The forward and the backward full annular rubbing dynamics of a coupled rotor-casing/foundation system

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Abstract In this paper the forward and the backward full annular rubs of a coupled rotor-casing/foundation system, by which the motion of the rotor is coupled with that of the casing/foundation even in the no rubbing state, are investigated analytically. First, the forward synchronous full annular rub solutions are solved analytically after proposing an equivalent form of the resultant contact force and their stability is analyzed to determine the parameter regions where the forward full annular rub occurs. Then, the existence condition and the whirl frequency of the dry friction backward whirl (the backward full annular rub) are derived analytically after considering the key features of the self-excited response. Finally, the examples of application are given to show the influence of the system parameters on the full annular rubs and to discuss the possible engineering implications.

Keywords Coupled rotor-casing/foundation system · Full annular rubs · Forward and backward whirl

List of Symbols

F	Resultant contact force;
$\hat{F}$	Non-dimensional resultant contact force;
M_{fr}	Mass ratio of foundation over rotor;
Y_r, Z_r, Y_f, Z_f	Non-dimensional horizontal and vertical deflection of rotor and foundation;
c_r, c_f	Damping of rotor and foundation;
e	Mass eccentricity of rotor;
k_r, k_f	Stiffness of rotor and foundation;
m_r, m_f	Mass of rotor and foundation;

r_r, r_f	Displacement of the geometric center of rotor and foundation;
Y_r, Z_r, Y_f, Z_f	Non-dimensional horizontal and vertical deflection of rotor and foundation;
t	time;
y_r, z_r, y_f, z_f	Horizontal and vertical deflection of rotor and foundation;
β_{fr}	Stiffness ratio of foundation over rotor, k_f/k_r ;
β_{cr}	Stiffness ratio of contact surface over rotor, k_c/k_r ;
δ	Clearance between rotor and stator on foundation;
λ	Eigenvalues of Jacobian matrix;
ρ_r, ρ_f	Amplitudes of rotor and foundation in rotating coordinate system;
μ	Friction coefficient;
τ	Non-dimensional time;
ω	Rotation speed of rotor;
ω_r	Natural frequency of rotor;
ω_w	Whirl frequency;
Δ	Non-dimensional clearance;
Ω	Non-dimensional rotation speed of rotor, ω/ω_r ;
Ω_{ci}	Non-dimensional natural frequencies of the system, $i = 1, 2$;
Ω_w	Non-dimensional whirl frequency;
ζ_r, ζ_f	Damping ratio of rotor and casing/foundation;

1 Introduction

Rotor-to-stator rubbing is a serious malfunction in the operation of rotating machinery and can seriously degrade the

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performance and even lead to the complete failure of the machines. Many researchers devoted to study the rub-related dynamical phenomena in order to make deep insight into the dynamic characteristics of rotor-to-stator rubbing. In this aspect three simplified rotor/stator models were studied. The first rotor/stator model consists of a Jeffcott rotor and a rigid stator that is rigidly fixed on a rigid foundation [1–3]. The second rotor/stator model still uses a Jeffcott rotor but the stator is modeled as an added stiffness. This model physically describes a rotor in contact with a non-rotating susceptible circumferential stator or a mechanical seal. There are a great number of papers in the literature studying on the rubbing behavior of this model. Various rubbing responses are found through theoretical analyses, numerical simulations and experimental investigations, such as the jump phenomenon [4], the synchronous full annular rubs [5–7], the partial rubs in sub- and super-synchronous whirl [8–10], the partial rubs in quasi-periodic whirl [11, 12], the chaotic motion [13] and dry friction backward whirl [14, 15]. The third rotor/stator model takes the stator as an elastically supported rigid ring with mass, and is also widely used in modeling the retainer bearings [16–20]. Most recently, an extension of the simple rotor/stator model by taking the stator to be an elastically supported *elastic* ring with mass was investigated numerically in [21] and analytically in [22–24]. It was shown that the general model can be reduced to all of the simpler rotor/stator models mentioned above by assigning the corresponding parameters to some specific values.

In all of the three rotor/stator models, the motions of the rotor and the stator become coupled only when the rubbing between them is established. This means that the casing/foundation, on which the rotor sits, is supposed to be rigidly supported and the influence of the motion of the casing/foundation on the dynamics of the rotor has been neglected. Generally, the casing/foundation is supported elastically in the real systems and its motion will couple with the motion of the rotor all the time. In the literature, only few research articles take into account the dynamics of the foun-

dation in studying the rubbing problems. However, Bently [6] found in experiments that the stiffness of foundation support has a strong effect on the occurrence of the backward full annular rub, where backward full annular rub easily occurred on a stiff concrete base while only forward annular rub could appear on a flexible base. Bartha [20] investigated the backward annular rub in a system considering the stator into two nested rings experimentally and showed that a flexible suspending stator will not acts in bring down the backward whirl frequency. A numerical investigation on the rub interaction of a flexible rotor/casing system with three disks under the excitation of rotor imbalance eccentricities and external base motion was reported in [25]. However, the rubbing behavior in the coupled rotor-casing/foundation system is not well understood.

In this paper, the forward and the backward full annular rubs of a coupled rotor-casing/foundation model are studied analytically. The solutions of the corresponding responses are derived and their stability is analyzed in order to investigate the influence of the system parameters on the occurrence of the full annular rubs. In Sect. 2, the mathematical model of the coupled rotor-casing/foundation system is introduced. In Sect. 3, the synchronous forward full annular rub solutions of the rotor-casing/foundation system are solved and their stability is analyzed. In Sect. 4 the existence condition of the backward full annular rub is derived and the characteristics of the whirl frequency are investigated. Examples are given in Sect. 5 to demonstrate the influence of the parameters of the casing/foundation on the full annular rubs. The conclusions of this work are drawn in Sect. 6.

2 The mathematical model

The coupled rotor-casing/foundation model studied in this paper is shown in Fig. 1. The rotor consists of a massless shaft, which has the effective transverse stiffness k_r and rotates at an angular speed ω , and a rigid disk of mass m_r ,

Fig. 1 The schematic plot of the model of the coupled rotor-casing/foundation system, by which the rotor coupled with the casing/foundation at the bearings

which is mounted at the midpoint of the shaft. The rotor possesses a mass eccentricity of e . The unit of casing/foundation is elastically supported isotropically in the vertical and the horizontal directions by springs with the effective stiffness k_f and the dampers with effective damping c_f . An annular stator is concentric with the disk of the rotor and is firmly fixed on the foundation to form the unit of the casing/foundation. The clearance between the rotor and the stator is denoted by δ and the mass of the casing/foundation by m_f . By considering the contact stiffness at the contact surfaces between the rotor and the stator on the casing/foundation, a symmetrical set of fictive springs with isotropic radial stiffness k_c is supposed to be laid in the inner ring of the stator to model the contact stiffness. In this model, the rotor is coupled with the casing/foundation through the bearings. So the motions of the rotor and the stator are coupled even when they are not in contact with each other. This feature is different from the rotor/stator models studied previously.

The equations that govern the motion of the coupled rotor-casing/foundation system can be written in the complex form as:

$$\begin{aligned} m_r \ddot{r}_r + c_r(\dot{r}_r - \dot{r}_f) + k_r(r_r - r_f) + \Theta F &= m_r e \omega^2 e^{j\omega t} \\ m_f \ddot{r}_f + c_f(\dot{r}_f - \dot{r}_r) + k_f(r_f - r_r) - c_r(\dot{r}_r - \dot{r}_f) - k_r(r_r - r_f) \\ &- \Theta F = 0 \\ F &= (1 + j\mu \text{sign}(v_{rel}))k_c(r_r - r_f - \delta \frac{r_r - r_f}{|r_r - r_f|}) \\ v_{rel} &= r_{disk}\omega + |r_r - r_f|\omega_w \end{aligned} \quad (1)$$

where $r_r = y_r + jz_r$ and $r_f = y_f + jz_f$ are respectively the complex deflections, and c_r and c_f the damping coefficients of the rotor and the casing/foundation respectively, F represents the resultant contact force at the contact point. v_{rel} is the relative velocity at the contact point with ω_w being the whirl frequency of the rotor and r_{disk} the radius of disk at the contact point. Θ is the Heaviside function defined as $\Theta = 1$, if $|r_r - r_f| \geq \delta$ and $\Theta = 0$, if $|r_r - r_f| < \delta$.

The model described by Eq. (1) is said to be general in the following three aspects: First, the important parameters in the rotor-to-stator contact, like the dynamics of the stationary parts as well as the friction and the contact stiffness at the contact surfaces are all taken into account; Secondly, the dry friction effect, which is the critical factor to initiate the self-excited dry friction backward whirl as pointed in [15], is also accounted for; Thirdly, the motion coupling between the rotor and the casing/foundation is also included in this model.

Equation (1) can be reformulated in the non-dimensional form:

$$\begin{aligned} \hat{r}_r'' + 2\zeta_r(\hat{r}_r' - \hat{r}_f') + (\hat{r}_r - \hat{r}_f) + \Theta \hat{F} &= \Omega^2 e^{j\Omega\tau} \\ M_{fr} \hat{r}_f'' + 2\bar{\zeta}_f \hat{r}_f' + \beta_{fr} \hat{r}_f - 2\zeta_r(\hat{r}_r' - \hat{r}_f') - (\hat{r}_r - \hat{r}_f) \\ &- \Theta \hat{F} = 0 \end{aligned}$$

$$\begin{aligned} \hat{F} &= (1 + j\mu \text{sign}(V_{rel}))\beta_{cr}(\hat{r}_r - \hat{r}_f - \Delta \frac{\hat{r}_r - \hat{r}_f}{|\hat{r}_r - \hat{r}_f|}) \\ V_{rel} &= R_{disk}\Omega + |\hat{r}_r - \hat{r}_f|\Omega_w \end{aligned} \quad (2)$$

where the apostrophe ' represents the differentiation with respect to the non-dimensional time $\tau = \omega_r t$ in which $\omega_r = \sqrt{k_r/m_r}$ is the natural frequency of the rotor. The other non-dimensional variables are defined as

$$\begin{aligned} \hat{r}_r &= \frac{r_r}{e}, \hat{r}_f = \frac{r_f}{e}, \hat{F} = \frac{F}{ek_r}, \Omega = \frac{\omega}{\omega_r}, \Omega_w = \frac{\omega_w}{\omega_r}, \\ M_{fr} &= \frac{m_f}{m_r}, \beta_{fr} = \frac{k_f}{k_r}, \beta_{cr} = \frac{k_c}{k_r}, \Delta = \frac{\delta}{e}. \end{aligned}$$

ζ_r and ζ_f are the damping ratios of the rotor and the casing/foundation respectively with $\bar{\zeta}_f = \zeta_f \sqrt{M_{fr}\beta_{fr}}$.

By setting the excitation term in the right-hand side of Eq. (2) to zero and letting $\Theta = 0$, we can derive the natural frequencies of the linear coupled rotor-casing/foundation system as

$$\begin{aligned} \Omega_{ci} &= \frac{(\beta_{fr} + M_{fr} + 1) \pm \sqrt{(\beta_{fr} + M_{fr} + 1)^2 - 4M_{fr}\beta_{fr}}}{2M_{fr}}, \\ i &= 1, 2 \end{aligned} \quad (3)$$

3 The synchronous full annular rub

When the rotor-casing/foundation system undertakes a synchronous full annular rub motion, the relative velocity at the contact point is always positive. So the *sign* function in Eqs. (1) and (2) always takes +1. As pointed in [17, 19], there is generally great difficulty in solving the analytical solutions for the rotor/stator systems that account for the dynamics of the stator as well as the friction and the contact stiffness at the contact surfaces. The model studied in this paper as given in Eq. (1) belongs to this category. However, the difficulty can be well released if an equivalent form of the resultant contact force is adopted as was done in [22]. The equivalent form of the resultant contact force in this case is defined as

$$F = (1 + j\mu)k_c \left(r_r - r_f - \delta \frac{1 - j\mu}{\sqrt{1 + \mu^2}} \frac{F}{|F|} \right) \quad (4)$$

For the coupled rotor-casing/foundation system given by Eq. (2) with the resultant contact force in the form of (4), the analytical steady-state periodic solutions, which have constant amplitudes and a frequency equal to the rotation speed of the rotor, are solvable. The solutions are the so-called *synchronous full annular rub solutions*.

The derivation of the solutions will become easier when Eq. (2) is transformed into a rotating coordinate system that rotates at a frequency equal to the rotation speed of the rotor.

To do this the transformations are adopted:

$$\hat{r}_r = \rho_r e^{j\Omega\tau}, \quad \hat{r}_f = \rho_f e^{j\Omega\tau}, \quad \hat{F} = \Phi e^{j\Omega\tau}$$

By substituting above formulae into Eq. (2), the governing equations in the rotating coordinates yield as:

$$\begin{aligned} \rho_r'' + 2(\zeta_r + j\Omega)\rho_r' + (1 - \Omega^2 + 2j\Omega\zeta_r)\rho_r - 2\zeta_r\rho_f' \\ - (1 + 2j\Omega\zeta_r)\rho_f = \Omega^2 - \Phi \\ M_{fr}\rho_f'' + 2(\zeta_r + \bar{\zeta}_f + j\Omega M_{fr})\rho_f' + [1 + \beta_{fr} - \Omega^2 M_{fr} \\ + 2j\Omega(\zeta_r + \bar{\zeta}_f)]\rho_f - 2\zeta_r\rho_r' - (1 + 2j\Omega\zeta_r)\rho_r = \Phi \\ (1 + j\mu)\beta_{cr}(\rho_r - \rho_s - \Delta \frac{1 - j\mu}{\sqrt{1 + \mu^2}} \frac{\Phi}{|\Phi|}) = \Phi. \end{aligned} \quad (5)$$

The synchronous solutions of Eq. (2) correspond to the fixed points of Eq. (5). So the terms that contain differentiation with respect to τ will be cancelled in order to get the time-independent solutions of Eq. (5). The algebraic equations of the deflection amplitudes of the rotor and the casing/foundation as well as the magnitude of the corresponding contact force are given in the form:

$$\begin{aligned} a_{11}\rho_r + a_{12}\rho_f = \Omega^2 - \Phi \\ a_{21}\rho_r + a_{22}\rho_f = \Phi \\ (1 + j\mu)\beta_{cr}(\rho_r - \rho_s - \Delta \frac{1 - j\mu}{\sqrt{1 + \mu^2}} \frac{\Phi}{|\Phi|}) = \Phi \end{aligned} \quad (6)$$

where $a_{11} := 1 - \Omega^2 + 2j\Omega\zeta_r$, $a_{22} := 1 + \beta_{fr} - \Omega^2 M_{fr} + 2j\Omega(\zeta_r + \bar{\zeta}_f)$, $a_{12} = a_{21} := -(1 + 2j\Omega\zeta_r)$.

The equations governing the motion of the rotor and the stator (casing/foundation) are linear when there is no contact between the rotor and the stator, namely, when $\Phi = 0$. The equations become nonlinear when the rubbing between the rotor and the stator occurs by introducing the contact force.

To solve Eq. (6) the complex amplitudes of the rotor ρ_r and the stator ρ_f can be first represented by the amplitude of contact force Φ as

$$\begin{aligned} \rho_r = C_1\Omega^2 - C_2\Phi \\ \rho_f = C_3\Phi - C_4\Omega^2. \end{aligned} \quad (7)$$

where $C_1 = a_{22}/D$, $C_2 = (a_{22} + a_{12})/D$, $C_3 = (a_{11} + a_{21})/D$, $C_4 = a_{21}/D$, $D = a_{11}a_{22} - a_{12}a_{21}$.

3.1 The synchronous no-rub solution

The synchronous no-rub solution can be solved by setting Φ to zero, so one gets

$$\rho_r = C_1\Omega^2, \quad \rho_f = -C_4\Omega^2 \quad (8)$$

It is worth to note that the stator (casing/foundation) will also vibrate when the rotor vibrates under the excitation of imbalance even though there is no contact between them.

When the solution (8) is substituted into the contact condition between the rotor and the stator

$$|\rho_r - \rho_f| = \Delta \quad (9)$$

a 4 th order polynomial equation of Ω . By solving the equation, the critical rotation speeds by which the rotor begins to contact with the casing/foundation can be determined. These are the boundaries for the existence regions of the synchronous no-rub motion.

3.2 The synchronous full annular rub solutions

The synchronous full annular rub solutions can be derived after substituting (7) into the third equation of Eq. (6) and doing some manipulation to obtain the magnitude of the contact force

$$|\Phi| = \frac{-(1 + R_1)K \pm \sqrt{[(1 + R_1)^2 + I_1^2](R_2^2 + I_2^2) - K^2 I_1^2}}{(1 + R_1)^2 + I_1^2} \quad (10)$$

where $K = \beta_{cr}\Delta\sqrt{1 + \mu^2}$ is a real number.

$$\begin{aligned} R_1 &= \Re[(1 + j\mu)\beta_{cr}(C_2 + C_3)], \\ R_2 &= \Re\left[(1 + j\mu)\beta_{cr}(C_1 + C_4)\Omega^2\right], \\ I_1 &= \Im[(1 + j\mu)\beta_{cr}(C_2 + C_3)], \\ I_2 &= \Im\left[(1 + j\mu)\beta_{cr}(C_1 + C_4)\Omega^2\right]. \end{aligned}$$

with $\Re(\cdot)$ and $\Im(\cdot)$ stand for the real and imaginary part of a complex number. The complex magnitude of the contact force is thus in the form

$$\Phi = \frac{(1 + j\mu)\beta_{cr}(C_1 + C_4)\Omega^2}{1 + \frac{K}{|\Phi|} + (1 + j\mu)\beta_{cr}(C_2 + C_3)}. \quad (11)$$

Now the complex amplitudes of the rotor ρ_r and the casing/foundation ρ_f in the rubbing case can be determined after substituting (11) back into Eq. (7).

To judge if the rotor-casing/foundation system really undertakes a synchronous full annular rub motion in the existence regions of the synchronous full annular rub solution, the stability of the solutions must be investigated. To do this, the complex amplitude of the rotor and the casing/foundation will be written in their component form

$$\hat{r}_r = Y_r + jZ_r, \quad \hat{r}_f = Y_f + jZ_f$$

After defining the state vector as

$$\underline{X} = \left\{ Y_r \ Z_r \ Y_f \ Z_f \ Y_r' \ Z_r' \ Y_f' \ Z_f' \right\}^T, \quad \text{Eq. (2) can be written as}$$

$$\underline{X}' = \underline{A}\underline{X} + \underline{g}(\underline{X}) = \underline{G}(\underline{X}) \quad (12)$$

where $\underline{A}$ and $\underline{g}(\underline{X})$ are respectively the coefficient matrix and the nonlinear vector:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -(1 + \beta_{cr}) & \mu\beta_{cr} & \beta_{cr} + 1 & -\mu\beta_{cr} & -2\zeta_r \\ -\mu\beta_{cr} & -(1 + \beta_{cr}) & \mu\beta_{cr} & \beta_{cr} + 1 & 0 \\ \frac{1}{M_{fr}} + B_{cr} & -\mu B_{cr} & -(B_{cr} + B_{fr} + \frac{1}{M_{fr}}) & \mu B_{cr} & \frac{2\zeta_r}{M_{fr}} \\ \mu B_{cr} & \frac{1}{M_{fr}} + B_{cr} & -\mu B_{cr} & -(B_{cr} + B_{fr} + \frac{1}{M_{fr}}) & 0 \end{bmatrix}$$

and

$$\mathbf{g}(\mathbf{X}) = \begin{Bmatrix} 0 & 0 & 0 & 0 & \Delta\beta_{cr}H_Y + \Omega^2 \cos(\Omega\tau) & \Delta\beta_{cr}H_Z \\ +\Omega^2 \sin(\Omega\tau) & -\Delta B_{cr}H_Y & -\Delta B_{cr}H_Z \end{Bmatrix}^T$$

with the variables in the above matrices are defined as followings:

$$B_{fr} = \beta_{fr}/M_{fr}, \quad B_{cr} = \beta_{cr}/M_{fr},$$

$$R = \sqrt{(Y_r - Y_f)^2 + (Z_r - Z_f)^2}$$

$$H_Y = \Delta/R[(Y_r - Y_f) - \mu(Z_r - Z_f)],$$

$$H_Z = \Delta/R[\mu(Y_r - Y_f) + (Z_r - Z_f)]$$

Denoting the synchronous full annular rub solutions by $\mathbf{X}_0$, which has the form

$$\mathbf{X}_0^T = \begin{pmatrix} B_r \cos(\varphi_\alpha) & B_r \sin(\varphi_\alpha) & B_s \cos(\varphi_\beta) & B_s \sin(\varphi_\beta) \\ -B_r \Omega \sin(\varphi_\alpha) & B_r \Omega \cos(\varphi_\alpha) \\ -B_s \Omega \sin(\varphi_\beta) & B_s \Omega \sin(\varphi_\beta) \end{pmatrix}^T$$

where $\varphi_\alpha = \Omega\tau + \alpha$, $\varphi_\beta = \Omega\tau + \beta$ and linearizing Eq. (12) about $\mathbf{X}_0$, it yields a linear differential equation

$$\delta\mathbf{X}' = \mathbf{J}(\mathbf{X}_0)\delta\mathbf{X} \quad (13)$$

In Eq. (13), $\delta\mathbf{X} = \mathbf{X} - \mathbf{X}_0$ is the perturbation solution to the full annular rub solution and

$$\mathbf{J}(\mathbf{X}_0)_{8 \times 8} = \frac{\partial \mathbf{G}}{\partial \mathbf{X}} \Big|_{\mathbf{X}=\mathbf{X}_0} = \mathbf{A} + \frac{\partial \mathbf{g}}{\partial \mathbf{X}} \Big|_{\mathbf{X}=\mathbf{X}_0} \quad (14)$$

is the so-called Jacobian matrix, whose expression is given in Appendix 1. Matrix (14) is periodically time-dependent and cannot be directly used to judge stabilities of the studied solution. The Jacobian matrix can be transformed into a time-independent matrix, $[\mathbf{J}_n]$ after some manipulation as shown in Appendix 2, whose eigenvalues will define the stability of the solution by solving the characteristic equation below:

$$\det[\mathbf{J}_n - \lambda \mathbf{I}_{8 \times 8}] = 0 \quad (15)$$

The studied solution is stable if all the real parts of its characteristic roots are less than zero. As shown in [7, 22], the region of the synchronous full annular rub is generally bounded by

the boundaries of the saddle-node and Hopf bifurcations of the synchronous full annular rub solutions.

4 The dry friction backward whirl

Dry friction backward whirl is a motion state of the rotor-casing/foundation systems, by which the rotor is in continuous full annular contact with the casing/foundation, rolling or slipping continuously on the contact surface and whirling backward at a super-synchronous frequency. Dry friction backward whirl is a self-excited motion induced by the dry friction effect [15].

Before we start to derive the existence condition and the whirl frequency of the dry friction backward whirl of the rotor-casing/foundation system, the governing equations will be first simplified based on the mechanism of the dry friction backward whirl so as to make an analytical study possible. From the definition, the rotor is in a full annular rub with the casing/foundation during the dry friction backward whirl. Thus, we shall study Eq. (2) only in the case that $\Theta = 1$. As can be seen from Eq. (2), the relative velocity at the contact point can be positive or negative during the dry friction backward whirl (with a negative Ω_w) depending upon the magnitude of the relative deflection of the rotor with respect to the casing/foundation, $|\hat{r}_r - \hat{r}_f|$. The direction of the friction force depends in turn on the sign of the relative velocity. By alternating its direction, the friction force dissipates or inputs the energy to the lateral vibration of the rotor and keeps the vibration amplitude of the system fluctuating around a constant value by which the relative velocity equals zero. Since the magnitude of the fluctuation is generally much smaller than the constant value by which the relative velocity equals zero as observed in tests and simulations, the relative deflection of the rotor with respect to the casing/foundation at the zero relative velocity ($V_{rel} = 0$) can serve as a good approximation to the real one of the dry friction backward whirl. In this way, the effect of dry friction is accounted for and the difficulty to deal with the alternation of the direction of the friction force according to the sign of the relative velocity

can be well released by omitting it from Eq. (2). So Eq. (2) can be simplified and rewritten as:

$$\begin{aligned} & \hat{r}_r'' + 2\zeta_r(\hat{r}_r' - \hat{r}_f') + (\hat{r}_r - \hat{r}_f) \\ & + (1 + j\mu)\beta_{cr} \left(\hat{r}_r - \hat{r}_f - \Delta \frac{\hat{r}_r - \hat{r}_f}{|\hat{r}_r - \hat{r}_f|} \right) = \Omega^2 e^{j\Omega\tau} \\ & M_{fr}\hat{r}_r'' + 2\bar{\zeta}_f\hat{r}_f' + \beta_{fr}\hat{r}_f - 2\zeta_r(\hat{r}_r' - \hat{r}_f') - (\hat{r}_r - \hat{r}_f) \\ & - (1 + j\mu)\beta_{cr} \left(\hat{r}_r - \hat{r}_f - \Delta \frac{\hat{r}_r - \hat{r}_f}{|\hat{r}_r - \hat{r}_f|} \right) = 0 \\ & |\hat{r}_r - \hat{r}_f| = -\frac{R_{disk}\Omega}{\Omega_w} \end{aligned} \quad (16)$$

where $\Omega_w < 0$ all the time to describe the backward whirl of the rotor in this case.

As observed in tests and simulations [6, 14], the response of the dry friction backward whirl is generally composed of a backward whirl motion with a negative super-synchronous whirl frequency and a forced forward whirl motion with a synchronous whirl frequency whose amplitude is usually negligible. Accordingly, the solutions of dry friction backward whirl of Eq. (16) can be supposed in the form

$$\begin{aligned} \hat{r}_i &= W_{bi} + W_{fi} \quad \text{with } W_{bi} = H_i e^{(\alpha + j\Omega_w)\tau + j\phi_i} \text{ and} \\ W_{fi} &= A_i e^{j\Omega\tau}, \quad i = r, f \end{aligned} \quad (17)$$

where W_{bi} represents the solution part of the backward whirl motion with $H_i e^{\alpha\tau}$ the time-varying amplitude and Ω_w the negative whirl frequency to be determined, and W_{fi} the solution part of the forced synchronous forward whirl motion with A_i to be determined. Without loss of generality, we take $\phi_r = 0$, $\phi_f = \phi$.

The solution part of the backward whirl defined in the form as given in (17) with $\alpha > 0$ shows the capability of the system to undertake a self-excited vibration. This point is critical in capturing the features of the dry friction backward whirl of the system since $\alpha > 0$ represents the capability that the relative deflection of the rotor with respect to the casing/foundation increases exponentially with time until the zero relative velocity is reached and the dry friction effect takes effect around it.

By substituting (17) and the third equation of (16) into the first two equations of Eq. (16) and setting the coefficients of the same harmonics to zero, we can decouple the solution part of the backward whirl motion W_{bi} with that of the force forward whirl motion W_{fi} , $i = r, f$ to get two sets of equations governing the two parts of motion, respectively. Since the present study concentrates mainly on the backward whirl motion and the forced forward whirl motion is, in the most cases, negligible on the final response of the dry friction backward whirl as shown in [14], only the equa-

tions governing the motion part of backward whirl are given below

$$\begin{aligned} & W_{br}'' + 2\zeta_r(W_{br}' - W_{bf}') + (W_{br} - W_{bf}) \\ & + (1 + j\mu)\beta_{cr} \left(1 + \frac{\Delta\Omega_w}{R_{disk}\Omega} \right) (W_{br} - W_{bf}) = 0 \\ & M_{fr}W_{bf}'' + 2\bar{\zeta}_fW_{bf}' + \beta_{fr}W_{bf} - 2\zeta_r(W_{br}' - W_{bf}') \\ & - (W_{br} - W_{bf}) - (1 + j\mu)\beta_{cr} \left(1 + \frac{\Delta\Omega_w}{R_{disk}\Omega} \right) \\ & (W_{br} - W_{bf}) = 0. \end{aligned} \quad (18)$$

To solve Eq. (18), the solutions of W_{bi} , $i = r, f$, in form of (17) are substituted into Eq. (18) to yield

$$\begin{aligned} & (\alpha + j\Omega_w)^2 H_r + 2\zeta_r(\alpha + j\Omega_w)(H_r - H_f e^{j\phi}) \\ & + (H_r - H_f e^{j\phi}) + \bar{F}(H_r - H_f e^{j\phi}) = 0 \\ & [M_{fr}(\alpha + j\Omega_w)^2 + 2\bar{\zeta}_f(\alpha + j\Omega_w) + \beta_{fr}] H_f e^{j\phi} \\ & - 2\zeta_r(\alpha + j\Omega_w)(H_r - H_f e^{j\phi}) - (H_r - H_f e^{j\phi}) \\ & - \bar{F}(H_r - H_f e^{j\phi}) = 0. \end{aligned} \quad (19)$$

where $\bar{F} = (1 + j\mu)\beta_{cr}(1 + C\Omega_w)$ and $C = \frac{\Delta}{R_{disk}\Omega}$ is the sole parameter containing the rotation speed.

By adding the two equations of Eq. (19), we get the relation between H_r and H_f

$$H_r = -\frac{[M_{fr}(\alpha^2 - \Omega_w^2) + 2\bar{\zeta}_f\alpha + \beta_{fr}] + j2\Omega_w(\alpha M_{fr} + \bar{\zeta}_f)}{(\alpha + j\Omega_w)^2} H_f e^{j\phi}. \quad (20)$$

Substituting Eq. (20) into the first equation of Eq. (19) and deleting the term of $H_f e^{j\phi}$, we get a complex equation of α and Ω_w as the function of the system parameters. Separating the real and the imaginary parts, it yields two real polynomial equations about the two unknown variables α and Ω_w :

$$\begin{aligned} & (\alpha^2 - \Omega_w^2)[M_{fr}(\alpha^2 - \Omega_w^2) + 2\bar{\zeta}_f\alpha + \beta_{fr}] - 4\alpha\Omega_w^2(\alpha M_{fr} + \bar{\zeta}_f) \\ & + [1 + 2\alpha\zeta_r + \beta_{cr}(1 + C\Omega_w)][(1 + M_{fr})(\alpha^2 - \Omega_w^2) \\ & + 2\alpha\bar{\zeta}_f + \beta_{fr}] - 2\Omega_w[2\Omega_w\zeta_r + \beta_{cr}(1 + C\Omega_w)] \\ & [\alpha(1 + M_{fr}) + \bar{\zeta}_f] = 0 \end{aligned} \quad (21)$$

$$\begin{aligned} & + 2\alpha\Omega_w[M_{fr}(\alpha^2 - \Omega_w^2) + 2\bar{\zeta}_f\alpha + \beta_{fr}] + [\beta_{cr}\mu(1 + C\Omega_w) \\ & + 2\zeta_r\Omega_w][(1 + M_{fr})(\alpha^2 - \Omega_w^2) + 2\bar{\zeta}_f\alpha + \beta_{fr}] \\ & + 2\Omega_w[1 + 2\alpha\zeta_r + \beta_{cr}(1 + C\Omega_w)][\alpha(1 + M_{fr}) + \bar{\zeta}_f] = 0. \end{aligned} \quad (22)$$

The existence condition of the dry friction backward whirl is the critical rotation speed by which the dry friction backward whirl starts to exist. As discussed above, the solution of the dry friction backward whirl should have an exponent

with a positive real part, namely $\alpha > 0$. The corresponding critical condition is thus to have a zero real part, namely, $\alpha = 0$. Applying this condition to Eqs. (21) and (22) yields two equations on Ω_w and C

$$\begin{aligned} & \beta_{cr} \Omega_w [-(1 + M_{fr}) \Omega_w^2 - 2 \Omega_w \mu \bar{\zeta}_f + \beta_{fr}] C \\ &= \Omega_w^2 (\beta_{fr} - M_{fr} \Omega_w^2) - (1 - \beta_{cr}) [\beta_{fr} - (1 + M_{sr}) \Omega_w^2] \\ &+ 2 \Omega_w \bar{\zeta}_f (\mu \beta_{cr} + 2 \zeta_r \Omega_w) \end{aligned} \quad (23)$$

$$\begin{aligned} & \{\mu \beta_{cr} \Omega_w [\beta_{fr} + (1 + M_{sr}) \Omega_w^2] + 2 \beta_{cr} \bar{\zeta}_f \Omega_w^2\} C \\ &= 2 \Omega_w \{\zeta_r [(1 + M_{fr}) \Omega_w^2 - \beta_{sr}] + \bar{\zeta}_f [\Omega_w^2 - (1 + \beta_{cr})]\} \\ &+ \mu \beta_{cr} [(1 + M_{fr}) \Omega_w^2 - \beta_{sr}] \end{aligned} \quad (24)$$

Ideally, one would like to delete Ω_w from above two equations and obtain an equation of C in order to solve the critical rotation speed Ω , by which the dry friction backward whirl starts to exist. However, it is almost impossible to delete Ω_w from Eqs. (23) and (24) explicitly. So we work in the following way: first deleting C from Eqs. (23) and (24) to get a polynomial equation of Ω_w , then solving the equation for Ω_w s, which will be called *the whirl frequencies at $\alpha = 0$* ; then, substituting the whirl frequencies back into either of Eqs. (23) or (24) to yield an equation of C ; finally, solving the equation of C to get the *critical rotation speeds* of the dry friction backward whirl.

5 Examples of application

From above investigation it is easy to find that the forward full annular rub and the backward full annular rub have quite different dynamic mechanism. The forward full annular rub is a forced vibration under imbalance excitation with its whirl frequency equal to the rotation speed. So it is also called as *the synchronous full annular rub*. On the other hand, the backward full annular rub is a self-excited vibration due to the dry friction effect between the contact surfaces. So it is also called *the dry friction backward whirl* with its whirl frequency usually exhibiting the character of two regimes defined as *dry whirl* and *dry whip*. Below examples are given with the aim to demonstrate respectively the characteristics of the two kinds of full annular rubs depending upon the system parameters.

5.1 The forward full annular rub

In this subsection, examples are presented in order to show the influence of the system parameters on the existence and the stability of the synchronous full annular rub solutions. In Fig. 2, the stability charts in the parameter plane of Ω - β_{fr} are drawn for three mass ratios of the casing/foundation over the rotor in order to examine the influence of the stiffness ratio and the mass ratio on the rubbing responses. In the plots

Fig. 2 The stability charts of the synchronous full annular rub solutions in the parameter plane of Ω - β_{fr} for different mass ratios, **a** $M_{fr} = 2.0$; **b** $M_{fr} = 1.0$; **c** $M_{fr} = 0.5$. The dashed lines indicate the natural frequencies of the rotor and the solid line the boundaries whereby the rotor begins to contact the stator. The shaded regions represent the existence regions of the synchronous full annular rub solutions with the dark shaded areas among them being the stable sub-regions. The other system parameters are given as: $\zeta_r = 0.05$, $\zeta_f = 0.05$, $\beta_{cr} = 200$, $\mu = 0.10$, $\Delta = 2.0$ and $R_{disk} = 20$

the other system parameters are fixed to: $\zeta_r = 0.05$, $\zeta_f = 0.05$, $\beta_{cr} = 200$, $\mu = 0.10$, $\Delta = 2.0$ and $R_{disk} = 20$.

In Fig. 2, the red dashed curves represent the natural frequencies of the coupled linear rotor-casing/foundation system computed from Eq. (3). The black solid curves, derived from the condition (9), define the boundaries of the synchronous no-rub motion or the rotation speeds by which the rotor begins to contact the stator under the excitation of mass eccentricity. So the synchronous no-rub motion of the

rotor exists outside the regions enclosed by the black solid curves.

The shaded regions in Fig. 2 are the existence regions of the synchronous full annular rub solutions. It can be seen that there are generally two existence regions of the solutions except for the range of small stiffness ratios, i.e., $\beta_{fr} < 1.0$ for $M_{fr} = 2.0$ in Fig. 2a, or for the range of large stiffness ratios, i.e., $\beta_{fr} > 15$ for the case of $M_{fr} = 0.5$ in Fig. 2c. Since only the stable synchronous full annular rub solutions have physical meaning that corresponds to the synchronous full annular rub motion, the stability analysis described in Sect. 3 is carried out to get the regions where the synchronous full annular rub solutions are stable, which are depicted by the dark shaded areas in Fig. 2. It is easily seen that the synchronous full annular rub occurs mainly in the first existence region. It is almost impossible to be detected in the second existence region with higher rotation speeds for the rotor-casing/foundation system. The borders between the dark shaded areas and the blank regions are the saddle-node bifurcation boundaries, and the borders between the dark shaded areas and the gray shaded areas are the Hopf bifurcation boundaries of the synchronous full annular rub solutions. When the former boundary is crossed, the synchronous full annular rub of the system ceases to exist. When the latter is crossed, the synchronous full annular rub of the system will change to the quasi-periodic partial rubs or the dry friction backward whirl, as pointed out in [14, 15]. From above discussion, the regions where the synchronous no-rub motion and the synchronous full annular rub motion exist and where the two responses coexist can be identified. However, it is not possible to exclude the coexistence of the other responses with the two responses.

The substantial influence of the mass ratio, M_{fr} , on the synchronous full annular rub is obvious as can be seen from Fig. 2. It is found that with the decrease of the mass ratio, the region of the synchronous full annular rub increases significantly, and will cover almost the whole area of the first existence region when $M_{fr} = 0.5$ (see Fig. 2c). Since the synchronous full annular rub belongs to relatively mild rub, in comparison with the partial rubs or dry whip which generally occur after the Hopf bifurcation of the synchronous full annular rub solutions (in the grey shaded areas of Fig. 2), it is beneficial for rotor rubbing when the mass of the rotor is larger than that of the casing/foundation as in the case of aero-engines.

5.2 The backward full annular rub

Below the examples of the influence of the system parameters on the existence boundaries of dry friction backward whirl are investigated. The existence boundaries are drawn in the parameter plane of $\Omega - \mu$ for six cases with combinations of the mass ratio, M_{fr} and the contact stiffness, β_{cr} (see Fig. 3).

In the study, the other system parameters are fixed to: $\zeta_r = 0.05$, $\zeta_f = 0.05$, $\beta_{fr} = 8.0$, $\Delta = 2.0$ and $R_{disk} = 20$.

For the cases of $\beta_{cr} = 200$: whereby the contact surfaces are relatively rigid, the influence of the mass ratios on the existence regions of the dry friction backward whirl is studied. As can be seen from Fig. 3a–c, there are total three existence boundaries given as the critical rotation speeds for the existence of dry friction backward whirl, following the procedure described in the Sect. 4. The first and the second boundaries form a narrow U-shaped region in a very low rotation speed range that represents the first existence region. The third boundary defines the border of the second existence region that covers a very large region with higher rotation speeds. The two existence regions are separated.

It is found that although the mass ratio changes greatly in an extent of an order, from $M_{fr} = 2.0$ to $M_{fr} = 0.2$, there is no qualitative change in the feature of the existence regions. However, substantial quantitative changes are detectable with the decrease of the mass ratio. In the case of $M_{fr} = 2.0$, the first U-shaped region is very narrow and might be hard to be identified in the real systems and the normal rotor response (no-rub response) exists also within the area. The second region begins to exist even when the rotation speed is only at about the three tenth of the first critical speed of the rotor when $\mu = 0.4$. It is not hard to imagine that the dry friction backward whirl may coexist with all other responses, like the synchronous full annular rub and partial rub as well as no-rub motions, of the system. Most importantly, it can be triggered even from a synchronous no-rub motion through an outside disturbance, or the quasi-periodic partial rub by the imbalance, as indicated in [14]. With the decrease of the mass ratio, the smallest critical rotation speed for the existence of the dry friction backward whirl has been shifted substantially higher referring to Fig. 3. So smaller mass ratio will be beneficial for avoidance of the destructive backward full annular rub.

The U-shaped region (first existence region) of the dry friction backward whirl in the $\Omega - \mu$ plane for the present model is similar to the one obtained in [16, 18] for the rotor/stator systems with an elastically supported rigid stator (namely, with an infinite contact stiffness). In fact, there is a second existence region of the dry friction backward whirl for the rotor/stator systems as indicated recently in [26] through numerical simulation. It can thus conclude from the above discussion that there should be two separate existence regions of the dry friction backward whirl for a rotor-to-stator contact system with large contact stiffness as also shown in [23, 24].

For the cases of $\beta_{cr} = 20$: that is, the contact surfaces are relatively soft. In this case, qualitative change on the existence regions of the dry friction backward whirl may occur with the variation of the mass ratio.

When $M_{fr} = 2.0$, it is seen from Fig. 3d that the feature of the existence regions is the same as the one demonstrated

Fig. 3 The existence regions of the dry friction backward whirl in the parameter plane of $\Omega - \mu$: When $\beta_{cr} = 200$, **a** $M_{fr} = 2.0$; **b** $M_{fr} = 0.5$; **c** $M_{fr} = 0.2$. When $\beta_{cr} = 20$, **d** $M_{fr} = 2.0$;

e $M_{fr} = 0.5$; **f** $M_{fr} = 0.2$. The other system parameters are given as: $\zeta_r = 0.05$, $\zeta_f = 0.05$, $\beta_{fr} = 8.0$, $\Delta = 2.0$ and $R_{disk} = 20$

above with two separate existence regions when the contact stiffness is large. From the whirl frequency of the dry friction backward whirl, Ω_w , it is more easily found that the whirl frequencies increase with the rotation speed and have different ranges of values in the two existence regions when $\mu = 0.25$. Actually, the magnitudes of the whirl frequencies in the two existence regions are limited respectively by the two natural frequencies of the coupled rotor-casing/foundation system with zero clearance as given by (3).

When $M_{fr} = 0.5$, the two existence regions become partly overlapped as shown by Fig. 3e. The overlapped region means that two responses of the dry friction backward whirl

with different backward whirl frequencies coexist. This feature is also well illustrated by the backward whirl frequencies in Fig. 4b when $\mu = 0.25$. It is found that the whirl frequency in the first existence region ceases to exist until about $\Omega = 1.15$, while the whirl frequency in the second existence region already begins to exist at about $\Omega = 0.8$. So the dry friction backward whirls with two different whirl frequencies coexist in the speed range of (0.8, 1.15). A very good experimental correspondence of this phenomenon can be found in [26].

If the mass ratio is further reduced, it is interesting to note that the second existence region of the dry friction backward

Fig. 4 The magnitudes of the whirl frequencies of the dry friction backward whirl in the two existence regions with $\beta_{cr} = 20$ and $\mu = 0.25$: **a** $M_{fr} = 2.0$; **b** $M_{fr} = 0.5$; **c** $M_{fr} = 0.2$. The other system parameters are given as: $\zeta_r = 0.05$, $\zeta_f = 0.05$, $\beta_{fr} = 8.0$, $\Delta = 2.0$ and $R_{disk} = 20$

whirl will be completely contained in the first one, as indicated by Fig. 3f when $M_{fr} = 0.2$. The magnitudes of the whirl frequencies in this case are drawn in Fig. 4c when $\mu = 0.25$. The whirl frequency in the first existence region shows obviously the character of the two regimes of dry whirl and dry whip as defined in [18]. Since the response of the dry friction backward whirl is relatively robust against the external disturbance, the dry friction backward whirl in the second existence region with a much higher whirl frequency, might be harder to be triggered in comparison with the one in the first existence region that exists over the whole speed range.

6 Conclusions

In this paper the forward and the backward full annular rubs of a coupled rotor-casing/foundation system, by which the motion of the rotor is in couple with that of the casing/foundation even in the no rub condition, are investigated analytically. The forward full annular rub, which is usually called as the synchronous full annular rub, is a forced vibration of the system under the excitation of imbalance. The backward full annular rub, also called as the dry friction backward whirl, is a self-excited vibration due to the effect of dry friction.

By studying the forward full annular rub, the synchronous full annular rub solutions are first derived by transforming the governing equations into the rotating coordinate system and using a proper form of the resultant contact force. The stability of the periodic solutions is then analyzed following the Floquet theory in order to determine the parameter regions where the synchronous full annular rub exists. It is found that although there are generally two existence regions for the synchronous full annular rub solutions, the stable regions of the solutions lie generally in the first one and depend heavily on the mass ratio as well as the friction coefficient. Small mass ratio and friction coefficient will benefit the system to avoid the loss of stability through Hopf bifurcation into the more dangerous responses, i.e., the partial rubs or the dry friction backward whirl.

By exploiting the backward full annular rub, the equations of motion that can well capture the features of the response are set up and the correct form of the solution is assumed so that the existence condition and the whirl frequency of the dry friction backward whirl are derived analytically. It is found that there are total three existence boundaries of the dry friction backward whirl that form two existence regions. The backward whirl frequencies in the two existence regions have different ranges of values that are, respectively, limited by the two natural frequencies of the coupled rotor-casing/foundation system at the zero clearance. Usually, the two existence regions are separate and will not overlap with each other when the contact stiffness or the mass ratio is sufficiently large. In the case of a small contact stiffness and/or a small mass ratio, the first existence region may overlap with the second one or even completely contain the second one. In the overlapped areas of the two existence regions, two responses of the dry friction backward whirl with two different whirl frequencies will coexist.

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7 Appendix 1

Expression of Jacobian matrix in (14).

$$\mathbf{J}(\mathbf{X}^0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - (1 + \beta_{cr}) + \lambda g_1 & \mu \beta_{cr} + \lambda g_2 & \beta_{cr} + 1 - \lambda g_1 & -\mu \beta_{cr} - \lambda g_2 & -2\zeta_r & 0 & 2\zeta_r & 0 & 0 & 0 \\ -\mu \beta_{cr} + \lambda g_3 & - (1 + \beta_{cr}) + \lambda g_4 & \mu \beta_{cr} - \lambda g_3 & \beta_{cr} + 1 - \lambda g_4 & 0 & -2\zeta_r & 0 & 0 & 2\zeta_r & 0 \\ \bar{c} + B_{cr} - \bar{\lambda} g_{11} & -\mu B_{cr} - \bar{\lambda} g_{22} & - (B_{cr} + B_{fr} + \bar{c}) + \bar{\lambda} g_{11} & \mu B_{cr} + \bar{\lambda} g_{22} & 2\bar{b} & 0 & -2\bar{q} - 2\bar{b} & 0 & 0 & 0 \\ \mu B_{cr} - \bar{\lambda} g_{33} & \bar{c} + B_{cr} - \bar{\lambda} g_{44} & -\mu B_{cr} + \bar{\lambda} g_{33} & - (B_{cr} + B_{fr} + \bar{c}) + \bar{\lambda} g_{44} & 0 & 2\bar{b} & 0 & 0 & -2\bar{q} - 2\bar{b} & 0 \end{bmatrix}$$

where

$$\bar{c} = \frac{1}{M_{fr}}, \bar{q} = \frac{\bar{\zeta}_f}{M_{fr}}, \bar{b} = \frac{\zeta_r}{M_{fr}}$$

$$\Psi = 0.5 \left(B_r^2 - 2B_r B_f \cos(\alpha - \beta) + B_f^2 \right)^{-1.5},$$

$$B_r = \sqrt{Y_r^2 + Z_r^2}, B_f = \sqrt{Y_f^2 + Z_f^2},$$

$$\lambda = \Psi \Delta \beta_{cr}, \bar{\lambda} = \Psi \Delta B_{cr}$$

$$\begin{aligned} g_1 &= B_r^2 - 2B_r B_f \cos(\alpha - \beta) + B_f^2 - B_r^2 \cos(2\Omega\tau + 2\alpha) \\ &\quad + 2B_r B_f \cos(2\Omega\tau + \alpha + \beta) - B_f^2 \sin(2\Omega\tau + 2\beta) \\ &\quad + \mu B_r^2 \cos(2\Omega\tau + 2\alpha) - 2\mu B_r B_f \sin(2\Omega\tau + \alpha + \beta) \\ &\quad + \mu B_f^2 \sin(2\Omega\tau + 2\beta) \end{aligned}$$

$$\begin{aligned} g_2 &= -\mu B_r^2 + 2B_r B_f \cos(\alpha - \beta) - \mu B_f^2 \\ &\quad - B_r^2 \sin(2\Omega\tau + 2\alpha) + 2B_r B_f \sin(2\Omega\tau + \alpha + \beta) \\ &\quad - B_f^2 \sin(2\Omega\tau + 2\beta) - \mu B_r^2 \cos(2\Omega\tau + 2\alpha) \\ &\quad + 2\mu B_r B_f \cos(2\Omega\tau + \alpha + \beta) - \mu B_f^2 \cos(2\Omega\tau + 2\beta) \end{aligned}$$

$$\begin{aligned} g_3 &= \mu B_r^2 - 2\mu B_r B_f \cos(\alpha - \beta) + \mu B_f^2 \\ &\quad - B_r^2 \sin(2\Omega\tau + 2\alpha) + 2B_r B_f \sin(2\Omega\tau + \alpha + \beta) \\ &\quad - B_f^2 \sin(2\Omega\tau + 2\beta) - \mu B_r^2 \cos(2\Omega\tau + 2\alpha) \\ &\quad + 2\mu B_r B_f \cos(2\Omega\tau + \alpha + \beta) - \mu B_f^2 \cos(2\Omega\tau + 2\beta) \end{aligned}$$

$$\begin{aligned} g_4 &= B_r^2 - 2B_r B_f \cos(\alpha - \beta) + B_f^2 - \mu B_r^2 \sin(2\Omega\tau + 2\alpha) \\ &\quad + 2\mu B_r B_f \sin(2\Omega\tau + \alpha + \beta) - \mu B_f^2 \sin(2\Omega\tau + 2\beta) \\ &\quad + B_r^2 \cos(2\Omega\tau + 2\alpha) - 2B_r B_f \cos(2\Omega\tau + \alpha + \beta) \\ &\quad + B_f^2 \sin(2\Omega\tau + 2\beta) \end{aligned}$$

$$\begin{aligned} g_{11} &= B_r^2 - 2B_r B_f \cos(\alpha - \beta) + B_f^2 - B_r^2 \cos(2\Omega\tau + 2\alpha) \\ &\quad + 2B_r B_f \cos(2\Omega\tau + \alpha + \beta) - B_f^2 \cos(2\Omega\tau + 2\beta) \\ &\quad + \mu B_r^2 \sin(2\Omega\tau + 2\alpha) - 2\mu B_r B_f \sin(2\Omega\tau + \alpha + \beta) \\ &\quad + \mu B_f^2 \sin(2\Omega\tau + 2\beta) \end{aligned}$$

$$g_{22} = -\mu B_r^2 + 2B_r B_f \cos(\alpha - \beta) - \mu B_f^2$$

$$\begin{aligned} &-B_r^2 \sin(2\Omega\tau + 2\alpha) + 2B_r B_f \sin(2\Omega\tau + \alpha + \beta) \\ &-B_f^2 \sin(2\Omega\tau + 2\beta) - \mu B_r^2 \cos(2\Omega\tau + 2\alpha) \\ &+ 2\mu B_r B_f \cos(2\Omega\tau + \alpha + \beta) - \mu B_f^2 \cos(2\Omega\tau + 2\beta) \end{aligned}$$

$$\begin{aligned} g_{33} &= \mu B_r^2 - 2\mu B_r B_f \cos(\alpha - \beta) + \mu B_f^2 \\ &\quad - B_r^2 \sin(2\Omega\tau + 2\alpha) + 2B_r B_f \sin(2\Omega\tau + \alpha + \beta) \\ &\quad - B_f^2 \sin(2\Omega\tau + 2\beta) - \mu B_r^2 \cos(2\Omega\tau + 2\alpha) \\ &\quad + 2\mu B_r B_f \cos(2\Omega\tau + \alpha + \beta) - \mu B_f^2 \cos(2\Omega\tau + 2\beta) \\ g_{44} &= B_r^2 - 2B_r B_f \cos(\alpha - \beta) + B_f^2 - \mu B_r^2 \sin(2\Omega\tau + 2\alpha) \\ &\quad + 2\mu B_r B_f \sin(2\Omega\tau + \alpha + \beta) - \mu B_f^2 \sin(2\Omega\tau + 2\beta) \\ &\quad + B_r^2 \cos(2\Omega\tau + 2\alpha) - 2B_r B_f \cos(2\Omega\tau + \alpha + \beta) \\ &\quad + B_f^2 \sin(2\Omega\tau + 2\beta) \end{aligned}$$

8 Appendix 2

To transform periodic time-dependent matrix $\mathbf{J}(\mathbf{X}^0)$ to the time-independent one, we employ following transformation:

$$\delta \underline{X} = [T] \delta \underline{U} \quad (\text{app. -1})$$

where

$$[T] = \begin{bmatrix} \bar{T} & & \\ & \bar{T} & \\ & & \bar{T} \end{bmatrix}, \quad \bar{T} = \begin{bmatrix} \cos(\Omega\tau + \alpha) & -\sin(\Omega\tau + \alpha) \\ \sin(\Omega\tau + \alpha) & \cos(\Omega\tau + \alpha) \end{bmatrix}$$

Substituting (app.-1) into (13) and after some manipulation, we get

$$\delta \underline{U}' = [J_n] \delta \underline{U}$$

$$\text{where } [J_n] = [T]^{-1} ([J][T] - [T]')$$

and is in the form

$$[J_n] = \begin{bmatrix} 0 & \Omega & 0 & 0 & 1 & 0 & 0 & 0 \\ -\Omega & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \Omega & 0 & 0 & 1 & 0 \\ 0 & 0 & -\Omega & 0 & 0 & 0 & 0 & 1 \\ J_{51} & J_{52} & J_{53} & J_{54} & -2\zeta_r & \Omega & 2\zeta_r \cos(\alpha - \beta) & 2\zeta_r \sin(\alpha - \beta) \\ J_{61} & J_{62} & J_{63} & J_{64} & -\Omega & -2\zeta_r & -2\zeta_r \sin(\alpha - \beta) & 2\zeta_r \cos(\alpha - \beta) \\ J_{71} & J_{72} & J_{73} & J_{74} & 2\bar{b} \cos(\alpha - \beta) & -2\bar{b} \sin(\alpha - \beta) & -2\bar{q} - 2\bar{b} & \Omega \\ J_{81} & J_{82} & J_{83} & J_{84} & 2\bar{b} \sin(\alpha - \beta) & 2\bar{b} \cos(\alpha - \beta) & -\Omega & -2\bar{q} - 2\bar{b} \end{bmatrix}$$

where

$$J_{51} = -1 - \lambda B_s^2 \cos(2\alpha - 2\beta) - \lambda \mu B_f^2 \sin(2\alpha - 2\beta)$$

$$+ 2\lambda \mu B_r B_f \sin(\alpha - \beta) - \beta_{cr} + \lambda B_f^2$$

$$J_{61} = -\mu \beta_{cr} + \lambda \mu B_s^2 + \lambda B_s^2 \sin(2\alpha - 2\beta)$$

$$- 2\lambda B_r B_f \sin(\alpha - \beta) - \lambda \mu B_f^2 \cos(2\alpha - 2\beta)$$

$$J_{71} = -\lambda B_r B_f + (\bar{c} + B_{cr}) \cos(\alpha - \beta)$$

$$+ \lambda B_r B_f \cos(2\alpha - 2\beta) - \lambda \mu B_r B_f \sin(2\alpha - 2\beta)$$

$$+ (2\lambda \mu B_f^2 - \mu B_{cr}) \sin(\alpha - \beta)$$

$$J_{81} = -\lambda \mu B_r B_f + \lambda \mu B_r B_f \cos(2\alpha - 2\beta)$$

$$+ \lambda B_r B_f \sin(2\alpha - 2\beta) + (\bar{c} + B_{cr}) \sin(\alpha - \beta)$$

$$+ B_{cr} \sin(\alpha - \beta) - 2\lambda B_f^2 \sin(\alpha - \beta)$$

$$+ \mu B_{cr} \cos(\alpha - \beta)$$

$$J_{52} = 4\lambda \mu B_r B_s \cos(\alpha - \beta) + \mu \beta_{cr} - 2\lambda \mu B_r^2 - \lambda \mu B_f^2$$

$$+ \lambda B_f^2 \sin(2\alpha - 2\beta) - 2\lambda B_r B_f \sin(\alpha - \beta)$$

$$- \lambda \mu B_f^2 \cos(2\alpha - 2\beta)$$

$$J_{62} = -1 + \lambda B_f^2 \cos(2\alpha - 2\beta) + \lambda \mu B_f^2 \sin(2\alpha - 2\beta)$$

$$- 2\lambda \mu B_r B_f \sin(\alpha - \beta) - \beta_{cr} - 4\lambda B_r B_s \cos(\alpha - \beta)$$

$$+ 2\lambda B_r^2 + \lambda B_f^2$$

$$J_{72} = -3\lambda \mu B_r B_f - \lambda \mu B_r B_f \cos(2\alpha - 2\beta)$$

$$+ \lambda B_r B_f \sin(2\alpha - 2\beta) + (2\lambda B_r^2 - (\bar{c} + B_{cr})) \sin(\alpha - \beta)$$

$$+ (2\lambda \mu B_r^2 + 2\lambda \mu B_f^2 - \mu B_{cr}) \cos(\alpha - \beta)$$

$$J_{82} = (-2\lambda B_f^2 - 2\lambda B_r^2 + \bar{c} + B_{cr}) \cos(\alpha - \beta)$$

$$+ 3\lambda B_r B_f + \lambda B_r B_f \cos(2\alpha - 2\beta)$$

$$- \lambda \mu B_r B_f \sin(2\alpha - 2\beta)$$

$$+ (-\mu B_{cr} + 2\lambda \mu B_r^2) \sin(\alpha - \beta)$$

$$J_{53} = (1 + \beta_{cr}) \cos(\alpha - \beta) - \lambda B_r B_f$$

$$+ \lambda \mu B_r B_f \sin(2\alpha - 2\beta)$$

$$+ (\mu \beta_{cr} - 2\lambda \mu B_r^2) \sin(\alpha - \beta) + \lambda B_r B_f \cos(2\alpha - 2\beta)$$

$$+ \mu \beta_{cr} \cos(\alpha - \beta) - \lambda B_r B_f \sin(2\alpha - 2\beta)$$

$$+ \lambda \mu B_r B_f \cos(2\alpha - 2\beta)$$

$$J_{73} = -2\lambda \mu B_r B_f \sin(\alpha - \beta) - \lambda B_r^2 \cos(2\alpha - 2\beta)$$

$$+ \lambda \mu B_r^2 \sin(2\alpha - 2\beta) - \bar{c} - B_{cr} - B_{fr} + \lambda B_r^2$$

$$J_{83} = 2\lambda B_r B_f \sin(\alpha - \beta) - \lambda \mu B_r^2 \cos(2\alpha - 2\beta)$$

$$- \lambda B_r^2 \sin(2\alpha - 2\beta) - \mu B_{cr} + \lambda \mu B_r^2$$

$$J_{54} = (2\lambda \mu B_r^2 - \mu \beta_{cr} + 2\lambda \mu B_f^2) \cos(\alpha - \beta) - 3\lambda \mu B_r B_f$$

$$+ (1 + \beta_{cr} - 2\lambda B_f^2) \sin(\alpha - \beta) - \lambda \mu B_r B_f \cos(2\alpha - 2\beta)$$

$$+ \lambda B_r B_f \sin(2\alpha - 2\beta)$$

$$J_{64} = (1 - 2\lambda B_r^2 - 2\lambda B_f^2 + \beta_{cr}) \cos(\alpha - \beta) + 3\lambda B_r B_f$$

$$+ \lambda \mu B_r B_f \sin(2\alpha - 2\beta) + (\mu \beta_{cr} - 2\lambda \mu B_f^2) \sin(\alpha - \beta)$$

$$+ \lambda B_r B_f \cos(2\alpha - 2\beta)$$

$$J_{74} = 2\lambda B_r B_f \sin(\alpha - \beta) - \lambda \mu B_r^2 \cos(2\alpha - 2\beta)$$

$$- \lambda B_r^2 \sin(2\alpha - 2\beta) + 4\lambda \mu B_r B_f \cos(\alpha - \beta) + \mu B_{cr}$$

$$- 2\lambda \mu B_f^2 - \lambda \mu B_r^2$$

$$J_{84} = -4\lambda B_r B_f \cos(\alpha - \beta) + 2\lambda \mu B_r B_f \sin(\alpha - \beta)$$

$$+ \lambda B_r^2 \cos(2\alpha - 2\beta) - \lambda \mu B_r^2 \sin(2\alpha - 2\beta)$$

$$- \bar{c} - B_{cr} - B_{fr} + \lambda (B_r^2 + B_f^2)$$

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