Polytopes

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Abstract Considering polyhedra as particular cases of objects known as polytopes, which are the analogies of polyhedra in spaces with dimensions greater than three, makes it possible to discover characteristics that were not immediately apparent before. The method used by Alicia Boole Stott to discover the sections of four-dimensional polytopes relied on these analogies. The method of sections also makes it possible to discover a great number of connections between the theory of polytopes and other branches of mathematics.

Keywords Polytopes · Alicia Boole Stott · Models

1 The generation of polyhedra and polytopes

Polyhedra are beautiful, interesting and concrete objects that have attracted the attention of mathematicians since Ancient Greece (see, for instance, [4]). They are among the mathematical objects most studied and most often considered by non-professional mathematicians as well, as hundreds of Internet sites about them show.

At first glance, polyhedra seem to constitute a chaotic, unorganised set of unconnected exemplars. There are, it is true, the regular polyhedra—Archimedean solids, Catalan solids, etc.—but the relationships between the different polyhedra and their various classes are often quite obscure.

One possible way to discover their connections is, as often happens in mathematics, that of considering them as particular cases of more general objects, the *polytopes*, which are the analogies of polyhedra in spaces with dimensions greater than three. The idea of considering higher-dimensional spaces, the so-called *hyperspaces*, made a very strong appearance on the scene in various fields of mathematics in the nineteenth century.

The 1800s was a century that witnessed a radical renovation of geometry: new disciplines were born, such as projective geometry and non-Euclidean geometry, and new possible geometric universes were conceived: a non-Euclidean universe, and one with space with more than three dimensions. Around 1850, Swiss mathematician Ludwig Schläfli conceived the idea of polytopes. Schläfli classified and studied regular polytopes. In three dimensions we know the regular tetrahedron, the octahedron, the cube (or hexahedron), the dodecahedron and the icosahedron. In four dimensions there are six regular polyhedra, and in dimensions higher than four there are only three for each dimension.

Polytopes can be defined analytically by assigning coordinates to the vertices in the space of the n -tuple real numbers, which constitutes the analytical model of the affine n -dimensional space. However, in order to be truly useful, polytopes can and must be conceived and intuited synthetically. Let us briefly explain how, beginning with a three-dimensional example.

A cube can be thought of as generated dynamically by the translation of a square along the direction perpendicular to its plane (Fig. 1).

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Analogously, we can think of a *hypercube* as being generated by translating the cube in the direction perpendicular to its space, that is, along a *fourth dimension* that

Fig. 1 A cube can be generated by translating the *red square* vertically (in the direction of the *green vectors*). The *red square*, in its translational movement to reach the *blue square*, generates the cube. Drawing: author (colour figure online)

Fig. 2 The hypercube is generated by translating the *red cube* in the direction perpendicular to its space (along the *green vectors*). The *red cube*, in its translational movement to reach the *blue cube*, generates the hypercube. Drawing: author (colour figure online)

Fig. 3 The hypercube is sections by a family of parallel hyperplanes. These two figures show the plane projections of two sections that are regular tetrahedra. Sectioning with more still more hyperplanes results in octahedra and then more tetrahedra. Alicia Boole Stott determined the hyperplane sections of all the regular four-dimensional polytopes. Drawing: author

we can imagine, but cannot see. The diagram in Fig. 2 represents the projection on the plane of the process of generating the hypercube, analogous to that seen in Fig. 1 for generating the cube.

To imagine more complicated polytopes, it is possible to consider a more general process of generation. Instead of translating a fixed polyhedron, we can imagine translating a variable polyhedron. As before, let's begin with a three-dimensional case. Let's first imagine selecting a cube by means of a family of parallel planes. We can then imagine reconstructing the cube by gluing together the plane sections that we obtained in the first step.

In the case of a polytope in four-dimensional space, while we cannot materially section an object that does not exist, we can still imagine such an object as being constructed by gluing together a suitable family of polyhedra that constitute its hyperplane sections. To understand this idea, it will be helpful to look at Fig. 3.

The problem of reconstructing an object from its sections is vividly described in the 1884 book by Abbott [1] with regard to the three-dimensional case and constitutes a very helpful educational exercise for developing geometrical intuition and the capability to work with three-dimensional objects. It is much more difficult to apply geometrical intuition to polytopes beginning with their hyperplane sections. Alicia Boole Stott studied the hyperplane sections of a polytope systematically [7].

2 The sections of regular polyhedra and polytopes in the work of Alicia Boole Stott

As we said, it is possible to obtain information about a solid (both in 3D and in 4D) by calculating its sections. In Alice Boole Stott's work of 1900 [3], she calculates the three-dimensional sections of six regular polytopes resulting from the intersection of a four-dimensional object with particular three-dimensional spaces. In order to clarify this

Fig. 4 The cube is sectioned with a family of parallel planes. Note that the different polygonal sections are not only different in size, but also in shape. We go from a *triangle* in the *first two figures* to a

hexagon in the *third*, to then go back to *triangles* as we continue to move the plane with which we perform the section parallel to itself. Drawing: author

Fig. 5 An unfolded tetrahedron and its parallel sections. Drawing: author

method, let us first look at its applications to the three-dimensional case.

The fundamental idea behind Alicia Boole Stott's method consists in unfolding an object in a lower dimension and working on the sections of the new drawing. For example, in a tetrahedron we can either calculate the sections parallel to one of its faces, in the way that we described previously for a cube (see Fig. 4), or we can unfold it in a lower dimension, as follows: let us consider a plane that goes through one of the faces of the tetrahedron. Clearly, the intersection of the plane and the tetrahedron will be a triangle of the same size as the face. In an unfolded tetrahedron (Fig. 5), that section is the triangle with vertex *a*. For the next section, the plane is moved parallel to this triangle until it passes through point *b*. In the unfolded figure, the edges of the triangle are moved parallel by the same distance until they pass through *b*, again forming a triangle but of a smaller size.

Although the application of this method to a three-dimensional example may appear trivial, it enormously facilitates the visualisation and calculation of the sections when we are working with four-dimensional polytopes. Alicia Boole Stott's method for calculating the three-dimensional sections of these polytopes is based on

unfolding them in the third dimension and considering the parallel nature of their cells where we can more easily use our intuition of three-dimensional space. For example, in order to calculate the three-dimensional sections of a 24-cell, Alicia Boole Stott proceeds as follows. She depicts part of an unfolded polytope (four octahedra in this case) and calculates the first parallel section, which is the result of the intersection with the three-dimensional space containing the octahedron ABCDEFGH (shown in red in Fig. 6, left). She then calculates the second section of the intersection of the 24-cell and a three-dimensional space which is parallel to the first and passes through point *a*. This gives a rectangle and a hexagon in the octahedron depicted (shown in green in Figs. 6 and 7). The central three-dimensional section is obtained in the same manner. We can find the remaining sections using the symmetry of the 24-cell.

The method of sections also makes it possible to discover a great number of connections between the theory of polytopes and other branches of mathematics. For example, if we identify the planes of a bundle of parallel planes with a real parameter *t*, we can consider how the volume *a(t)* of the corresponding hyperplane section of a fixed polytope varies as *t* varies. The functions *a(t)* are *spline functions*, and are characterised by very interesting combinatorial properties, of great usefulness in numerical analysis. In the case of the hypercube, these spline functions are *box splines* [5].

The geometric link between polytopes and hyperplane sections has also been amply exploited to explore the combinatorial relationships, which have numerous applications in number theory [6]. Many of these relationships, at least in an initial form, are accessible to high school students, and suggest interesting educational pathways for deeper investigation, and for introducing contemporary mathematics [2].

Fig. 6 Calculation of the three-dimensional sections of a 24-cell. Original drawing by Alicia Boole Stott [1900] with colour overlays by the author (colour figure online)

Fig. 7 Second section of the 24-cell

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