Linearization of second-order jump-diffusion equations

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Abstract We give the exact linearization criterion for the second-order jump-diffusion equations. We also present several illustrative examples.

Keywords Linearization · Jump-diffusion · Second-order sde · Solutions

1 Introduction

Exact solutions (stochastic processes) are important when studying the controllability issues of stochastic differential equations. However, there is no general method to solve these equations. Transforming a nonlinear stochastic differential equations into linear ones via invertible stochastic mappings prove to be useful in obtaining exact solutions [1, 2, 5, 7].

Second-order stochastic jump-diffusion equation reads as

$$\ddot{X} = f(t, X, \dot{X}) + g(t, X, \dot{X})\dot{W} + j(t, X, \dot{X})\dot{N} \quad (1)$$

where prime denotes differentiation with respect to t . One can write the scalar second-order stochastic jump-diffusion Eq. (1) in differential form as

$$d\dot{X} = f(t, X, \dot{X})dt + g(t, X, \dot{X})dW(t) + j(t, X, \dot{X})dN(t). \quad (2)$$

Here, $dW(t)$ is the infinitesimal increment of the Wiener process [6] and $dN(t)$ is the infinitesimal increment of the

Poisson process [3]. We assume that the coefficient functions f , g and j satisfy the Ikeda–Watanabe conditions for the existence and the uniqueness of the solutions. The outline of the paper is as follows. In Sect. 2, we derive the linearization criteria imposed on $f(t, X, \dot{X})$, $g(t, X, \dot{X})$ and $j(t, X, \dot{X})$. We also determine the conditions for the invertible transformation which linearize the jump-diffusion Eq. (2) in a theorem. The latter extends the previous works [1, 2, 5, 7]. In Sect. 3, we present some linearizable equations and also, we present a special case of Eq. (2) where the coefficient functions are given as $f(t, \dot{X})$, $g(t, \dot{X})$ and $j(t, \dot{X})$. To linearize this type of equations, we use the linearization conditions which have been obtained in [7]. Finally, concluding remarks are made in Sect. 4.

2 Linearization criteria for second order jump-diffusion equations

Our goal is to find a change of the dependent variable $y = h(t, x)$ which transforms Eq. (2) to a linear stochastic differential equation of the form

$$d\dot{Y} = \left(a_1(t)Y + a_2(t)\dot{Y} + a_3(t) \right) dt + \left(b_1(t)Y + b_2(t)\dot{Y} + b_3(t) \right) dW + \left(c_1(t)Y + c_2(t)\dot{Y} + c_3(t) \right) dN. \quad (3)$$

We can rewrite (2) as a system of first-order stochastic differential equations

$$dX = Vdt \\ dV = f(t, X, V)dt + g(t, X, V)dW + j(t, X, V)dN. \quad (4)$$

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Similarly, the linear system corresponding to (3) can be given as

$$\begin{aligned} dY &= Zdt \\ dZ &= (a_1(t)Y + a_2(t)Z + a_3(t))dt + (b_1(t)Y + b_2(t)Z \\ &\quad + b_3(t))dW + (c_1(t)Y + c_2(t)Z + c_3(t))dN. \end{aligned} \quad (5)$$

We now seek invertible mappings to linearize the nonlinear jump-diffusion given in Eq. (2). Invertible mapping $h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ sought

$$y = h(t, x), \quad \frac{\partial h}{\partial x} \neq 0 \quad (6)$$

transforms (2) into the linear Eq. (3). Also, we assume here that h is three times continuously differentiable. Applying the Itô formula [3, 4] to the system of stochastic differential Eq. (4), one obtains

$$\begin{aligned} dY &= \left(\frac{\partial h}{\partial t} + V \frac{\partial h}{\partial x} \right) dt \\ dZ &= \left(\frac{\partial Z}{\partial t} + V \frac{\partial Z}{\partial x} + f \frac{\partial Z}{\partial V} + \frac{1}{2} g^2 \frac{\partial Z}{\partial W} \right) dt + g Z_V dW \\ &\quad + [Z(t, X, V + j) - Z(t, X, V)] dN \\ &= \left(\frac{\partial^2 h}{\partial t^2} + 2V \frac{\partial^2 h}{\partial t \partial x} + V^2 \frac{\partial^2 h}{\partial x^2} + f \frac{\partial h}{\partial x} \right) dt + g \frac{\partial h}{\partial x} dW \\ &\quad + \left[\frac{\partial h}{\partial t} + (V + j) \frac{\partial h}{\partial x} - \left(\frac{\partial h}{\partial t} + V \frac{\partial h}{\partial x} \right) \right] dN. \end{aligned} \quad (7)$$

From (5) and (7) we obtain the following partial differential equations (PDEs):

$$\begin{aligned} v^2 \frac{\partial^2 h}{\partial x^2} + v \left(2 \frac{\partial^2 h}{\partial t \partial x} - a_2 \frac{\partial h}{\partial x} \right) + \frac{\partial^2 h}{\partial t^2} - a_2 \frac{\partial h}{\partial t} \\ + f \frac{\partial h}{\partial x} - a_1 h - a_3 = 0 \end{aligned} \quad (8)$$

$$g \frac{\partial h}{\partial x} = b_2 \left(\frac{\partial h}{\partial t} + v \frac{\partial h}{\partial x} \right) + b_1 h + b_3 \quad (9)$$

$$j \frac{\partial h}{\partial x} = c_2 \left(\frac{\partial h}{\partial t} + v \frac{\partial h}{\partial x} \right) + v \frac{\partial h}{\partial x} + c_1 h + c_3 \quad (10)$$

Thus, (8)–(10) are necessary and sufficient conditions for stochastic differential Eq. (2) to be linearizable. We now derive the linearization criteria imposed on the coefficient functions f , g and j in detail. Differentiating (9) and (10) with respect to v , we obtain that $\frac{\partial g}{\partial v} = b_2$ and $\frac{\partial j}{\partial v} = c_2$, hence the functions $g(t, x, v)$ and $j(t, x, v)$ have to be of the form

$$g = b_2 v + \psi \quad \text{and} \quad j = c_2 v + \phi, \quad (11)$$

where $\psi = \psi(t, x)$ and $\phi = \phi(t, x)$. In particular, $\frac{\partial^2 g}{\partial x \partial v} = 0$, $\frac{\partial^2 g}{\partial v \partial v} = 0$, $\frac{\partial^2 j}{\partial x \partial v} = 0$, $\frac{\partial^2 j}{\partial v \partial v} = 0$. Differentiating (8) three

times with respect to v to obtain

$$\begin{aligned} 2 \frac{\partial^2 h}{\partial x^2} v + 2 \frac{\partial^2 h}{\partial t \partial x} - a_2 \frac{\partial h}{\partial x} + \frac{\partial h}{\partial x} \frac{\partial f}{\partial v} &= 0 \\ 2 \frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial x} \frac{\partial^2 f}{\partial v^2} &= 0 \\ \frac{\partial h}{\partial x} \frac{\partial^3 f}{\partial v^3} &= 0 \end{aligned}$$

Since $\frac{\partial h}{\partial x} \neq 0$, one finds that $\frac{\partial^3 f}{\partial v^3} = 0$ or

$$f = f_2 v^2 + f_1 v + f_0, \quad (12)$$

where $f_i = f_i(t, x)$ ($i = 0, 1, 2$). Notice that knowledge of the functions $j(t, x, v)$, $g(t, x, v)$ and $f(t, x, v)$ uniquely determines the functions $b_2(t)$, $c_2(t)$, $\psi(t, x)$, $\phi(t, x)$ and $f_i(t, x)$. We first substitute (12) into (8) and then we rearrange to obtain

$$\begin{aligned} \frac{\partial^2 h}{\partial x^2} &= -f_2 \frac{\partial h}{\partial x}, \quad \frac{\partial^2 h}{\partial t \partial x} = \frac{a_2 - f_1}{2} \frac{\partial h}{\partial x}, \\ \frac{\partial^2 h}{\partial t^2} &= a_2 \frac{\partial h}{\partial t} - f_0 \frac{\partial h}{\partial x} + a_1 h + a_3. \end{aligned} \quad (13)$$

Differentiating $\frac{\partial^2 h}{\partial t \partial x}$ and $\frac{\partial^2 h}{\partial t^2}$ with respect to t and x , respectively, and using (13) and $\frac{\partial}{\partial t} \left(\frac{\partial^2 h}{\partial t \partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial^2 h}{\partial t^2} \right) = 0$, one obtains

$$a_1 = \frac{(2b_1' + 4 \frac{\partial f_0}{\partial x} - 2 \frac{\partial f_1}{\partial t} - a_2^2 - 4f_0 f_2 + f_1^2)}{4} \quad (14)$$

We next differentiate $\frac{\partial^2 h}{\partial t \partial x}$ and $\frac{\partial^2 h}{\partial x^2}$ with respect to x and t , respectively, and again using the above identities, we get

$$\frac{\partial f_1}{\partial x} = 2 \frac{\partial f_2}{\partial t} \quad (15)$$

Since a_1 is independent of x , differentiation of (14) with respect to x and substitution of (15) yield

$$\frac{\partial^2 f_0}{\partial x^2} = f_2 \frac{\partial f_0}{\partial x} + \frac{\partial^2 f_2}{\partial t^2} - f_1 \frac{\partial f_2}{\partial t} + f_0 \frac{\partial f_2}{\partial x} \quad (16)$$

Suppose now, that Eq. (2) can be linearized. As the form of the coefficient g and j is known, Eqs. (9) and (10) becomes

$$\psi \frac{\partial h}{\partial x} - b_2 \frac{\partial h}{\partial t} - b_1 h - b_3 = 0 \quad (17)$$

$$\phi \frac{\partial h}{\partial x} - c_2 \frac{\partial h}{\partial t} - c_1 h - c_3 = 0 \quad (18)$$

Differentiating Eqs. (17) and (18) with respect to x and using (13), we obtain

$$b_1 = \frac{1}{2} \left(2 \frac{\partial \psi}{\partial x} - a_2 b_2 + b_2 f_1 - 2 f_2 \psi \right) \quad (19)$$

$$c_1 = \frac{1}{2} \left(2 \frac{\partial \phi}{\partial x} - a_2 c_2 + c_2 f_1 - 2 f_2 \phi \right). \quad (20)$$

Since $\frac{\partial b_1}{\partial x} = 0$, $\frac{\partial c_1}{\partial x} = 0$ and because of (15), one has

$$\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} - f_2 \psi \right) + b_2 \frac{\partial f_2}{\partial t} = 0, \quad (21)$$

$$\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} - f_2 \phi \right) + c_2 \frac{\partial f_2}{\partial t} = 0. \quad (22)$$

By the above equations one can get

$$c_2 \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} - f_2 \psi \right) = b_2 \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} - f_2 \phi \right) \quad (23)$$

Since $\frac{\partial b_2}{\partial x} = 0$ and $\frac{\partial c_2}{\partial x} = 0$, we can integrate (23) to obtain

$$\frac{\partial}{\partial x} (c_2 \psi - b_2 \phi) - f_2 (c_2 \psi - b_2 \phi) = \alpha(t). \quad (24)$$

In order to solve this differential equation, we can use the integrating factor e^β where $\beta(t, x) = -\int f_2(t, x) dx$. Then the solution becomes

$$c_2 \psi - b_2 \phi = e^{-\beta(t, x)} \left(\alpha(t) \int e^{\beta(t, x)} dx + \gamma(t) \right). \quad (25)$$

This leads to

$$\phi = \frac{c_2}{b_2} \psi - \frac{e^{-\beta(t, x)}}{b_2} \left(\alpha(t) \int e^{\beta(t, x)} dx + \gamma(t) \right). \quad (26)$$

Using (19) and (20) in (17) and then in (18), we obtain

$$-h \frac{\partial \psi}{\partial x} - b_2 \frac{\partial h}{\partial t} + \psi \frac{\partial h}{\partial x} + \frac{a_2 - f_1}{2} b_2 h + f_2 \psi h - b_3 = 0 \quad (27)$$

$$-h \frac{\partial \phi}{\partial x} - c_2 \frac{\partial h}{\partial t} + \phi \frac{\partial h}{\partial x} + \frac{a_2 - f_1}{2} c_2 h + f_2 \phi h - c_3 = 0 \quad (28)$$

We now substitute (26) into (28) to obtain

$$\begin{aligned} & \frac{c_2}{b_2} \left(-h \frac{\partial \psi}{\partial x} - b_2 \frac{\partial h}{\partial t} + \psi \frac{\partial h}{\partial x} + \frac{(a_2 - f_1)}{2} b_2 h + f_2 \psi h \right) \\ & + \frac{e^{-\beta(t, x)}}{b_2} \left(\alpha h - \left[\gamma + \alpha \int e^{\beta(t, x)} dx \right] \frac{\partial h}{\partial x} \right) - c_3 = 0 \end{aligned}$$

Using (27), we get

$$\begin{aligned} & \left[e^{-\beta(t, x)} \left(\gamma + \alpha \int e^{\beta(t, x)} dx \right) \right] \frac{\partial h}{\partial x} - \alpha h \\ & - (c_2 b_3 - b_2 c_3) = 0 \end{aligned} \quad (29)$$

Thus far, we have shown that identities (11), (12), (15), (16), (21) and (22) are necessary conditions for linearizability of Eq. (2). The value of the coefficient b_2 and c_2 is already determined by (11), while the coefficients a_1 , b_1 , c_1 and a_2 of the linearized equation are required to satisfy the relations (14), (19) and (20). In order to obtain sufficiency conditions, assume that all the identities listed above are satisfied. The relations (14), (19) and (20) leave three degrees of freedom for the choice of coefficients of the linearized equation. Observe that Eqs. (8), (9) and (10) are equivalent

to (13), (17) and (18), respectively, and first equation of (13) is equivalent to (29), further analysis of compatibility of this overdetermined system of equations depends on the value of $\alpha(t)$ and $\gamma(t)$ and then the values of b_2 and c_2 as determined by (11). Hence we have proven following theorem.

Theorem 1 A second-order stochastic jump-diffusion equation

$$d\dot{X} = f(t, X, \dot{X})dt + g(t, X, \dot{X})dW + j(t, X, \dot{X})dN$$

is linearizable via an invertible transformation if and only if

$$f(t, x, v) = f_2(t, x)v^2 + f_1(t, x)v + f_0(t, x),$$

$$g(t, x, v) = b_2(t)v + \psi(t, x),$$

$$j(t, x, v) = c_2(t)v + \phi(t, x),$$

where the functions f_i , g , j , ψ , ϕ , b_2 and c_2 satisfy the conditions (15), (16), (21) and (22).

$$\frac{\partial f_1}{\partial x} = 2 \frac{\partial f_2}{\partial t}, \quad \frac{\partial^2 f_0}{\partial x^2} = f_2 \frac{\partial f_0}{\partial x} + \frac{\partial^2 f_2}{\partial t^2} - f_1 \frac{\partial f_2}{\partial t} + f_0 \frac{\partial f_2}{\partial x} \quad (30)$$

$$\begin{aligned} & \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} - f_2 \psi \right) + b_2 \frac{\partial f_2}{\partial t} = 0, \quad \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} - f_2 \phi \right) \\ & + c_2 \frac{\partial f_2}{\partial t} = 0. \end{aligned} \quad (31)$$

3 Applications

We now consider some linearizable nonlinear second-order jump-diffusion equations. Then we give a special case where the functions f , g and j depend only t and $\dot{x}$. For this case we will use the linearization criteria that is given in [7]. Our first example reads,

$$\begin{aligned} d\dot{X} = & \left[p(t)\dot{X}^2 + (q(t) + 2p'(t)X)\dot{X} + f_0(t, X) \right] dt \\ & + (\sigma \dot{X} + \psi(t, X))dW + \eta dN \end{aligned} \quad (32)$$

where

$$f_0 = r(t)e^{px} + \frac{(p')^2}{p}x^2 + \left(\frac{qp' - p''}{p} + 2\left(\frac{p'}{p}\right)^2 \right)x + s(t)$$

and

$$\psi = \sigma(1 + px) \frac{p'}{p^2}.$$

It is seen that the functions f_i , g , j , ψ , ϕ , b_2 and c_2 satisfy the linearization conditions and from (24) and (25), we have $\alpha(t) = \sigma \eta p$ and $\gamma(t) = 0$. A linearizing transformation is given by

$$y = h(t, x) = \frac{1}{p(t)} e^{\int \frac{-q(t)}{2} dt - p(t)x}$$

which yields an equivalent linear stochastic differential equation

$$d\dot{Y} = \left(a_1(t)Y - r(t)e^{\int \frac{-q(t)}{2} dt} \right) dt + \left(\frac{\sigma}{2} q(t)Y + \sigma \dot{Y} \right) dW - \eta p(t)Y dN$$

where

$$a_1 = \left(\frac{qp' - p''}{p} + 2 \left(\frac{p'}{p} \right)^2 \right) - \frac{q'}{2} + \frac{q^2}{4} - sp.$$

The second example reads as

$$d\dot{X} = \left[p(x)\dot{X}^2 + q(t)\dot{X} + f_0(t, X) \right] dt + \sigma \dot{X} dW + \eta \dot{X} dN \quad (33)$$

where

$$f_0 = \frac{1}{\Gamma(x)} \left[r(t) + s(t) \int \Gamma(x) dx \right]$$

with

$$\Gamma = e^{-\int q(x) dx}.$$

It is easy to verify that the linearization conditions are satisfied. The transformation

$$y = h(t, x) = \int \Gamma(x) dx$$

maps the nonlinear equation to the equation

$$d\dot{Y} = \left(s(t)Y + q(t)\dot{Y} + r(t) \right) dt + \sigma \dot{Y} dW + \eta \dot{Y} dN$$

The third example we consider here is a special case of the Eq. (2) where the functions f , g and j depend on t and $\dot{x}$. For this case, we will use the linearization conditions given in [7]. It is

$$d\dot{X} = \left[a\dot{X}^{3/4} + \frac{3}{8}b^2\dot{X}^{1/2} \right] dt + b\dot{X}^{3/4} dW + \left[4b\dot{X}^{3/4} + 6b^2\dot{X}^{2/4} + 4b^3\dot{X}^{1/4} + b^4 \right] dN \quad (34)$$

It can be shown that the Eq. (34) satisfy the linearization conditions that are given in [7]. Hence, the transformation

$$y = h = \frac{4}{b}\dot{x}^{1/4}$$

linearizes the Eq. (34) into

$$d\dot{Y} = \frac{a}{b}dt + dW + 4dN.$$

Integration gives

$$\dot{Y} = \frac{a}{b}t + W + 4N.$$

Hence,

$$\dot{X} = \left(\frac{a}{b}t + W + 4N \right)^4.$$

Then we obtain the solution process as

$$X(t) = \int_0^t \left(\frac{a}{b}s + W(s) + 4N(s) \right)^4 ds.$$

4 Conclusions

In this paper, we have shown that when certain conditions are satisfied, nonlinear second-order jump-diffusion equations are linearizable via invertible transformations. Certain nonlinear equations are shown to be linearizable via invertible transformations.

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