\quad - \zeta_1 \Omega (z_1' - z_2') - 3g^* y_{12}^2 (t - \tau) z_1 (t - \tau)], \\ z_2'' + z_2 &= \frac{1}{\Omega^2} [-(2\Omega\sigma_2 + \sigma_2^2)z_2 - \zeta_2 \Omega z_2' \\ &\quad - \gamma \omega_1^2 (z_2 - z_1) - \gamma \zeta_1 \Omega (z_2' - z_1') - 3\alpha_1 y_{22}^2 z_2 \\ &\quad + 3\gamma g^* y_{12}^2 (t - \tau) z_1 (t - \tau)], \end{aligned} \quad (30)$$

where y_{12}, y_{22} are the second approximations. Corresponding to the Floquet theory, the solution of Eq. (30) can be written in the following form

$$z_i(t) = \exp(\rho t) Z_i(t), \quad i = 1, 2. \quad (31)$$

and Eq. (30) can change to

$$\begin{aligned} Z_1'' + Z_1 &= \left(-\rho^2 - \frac{\sigma_1^2}{\Omega^2} + \frac{2\sigma_1\sigma_2}{\Omega^2} - \frac{\sigma_2^2}{\Omega^2} - \frac{\zeta_1\rho}{\Omega} - \frac{2\sigma_1}{\Omega} \right. \\ &\quad \left. + \frac{2\sigma_2}{\Omega} \right) Z_1 - \frac{3\exp(-\rho\tau)g^*}{\Omega^2} (a_0 + a_2 \cos(2t) \\ &\quad + a_4 \cos(4t) + a_6 \cos(6t) + b_2 \sin(2t) \\ &\quad + b_4 \sin(4t) + b_6 \sin(6t)) Z_1(t - \tau) + \left(\frac{\zeta_1\rho}{\Omega} \right. \\ &\quad \left. + \frac{\omega_1^2}{\Omega^2} \right) Z_2(t) + \left(-2\rho - \frac{\zeta_1}{\Omega} \right) Z_1'(t) + \frac{\zeta_1\rho}{\Omega} Z_2'(t), \end{aligned} \quad (32)$$

$$\begin{aligned}
Z_2'' + Z_2 = & \left(\frac{\gamma \zeta_1 \rho}{\Omega} + \frac{\gamma \omega_1^2}{\Omega^2} \right) Z_1(t) + \frac{3 \exp(-\rho \tau) g^* \gamma}{\Omega} \left(a_0 \right. \\
& + a_2 \cos(2t) + a_4 \cos(4t) + a_6 \cos(6t) \\
& + b_2 \sin(2t) + b_4 \sin(4t) + b_6 \sin(6t) \Big) Z_1(t - \tau) \\
& + \left(-\rho^2 - \frac{3c_0 \alpha_1}{\Omega^2} - \frac{\sigma_2^2}{\Omega^2} - \frac{\gamma \zeta_1 \rho}{\Omega} - \frac{\zeta_2 \rho}{\Omega} + \frac{2\sigma_2}{\Omega} \right. \\
& - \frac{\gamma \omega_1^2}{\Omega^2} - \frac{3c_2 \alpha_1 \cos(2t)}{\Omega^2} - \frac{3c_4 \alpha_1 \cos(4t)}{\Omega^2} \\
& - \frac{3c_6 \alpha_1 \cos(6t)}{\Omega^2} - \frac{3\alpha_1 s_2 \sin(2t)}{\Omega^2} - \frac{3\alpha_1 s_4 \sin(4t)}{\Omega^2} \\
& \left. - \frac{3\alpha_1 s_6 \sin(6t)}{\Omega^2} \right) Z_2(t) + \frac{\gamma \zeta_1}{\Omega} Z_1'(t) + \left(-2\rho \right. \\
& \left. - \frac{\gamma \zeta_1}{\Omega} - \frac{\zeta_2}{\Omega} \right) Z_2'(t). \quad (33)
\end{aligned}$$

The first approximation of Eqs. (32) and (33) can be expressed as

$$\begin{aligned}
Z_1(t) &= p_0 + p_1 \cos t + q_1 \sin t, \\
Z_2(t) &= u_0 + u_1 \cos t + v_1 \sin t. \quad (34)
\end{aligned}$$

Substituting (34) into (32), (33) and equating the coefficients of same harmonic terms, we obtain a set of linear homogeneous algebraic equations governing the coefficients $p_0, p_1, q_1, u_0, u_1, v_1$. Since these coefficients are not all zero, the determinant of the coefficients matrix, which is the so-called Hill's determinant, must be zero. Expanding the determinant, one can get a formula in the following form

$$\begin{aligned}
\rho^{12} + f_1(\exp(-\Omega \rho \tau), \exp(-2\Omega \rho \tau), \exp(-3\Omega \rho \tau)) \rho^{11} \\
+ \dots + f_{11}(\exp(-\Omega \rho \tau), \exp(-2\Omega \rho \tau), \\
\exp(-3\Omega \rho \tau)) \rho + f_{12}(\rho) = 0 \quad (35)
\end{aligned}$$

Equation (35) can be solved numerically. If the real parts of all characteristic exponents are negative, the periodic solution is asymptotically stable. On the other hand, if the real part of at least one characteristic exponent is positive, the periodic solution is unstable. The solution subsequent to the bifurcation depends on the manner in which the eigenvalues cross from the left half-plane to the right (see [18, 19]).

6 Time delay-response curves

In this section, the effect of feedback gain on the amplitude-delay response curve of the main system is investigated. Figure 3 shows the variation of amplitude of periodic motions with time delay obtained by the integral equation method for six different positive feedback gains, i.e. $g^* \in \{0.01, 0.1, 0.133, 0.15, 0.2, 0.5\}$ while Fig. 4 shows that for the negative feedback gains, i.e. $g^* \in \{-0.01, -0.1, -0.133, -0.15, -0.2, -0.5\}$. The amplitude of the main system depends on the feedback gain g^* and the time delay τ . It is obvious to see that the effect of time

Fig. 3 Amplitude-delay response curves of the main system with different positive feedback coefficients g^* . **a** Points stand for unstable solutions, solid line for stable solutions, blue for $g^* = 0.01$, red for $g^* = 0.1$, orange for $g^* = 0.133$, black for $g^* = 0.15$, purple for $g^* = 0.2$, **b** $g^* = 0.5$. (Color figure online)

delay is periodic with periodic $2\pi/\Omega$. When $|g^*| = 0.01$, we notice that the amplitude-delay response curves are always stable. However, the efficiency of τ to suppress the vibration of the main system is inconspicuous. As $|g^*|$ increases, the peak amplitude of the main system increases gradually. The minimum amplitude of the main system decreases first as $|g^*|$ increases from 0.01 to 0.133, and then increases as $|g^*|$ increases further from 0.133 to 0.2. In this process, the stable region of the amplitude-delay curves shrinks. When $0.01 \leq |g^*| \leq 0.2$, there always exists a tunable range of τ to suppress the vibration. However, the effect of vibration suppression becomes inconspicuous when $|g^*| > 0.2$. When $g^* = 0.2$, the periodic solution on the amplitude-delay curve corresponding to $\tau = 7.85$ is unstable and the optimal value τ for vibration suppression is 7. Similarly when $g^* = -0.2$, the periodic solution corresponding to $\tau = 11$ is also unstable and the optimal value of τ becomes 10.6. Therefore, the efficiency for vibration suppression becomes slight when $|g^*| = 0.2$. When $g^* = 0.5$, vibration reduction is impossible as the amplitude (stable) with time delay is not less than that without. When $g^* = -0.5$, the effect of vibration reduction is very slight. When $g^* \geq 0.55$ or $g^* \leq -2.1$,

Fig. 4 Amplitude-delay response curves of the main system with different negative feedback coefficients g^* . **a** Points stand for unstable solutions, *solid line* for stable solutions, *blue* for $g^* = -0.01$, *red* for $g^* = -0.1$, *orange* for $g^* = -0.133$, *black* for $g^* = -0.15$ *purple* for $g^* = -0.2$, **(b)** $g^* = -0.5$. (Color figure online)

the amplitude-delay response becomes fully unstable. For the positive delayed feedback control, it follows from Eqs. (28a) and (28b) that the minimum amplitudes occur when $\tau = 1.57$ and $\tau = 7.85$. However, the amplitude curves are unstable at $\tau = 1.57$. For the negative delayed feedback, the minimum amplitudes occur when $\tau = 4.71$ and $\tau = 11$. Similarly, the periodic solutions are unstable at $\tau = 4.71$.

First, we make a comparison of the analytical results obtained by the integral equation method and the numerical simulations when $g^* = 0.11$ and $g^* = -0.11$ in Fig. 5a, b, respectively. It is observed that the analytical results are in good agreement with the numerical simulations, which confirms that the above analysis is valid.

Figure 6 shows that the maximum percentage of vibration reduction of the main system when g^* vary from -0.24 to 0.28 with an appropriate time delay. The solid blue line denotes the analytical results and the symbol “+” denotes the numerical results. From this figure, we observe that the amplitude of the main system with time delay can be suppressed at different levels. For g^* varying from 0 to 0.153 , the time delay should be fixed at the value of 7.85 . For g^*

Fig. 5 Comparison of amplitude-delay response curves between numerical solutions and analytical solutions obtained by the integral equation method. **a** $g^* = 0.11$, **b** $g^* = -0.11$. Points stand for unstable solutions, *solid line* for stable solutions and “+” for numerical simulation

Fig. 6 The maximum percentage of the vibration reduction of the main system when g^* varies from -0.24 to 0.28 with an appropriate time delay. The *solid line* denotes the analytical results and the “+” denotes the numerical simulation

varying from -0.18 to 0 , the time delay should be fixed at $\tau = 11$. For other range of g^* , the periodic solutions corresponding to $\tau = 7.85$ and $\tau = 11$ become unstable and the time delay should be chosen near to $\tau = 7.85$, $\tau = 1.57$

Fig. 7 Time history of the main system for **a** $g^* = 0.133$, $\tau = 0$, **b** $g^* = 0.133$, $\tau = 7.85$

Fig. 8 Time history of the main system for **a** $g^* = -0.133$, $\tau = 0$, **b** $g^* = -0.133$, $\tau = 11$

(for the positive feedback control) and $\tau = 11$, $\tau = 4.71$ (for the negative feedback control). For $g^* \in [0.154, 0.202]$, the time delay should be chosen near $\tau = 7.85$ while for $g^* \in [0.203, 0.28]$, it should be chosen near $\tau = 1.57$. For $g^* \in [-0.24, -0.181]$, the time delay should be chosen near $\tau = 11$. From Fig. 6, the vibration can be reduced by nearly 100 % when $g^* = 0.133$, $\tau = 7.85$ and $g^* = -0.133$, $\tau = 11$. Figure 7a, b show the time history of the main system for $\tau = 0$ and $\tau = 7.85$, respectively when $g^* = 0.133$. It is seen that the amplitude of the main system can be reduced by 98.53 % from its value without time delay. For $g^* = -0.133$, the efficiency of vibration suppression by time delay amounts to 97.83 % as shown in Fig. 8. Therefore, the analytical prediction is in excellent agreement with the numerical simulations as shown in Figs. 7 and 8. Combining Figs. 3, 4 and 6, we observe that the best performance of vibration reduction can be found when $g^* = 0.133$, $\tau = 7.85$ and $g^* = -0.133$, $\tau = 11$. This result is in excellent agreement with the analysis in Sect. 4.

7 Force-response curves

In Sect. 6, we observe that the optimal vibration reduction occurs when $\tau = 7.85$ for positive feedback gain, and $\tau = 11$ for negative feedback gain. In this section, the time delay $\tau = 7.85$ or $\tau = 11$ is kept fixed.

The effect of varying the feedback gain on the force-response curves of the main system and controller is shown

in Fig. 9. Here, four different feedback gains, i.e. $g^* = \pm 0.1$, $g^* = \pm 0.133$ are considered. From Fig. 9, a tunable range of F to suppress the vibration is obtained when the force amplitude is located in $[0.05, 0.12]$. The best vibration suppression points can be obtained when $F = 0.12$ for $g^* = \pm 0.1$, and $F = 0.1$ for $g^* = \pm 0.133$. Figures 10 and 11 show the time responses of the main system without time delay and with time delay. It can be seen that the amplitude of the main system with appropriate time delay can be reduced by 91 % from its value without time delays when $|g^*| = 0.1$. Figure 9 also shows that the stability zone decreases under the time-delayed feedback. The parameter F should be chosen in the region $[0.05, 0.12]$. In the region $[0.19, 0.205]$ on Fig. 9b, varying F for the force-response curve with $\tau = 0$ leads to the jump phenomenon, which results in the coexistence of two stable solutions and one unstable solution. The jump phenomenon also occurs in the region $[0.176, 0.188]$ in Fig. 9d for $g^* = -0.13$ when $\tau = 0$, such phenomenon is confirmed from the coexistence of stable solutions obtained in these regions as shown in Figs. 12 and 13.

8 Frequency-response curves

In this section, we investigate the influence of the feedback gain and time delay on the frequency-response curves of the main system. In fact, the stability of the motion, the bifurcation points and the position for the minimum amplitude depend on time delay.

Fig. 9 Effect of varying the feedback gain on the force-response curves for
a $g^* = 0.1$, **b** $g^* = -0.1$,
c $g^* = 0.133$, **d** $g^* = -0.133$

Fig. 10 Time history of the main system for
 $F = 0.12$, $g^* = 0.1$.
a $\tau = 0$, **b** $\tau = 7.85$

Fig. 11 Time history of the main system for
 $F = 0.12$, $g^* = -0.1$.
a $\tau = 0$, **b** $\tau = 11$

Figure 14a shows a comparison of frequency-response curve of the main system for $\tau = 0$ and $\tau = 7.85$ when $g^* = 0.13$. We observe that the frequency-response curve is always stable in the frequency interval $[-0.5, 0.5]$ for $\tau = 0$. The minimum steady-state amplitude of the main system occurs for the detuning parameter $\sigma_2 = 0.03$. For $\tau = 7.85$,

the minimum amplitude of the main system occurs when $\sigma_2 = 0$. The frequency-response curve is stable for large negative value of σ_2 . At $\sigma_2 = -0.18$, the curve becomes unstable due to a saddle-node bifurcation. However, the solution becomes stable again when $\sigma_2 = -0.02$. For σ_2 varying in the region $[0.036, 0.075]$, it has multiple solutions and

Fig. 12 Two coexisting stable solutions when $g^* = -0.1$, $\tau = 0$, $F = 0.2$ for different initial conditions given by **a** $y_1(0) = 0.1$, $y_2(0) = 0.2$, $\dot{y}_1(0) = 0.0001$, $\dot{y}_2(0) = 0.0001$; **b** $y_1(0) = 0.0001$, $y_2(0) = 0.0001$, $\dot{y}_1(0) = 0.0001$, $\dot{y}_2(0) = 0.0001$

Fig. 13 Two coexisting stable solutions when $g^* = -0.133$, $\tau = 0$, $F = 0.18$ for different initial conditions given by **a** $y_1(0) = 0.0001$, $y_2(0) = 0.0001$, $\dot{y}_1(0) = 0.0001$, $\dot{y}_2(0) = 0.0001$; **b** $y_1(0) = 0.1$, $y_2(0) = 1$, $\dot{y}_1(0) = 0$, $\dot{y}_2(0) = 0$

Fig. 14 Frequency-response curves of the main system for **a** $g^* = 0.133$, **b** $g^* = -0.133$. Box denotes Hopf bifurcation, “+” denotes numerical simulation for $\tau = 0$ and “*” denotes numerical simulation for $\tau = 7.85$ or $\tau = 11$, black line for the original 1 DoF system

Fig. 15 Quasi-periodic motion of the main system for $g^* = 0.133$, $\tau = 7.85$, $\sigma_2 = 0.1268$ **a** phase plane, **b** Poincaré section

the smallest solution is stable. On the other hand, when σ_2 decreases from 0.5, the frequency-response curve is initially stable and becomes unstable at $\sigma_2 = 0.1271$ due to a Hopf bifurcation. As σ_2 decreases further, a quasi-periodic solution comes into existence at $\sigma_2 = 0.1269$ due to a secondary Hopf

Bifurcation. As shown in Fig. 15, a quasi-periodic solution is obtained by numerical simulation at $\sigma_2 = 0.1268$.

Figure 14b shows a comparison of frequency-response curve of the main system for $\tau = 0$ and $\tau = 11$ when $g^* = -0.133$. For $\tau = 0$, a jump phenomenon occurs in

Fig. 16 Quasi-periodic motion of the main system for $g^* = -0.133$, $\tau = 11$, $\sigma_2 = -0.058$
a phase plane, **b** Poincaré section

Fig. 17 Quasi-periodic motion of the main system for $g^* = -0.133$, $\tau = 11$, $\sigma_2 = 0.101$
a phase plane, **b** Poincaré section

Fig. 18 Effect of varying the positive internal detuning parameter σ_1 on the frequency-response curves of the main system. Blue for $\sigma_1 = 0$, red for $\sigma_1 = 0.1$, orange for $\sigma_1 = 0.2$, purple for $\sigma_1 = 0.3$, black for the original 1 DoF system.
a $(g^*, \tau) = (0.1, 7.85)$,
b $(g^*, \tau) = (-0.1, 11)$,
c $(g^*, \tau) = (0.133, 7.85)$,
d $(g^*, \tau) = (-0.133, 11)$.
 (Color figure online)

the region $[-0.4, -0.25]$, which implies the coexistence of two unstable solutions and one stable solution. The minimum amplitude of the main system occurs when $\sigma_2 = -0.03$. For $\tau = 11$, the jump phenomenon disappears. The frequency-response curve has three stable segments: $\sigma_2 < -0.17$, $\sigma_2 \in$

$[-0.056, 0.08]$ and $\sigma_2 > 0.101$. Hopf bifurcation occurs at $\sigma_2 = -0.056$ and $\sigma_2 = 0.101$ while saddle-node bifurcation at $\sigma_2 = -0.17$ and $\sigma_2 = 0.08$. Three solutions coexist in the region $[0.028, 0.08]$, and the lowest branch is stable. Quasi-periodic solutions are observed in the unstable segments.

Fig. 19 Time history of the main system for

- a** $\sigma_1 = 0.1$, $\sigma_2 = 0.25$,
b $\sigma_1 = 0.25$, $\sigma_2 = 0.25$,
c $\sigma_1 = 0.1$, $\sigma_2 = 0.1$,
d $\sigma_1 = \sigma_2 = 0$

Figures 16 and 17 show such solutions at $\sigma_2 = -0.058$ and $\sigma_2 = 0.101$, respectively.

Based on the frequency-response curves in Fig. 14, we can conclude that time delays can change the position where the minimum amplitude of the main system occurs. It is found that the frequency-response curves for $\tau \neq 0$ are lower than that for $\tau = 0$ near $\sigma_2 = 0$, which means that there exists a tunable range of σ_2 for the vibration suppression near $\sigma_2 = 0$. The time delay may result in the disappearance of the jump phenomenon and the occurrence of Hopf Bifurcations.

Figure 18 shows the effect of varying the positive internal detuning parameter σ_1 on the frequency-response curves for the four cases $(g^*, \tau) \in \{(0.1, 7.85), (-0.1, 11), (0.133, 7.85), (-0.133, 11)\}$ for $\sigma_1 \in \{0, 0.1, 0.2, 0.3\}$. For all these cases, we observe that the minimum amplitude of the main system occurs near $\sigma_1 = \sigma_2$. This result can be explained from the optimal solution obtained from (29). Recall that we choose $\omega_1 = \omega_2 = \Omega = 1$ for our investigation. It follows from (10) and (21) that $\sigma = \sigma_1 - \sigma_2 = 0$. In Fig. 18, since the control parameters (g^*, τ) satisfy (29), the minimum amplitude of the main system occurs when $\sigma = 0$, i.e. $\sigma_1 = \sigma_2$. Therefore, it is important to tune the natural frequency ω_1 of the controller to near the excitation frequency Ω . This results are in excellent agreement with previous work [20]. Figure 19 shows the time history of the main system for the four cases (a) $\sigma_1 = 0.1$, $\sigma_2 = 0.25$, (b) $\sigma_1 = 0.25$, $\sigma_2 = 0.25$, (c) $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, (d) $\sigma_1 = 0$, $\sigma_2 = 0$ when $g^* = 0.1$, $\tau = 7.85$. It is clear that the amplitude of the main system can be much reduced by setting $\sigma_1 = \sigma_2$. Moreover, the best performance of the vibration

reduction occurs when $\sigma_1 = \sigma_2 = 0$ for $g^* = 0.1$, $\tau = 7.85$. The numerical simulations in Fig. 19 are in good agreement with the above analysis from Fig. 18.

9 Conclusions

A nonlinear time delayed feedback controller is designed to reduce the vibration of the main system in a two-degree-of-freedom nonlinear system with external excitation. The integral equation method is applied to derive four amplitude equations of the system when the primary resonance and 1:1 internal resonance occur simultaneously. Then, the stability of the system is analyzed using the Floquet theory to assess the performance of the feedback gain and time delay on vibration suppression.

Although the amplitude equations (17)–(20) look very complicated, analytical expression of the optimal feedback gain g^* and time delay τ can be obtained such that the amplitude of the main system vanishes. Two values of optimal feedback gain are found, which are in opposite sign of the same magnitude ($\pm g^*$). For the system parameters we have considered, excellent vibration suppression can be achieved. Applying the positive optimal feedback gain ($g^* = 0.133$), the amplitude of the main system can be reduced by 98.53 % in comparison with the amplitude without time delay. On the other hand, applying the negative optimal feedback gain ($g^* = -0.133$), the amplitude reduction is 97.83 %. However, if the feedback gain deviates from the optimal value, the

percentage of amplitude reduction decreases rapidly. Therefore, finding the optimal feedback gain and time delay is crucial in the vibration control of this system.

When the time delay is fixed at the optimal value obtained from the integral equation method, it is important to tune the natural frequency ω_1 of the controller to near the external excitation frequency Ω for effective vibration reduction. If ω_1 deviates much from Ω , Hopf and even second Hopf bifurcations may occur which ruins the purpose of vibration control.

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