“Reasoning well from badly drawn figures”: the birth of algebraic topology

Claudio Bartocci

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Abstract In this paper the emergence of Poincaré’s “analysis situs” is described by means of an overview of the original memoir and its supplements. In particular, the genesis of the celebrated “Poincaré conjecture” is discussed.

Keywords Henri Poincaré · Homology · Fundamental group · Poincaré conjecture

1 Reasoning with figures

In December 1908 Henri Poincaré sent a few lines in answer to a letter from Doctor Édouard Toulouse (1867–1947), an ‘alienist’ and psychologist: ‘Dear Sir, I happily authorise you to publish the observations you have made about me’. These ‘observations’ had been conducted 12 years previously in the context of research that Toulouse was carrying out to ‘clarify the relationships of intellectual superiority to neuropathy’ [44]. The method used consisted in subjecting several ‘men of genius’—to use an expression of the Italian criminologist Cesare Lombroso, at the height of his fame at the time—to a series of anthropometric measurements and physiological and psychological tests. Among the subjects studied by Toulouse, a careful and attentive experimenter, were Émile Zola, Pierre Berthelot and, of course, Poincaré. During the numerous tests of memory, Poincaré was asked to reproduce a simple geometric figure: after having observed it for 5 s, the ‘living brain of the rational sciences’—as Paul

Painlevé once defined him—could do no better than scribble down two clumsy attempts (Fig. 1).

Poincaré was never brilliant in drawing: according to the accounts of some of his biographers, his lack of skill might even have jeopardised his own admittance to the École Polytechnique (he earned a perfect 0 in the test of *lavis*) had it not been for the farsightedness of his examiners who, impressed by the results obtained by the candidate in other subjects, changed the 0 into a 1. In spite of this (or perhaps even because of it), figures—and in particular those ‘drawn badly’—play an essential role in the articulation and development of Poincaré’s geometric thinking, as he himself explains, not without a note of self-irony, in the introduction to his memoir ‘Analysis situs’ [7]:

We know how useful geometric figures are in the theory of imaginary functions and integrals evaluated between imaginary limits, and how much we desire their assistance when we want to study, for example, functions of two complex variables.

If we try to account for the nature of this assistance, figures first of all make up for the infirmity of our intellect by calling on the aid of our senses; but not only this. It is worth repeating that geometry is the art of reasoning well from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset.

The use of figures is, above all, then, for the purpose of making known certain relations between the objects that we study, and these relations are those which occupy the branch of geometry that we have called Analysis situs, and which describes the relative

C. Bartocci (✉)
Dipartimento di Matematica, Università di Genova,
Via Dodecaneso 35, 16146 Genoa, Italy
e-mail: bartocci@dima.unige.it

Fig. 1 On the *left*, the figure shown to Poincaré by Toulouse; in the *centre* and *right*, Poincaré's two attempts at reproducing it

situation of points and lines on surfaces, without consideration of their magnitude.

The fact that relations of the same nature hold between the objects of hypersurface, so that there is then an *Analysis situs* of more than three dimensions, is due, as we have shown, to Riemann and Betti.

This science enables us to know the nature of these relations, although this knowledge is less intuitive, since it lacks a counterpart in our senses. Indeed, in certain cases it renders us the service that we ordinarily demand of geometrical figures [7, p. 2; Engl. trans. p. 6].

In effect, looking at Poincaré's vast and multifaceted production as a whole, it is striking how often the French mathematician resorts to figures—above all in works of analysis—both as useful auxiliaries for studying certain particular cases, and as efficacious tools for proof (as, for example, in the theory of limit cycles developed in his '*Mémoire sur les courbes définies par une équation différentielle (deuxième partie)*':... *l'inspection de la figure le démontre* ('...the inspection of the figure proves...') [2, p. 56]. On some occasions these two functions meld into one (as in the work '*Théorie des groupes fuchsien*s' [3]), while on others they remain separate for the precise reason that the author has not yet found the key to unravelling the tangled skein of specific examples:

What embarrasses me is the fact that I will be forced to insert many figures, precisely because I have not yet been able to obtain a general rule, but have only accumulated the particular solutions.¹

Finally, there is at least one case in which the figure turns out to be impossible to visualise—even for the most visionary geometric imagination of the time—and thus impossible to draw. In the third and final volume of *Les méthodes nouvelles de la mécanique céleste* [10], studying the intersections of two curves belonging to what we today call stable and unstable manifolds of a periodic solution, Poincaré runs up against a behaviour that is typically chaotic:

Let us attempt to have an idea of the figure formed by these two curves and their intersections, which are infinite in number and each of which corresponds to a solution that is doubly asymptotic; these intersections form a type of trellis, tissue or grid with an infinitely dense mesh; neither of the two curves must intersect itself, but must fold over on itself in a very complex manner so as to intersect all of the meshes of the grid. The complexity of this figure, which I will not even attempt to draw, is striking [10, p. 389].

2 In search of invariants

In the first half of the 1880s, in order to study curves defined by differential equations (the results appeared in [1, 2, 4]), Poincaré adopted a 'new point of view', which he himself defined 'qualitative'. Thanks to this change in perspective, he was able to obtain results that, using a classical quantitative approach, would have been almost impossible even to imagine: think, for example, of the 'index theorem', according to which, for a vector field on a surface of genus g , the number N of the nodes, the number F of the foci, and the number C of the saddle points fulfil the relation:

$$N - F + C = 2g - 2.$$

This same qualitative strategy also allowed him, just a few years later, to open a breach—while studying the perturbations of periodic orbits—in the otherwise impregnable fortress of the three body problem.²

It is therefore no coincidence that in the informal definition of *analysis situs* that Poincaré gives in the '*Analyse de ses travaux...*' [16] we find precisely the adjective 'qualitative':

Analysis situs is the science that allows us to know the qualitative properties of geometric figures, not only in ordinary space but in space of more than three dimensions.³

While *analysis situs* in three dimensions is, in Poincaré's judgment, 'an almost intuitive knowledge',⁴ enormous difficulties present themselves in the extension of it concepts in higher dimensions: to attempt to overcome these it is thus necessary 'to be profoundly convinced of the great

¹ See the letter from Poincaré to Guccia, 9 December 1911, in [35, p. 296].

² We refer the reader to J.-C. Yoccoz, 'Une erreur féconde du mathématicien Henri Poincaré', in this same issue.

³ 'L'Analysis Situs est la science qui nous fait connaître les propriétés qualitatives des figures géométriques non seulement dans l'espace ordinaire, mais dans l'espace à plus de trois dimensions' [16, p. 100].

⁴ 'une connaissance presque intuitive' [16, p. 100].

⁵ 'il faut pour tenter de les surmonter être bien persuadé de l'extrême importance de cette science' [16, p. 100].

importance of this discipline'.⁵ In Poincaré there is certainly no lack of such a profound conviction:

As for me, all of the diverse paths on which I was successively engaged led me to *analysis situs*. I had need of the ideas of this science to pursue my studies on curves defined by differential equations ... and in particular for those of the three body problem. I had need of it for the study on multivalued functions of two variables. I had need of it for the study of periods of multiple integrals and for the application of this study to the development of the perturbation function. Finally I glimpsed in *Analysis Situs* a means of attacking an important problem in the theory of groups, the search for discrete or finite groups contained in a given continuous group.⁶

But what are, in concrete terms (if possible), the objects that *analysis situs* is supposed to deal with? A geometry—Poincaré had previously observed in a paper of 1880 (completely independent, it is worthwhile underlining, of Klein's Erlangen program)—is 'the study of a group of operations comprised of the displacements to which a figure can be subject without deforming it'.⁷ Analogously, *analysis situs* is the 'science whose object of study is the group (of the homeomorphisms)' [7, p. 7; Eng. trans. p. 9], where by 'homeomorphisms' is meant an application between two manifolds that is bijective, differentiable and whose inverse is differentiable (that is, what today is called a 'diffeomorphism'). In other words, the crux of the question lies in being able to determine, for each 'variety', suitable quantities associated with it that remain 'invariant' when the variety undergoes a differentiable deformation.

In the case of surfaces that are closed⁸ and oriented, the problem of the invariants had been solved for some time. In 1863 August Ferdinand Möbius—at the respectable age of 73 years—studied closed surfaces in Euclidean space [33], sectioning them into 'primitive forms' (that is, disks,

cylinders or unions of cylinders) by means of parallel planes; between two surfaces it is then possible to establish an 'elementary correlation' (*elementar Verwandtschaft*, that is, more or less, diffeomorphism) if and only if they belong to the same class, which is identified by a non-negative integer. This integer g corresponds to the number of holes of the surface and is connected to the surface's Euler characteristic χ (which can be more easily defined by choosing any triangulation whatsoever) according to the formula $\chi = 2 - 2g$. Some years earlier, Bernhard Riemann, in his doctoral thesis (*Inaugural dissertation*) entitled 'Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse' [38, pp. 35–80] had introduced a completely original geometric concept—the Riemann surfaces—establishing new and unexpected relations between complex analysis, topology and the theory of algebraic curves. In effect, every algebraic curve can be associated with a Riemann surface, that is, the branched cover of the complex plane determined by the given curve: the genus of the curve (defined in terms of the meromorphic functions on it) coincides with the 'order of connectivity' of the curve (which is instead defined in topological terms).⁹ If Möbius had been aware of Riemann's results,¹⁰ he would not have failed to see that his class number corresponded exactly to Riemann's 'order of connectivity'.

Classification theorems for the surfaces equivalent to that of Möbius were obtained, with innovative strategies for proving them, by Camille Jordan [29] and, successively, by Clifford [20]. In a different line of research, Enrico Betti, profoundly influenced by Riemann, generalised the concept of order of connectivity to spaces of higher dimensions: in the memoir 'Sopra gli spazi di un numero qualunque di dimensioni' [19], to every space of dimension n are associated $n - 1$ integers $p_1, \dots, p_{n-1}$, 'orders of connectivity', which later mathematicians would call 'Betti numbers'.

Are Betti numbers 'invariants' in the same sense in which Riemann's order of connectivity is? If so, can they be used to obtain classification theorems for higher-dimensional geometric spaces analogous to the classification theorem for surfaces? These two questions, completely natural in the theoretical context of geometry in the early 1880s, were formulated with crystalline clarity in 1884 by Walther von Dick for the case of dimension 3:

The object is to determine certain characteristic numbers for closed threedimensional spaces, analogous to those introduced by Riemann in the theory of his surfaces, so that their identity shows the possibility of (establishing a) 'one-to-

⁵ 'il faut pour tenter de les surmonter être bien persuadé de l'extrême importance de cette science' [16, p. 100].

⁶ 'Quant à moi, toutes les voies diverses où je m'étais engagé successivement me conduisaient à l'Analysis Situs. J'avais besoin des données de cette science pour poursuivre mes études sur les courbes définies par les équations différentielles ... et en particulier à celles du problème des trois corps. J'en avais besoin pour l'étude des fonctions non uniformes de 2 variables. J'en avais besoin pour l'étude des périodes des intégrales multiples et pour l'application de cette étude au développement de la fonction perturbatrice. Enfin j'entrevois dans l'Analysis Situs un moyen d'aborder un problème important de la théorie des groupes, la recherche des groupes discrets ou des groupes finis contenus dans un groupe continu donné' [16, p. 101].

⁷ The quote is taken from the first of the three *Suppléments* (unpublished) to the memoir written for the Grand Prix des Sciences Mathématiques [17] and cited in [24, p. 225].

⁸ A surface (that is, a topological manifold of dimension 2) is said to be closed if it is compact and has no boundary.

⁹ The result is proved in [38]. For further details, we refer the reader to [31] and [18].

¹⁰ That this was not the case is argued in a completely convincing way in [37, p. 97].

one geometrical correspondence' (between the two spaces).¹¹

An analogous question is found expressed in the first work by Poincaré that was explicitly devoted to the subject of *Analysis situs*—a note barely four pages long presented in 1892 to the Académie des Sciences:

One may ask whether the Betti numbers suffice to determine a closed surface from the viewpoint of *Analysis situs*. That is, given two surfaces with the same Betti numbers, we ask whether it is possible to pass from one to the other by a continuous deformation [6, pp. 189–190; English trans. p. 15].

Poincaré also gives the answer: no. He is in fact able to construct a family of three-dimensional spaces (manifolds) whose Betti numbers can only assume whole values that are less than or equal to 4, and which nonetheless contain an infinite number of non-diffeomorphic manifolds: this fact is proven by recourse to a new invariant, the 'fundamental group'.

3 Analysis situs: an overview

The intuitions contained in his note of 1892 [6] were developed by Poincaré in a monumental memoir of more than a hundred pages, 'Analysis Situs', published in 1985 in the *Journal de l'École Polytechnique* [7], a work whose extraordinary originality never ceases to amaze, even today: Jean Dieudonné, certainly not one well-disposed towards a mathematician who represented the antithesis of the Bourbakist ideal, called it a 'fascinating and exasperating paper' [21, p. 28]. Here I will not provide a detailed summary, but will rather confine myself to highlighting some of the key concepts contained in this memoir.¹²

A. In §§1–4 (sections that are certainly not perspicuous) Poincaré defines the spaces that he proposes to study. In effect, he gives two different definitions. According to the first, manifolds are suitable subspaces of $\mathbf{R}^N$, determined by systems of equations and inequalities; instead, according to the second, the spaces to be studied are 'continuous networks [*réseaux*] of manifolds'. According to him, the second definition is 'broader than the first' [7, p. 11; Eng. trans. p. 11], because it makes it possible to consider non-orientable spaces as well. In the second section he introduces the concept of homeomorphism, which is implicitly extended in the 'continuous network of manifolds'.

B. §§5 and 6 provide the concept of homology and the definition of Betti numbers. Following the formulation of Riemann and Betti, Poincaré writes that the manifolds $v_1, \dots, v_\lambda$ satisfy the 'homology'

$$v_1 + \dots + v_\lambda \sim 0$$

if they constitute the 'complete boundary' of dimension $q - 1$ of a manifold W of dimension q that is part of manifold V of dimension p ($p \geq q$). The next step—which follows *ex abrupto*, but effectively constitutes a significant conceptual leap—consists in considering 'homologies' of the type

$$k_1 v_1 + k_2 v_2 \sim k_3 v_3 + k_4 v_4,$$

where the coefficients k are positive or negative integers. This relation means that:

There exists a manifold W of dimension q forming part of V , the [complete] boundary of which is composed of k_1 manifolds similar to v_1 , k_2 manifolds similar to v_2 , k_3 manifolds similar to v_3 but oppositely oriented and k_4 manifolds similar to v_4 but oppositely oriented [7, p. 18–19; Eng. trans. p. 13].¹³

A finite set of manifolds are said to be 'linearly independent' if they are not bound by any relation of homology. From this notion derives the 'definition' of order of connectivity:

If there exist P_{m-1} closed manifolds of m dimensions which are linearly independent and form part of V , but not more than P_{m-1} , then we shall say that the connectivity of V with respect to manifolds of m dimensions is equal to P_m [7, p. 19; Eng. trans., p. 15].

These numbers P_k are called 'Betti numbers'; in actual fact, as emerges from the critical observations of the then-young Danish mathematician Poul Heegaard, Poincaré's definition does not coincide with that given by Betti in his article of 1871 [19].

C. §7 provides an interpretation of Betti numbers as the maximum number of 'periods' of multiple integrals that satisfy certain conditions of integrability (set forth in the memoir entitled 'Sur les résidus des intégrales doubles' [5]).¹⁴

¹³ The opposite manifold of an oriented manifold v is v itself with the opposite orientation. Instead, Poincaré does not specify what is meant by the expression 'manifold similar to'; in the language of modern algebraic topology, we would be tempted to interpret this expression in the sense of 'manifold isotopic to', but we are well aware of how insidious these exegetic exercises are.

¹⁴ Poincaré's results can easily be reformulated in the language of differential geometry today by means of the concept of closed (and not exact) differential form: in fact, these ideas would remain dormant until Élie Cartan's *Leçons sur les invariants intégraux* of 1922 and the work of Georges de Rham of 1931 (so much so that we speak of de Rham cohomology and not Poincaré cohomology, as is perhaps more correct); see [30].

¹¹ Quoted in [43, p. 556].

¹² For further details, see [41], [21, chap. I] and [40].

D. The famous duality theorem is stated and ‘proved’ in §9:

For a closed manifold the Betti numbers equally distant from the ends of the sequence are equal [7, p. 45; Eng. trans. p. 30].

Poincaré’s argumentation, which is based on the notion of the ‘intersection number’ of two submanifolds that intersect (transversally) in a finite number of points, was in any case difficult to understand, even obscure (‘totally unconvincing’, in the words of Dieudonné [21, p. 22]), and did not fail to draw the criticism of Heegaard.¹⁵

E. In the earlier memoir entitled ‘Théorie des groupes fuchsien’ [3] Poincaré had constructed closed surfaces by ‘gluing together’ in a suitable way the sides of curvilinear polygons. In §10 of ‘Analysis situs’ he extends this procedure with the aim of ‘representing’¹⁶ three-dimensional manifolds as ‘quotient spaces’ (as we would say today) of convex polyhedra modulo equivalence relations. Poincaré examines in detail—almost with the eye of a naturalist of geometric forms—five examples: the first is the three-dimensional torus obtained by gluing, without twisting or reflecting, the opposite faces of a cube, while the last is real projective space constructed beginning with a regular octahedron.¹⁷ ‘Another mode of representation’ of a three-dimensional manifold that ‘can be applied in certain cases’ consists of *gluing together* (he uses the French verb *coller*, literally ‘to glue’) the faces of a polyhedron with recourse to a ‘properly discontinuous group’ of substitutions (§11) [7, p. 56; Eng. trans. p. 36]. In this case as well, Poincaré observes, ‘the analogy with the theory of fuchsian groups is too evident to need stressing’. The single example described is the same that was briefly in the note of 1892 [6]. Let us consider the transformations:

$$(x, y, z) \rightarrow (x + 1, y, z)$$

$$(x, y, z) \rightarrow (x, y + 1, z)$$

$$(x, y, z) \rightarrow (\alpha x + \beta y, \gamma x + \delta y, z + 1),$$

with $\alpha, \beta, \gamma, \delta$ being integers such that $\alpha\delta - \beta\gamma = 1$,¹⁸ and imagining them being used to glue together the faces of the cube in $\mathbf{R}^3$ with vertices in the origin $(0, 0, 0)$ and in the

point $(1, 1, 1)$. Performing the identifications defined by the first two ‘substitutions’ (which are independent of z) we obtain a family of tori parametrized by the segment $0 \leq z \leq 1$, in other words, a three-dimensional cylinder of two-dimensional tori. The third substitution identifies the two bases of this cylinder by means of the transformation defined by the parameters $\alpha, \beta, \gamma, \delta$: the three-dimensional manifold (Poincaré’s sixth example) that results can be thought of as a family of two-dimensional tori parametrized by a circle.¹⁹

F. The crucial strategic importance of the sixth example emerges, step by step, in the three sections that follow, §§ 12, 13 and 14, which in themselves occupy some twenty pages. In §12 is introduced the notion of fundamental group of a manifold, defined in terms of closed paths (*contours fermés*) based in an initial point M_0 : on the set of these paths, which are identified with zero if they can be reduced by deformation to a trivial ‘round trip’ course along a single path,²⁰ are given natural operations of composition and inverse. The group that thus results is considered to be a group of substitutions acting on a suitable set of functions F that are not necessarily ‘uniform’ on the manifold:

When the point M , leaves an initial point M_0 , and returns to that position after traversing an arbitrary [closed] path, it may happen that the functions F do not return to their initial values [7, pp. 61; English trans. p. 38].²¹

By means of a long and tortuous argumentation developed in §14, Poincaré proves that it is possible to choose infinite integers $\alpha, \beta, \gamma, \delta$ (such that $\alpha\delta - \beta\gamma = 1$) for which the associated three-dimensional manifolds of the sixth example have fundamental groups that are not isomorphic, and thus, are not diffeomorphic. However, the first Betti number of these manifolds (in keeping with the calculations made in §13) is always less than or equal to four. Poincaré therefore concludes:

Thus for two closed manifolds to be homeomorphic, it does not suffice for them to have the same Betti numbers [7, p. 83; English trans. p. 52].

4 Poincaré’s conjectures

In 1898 the Danish mathematician Poul Heegaard (1871–1948), who had studied in Paris (without, however,

¹⁵ As we shall see, Poincaré later provided a completely different proof of the duality theorem (based on the fact that every manifold admits a triangulation) in the first supplement to Analysis situs [9].

¹⁶ ‘There is a manner of representing manifolds of three dimensions situated in a space of four dimensions which considerably facilitates their study’ [7, p. 46; Eng. trans. p. 31].

¹⁷ It must be noted that Poincaré (inexplicably) does not give any name to the manifolds he constructs.

¹⁸ The matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is thus an element of the modular group $SL(2, \mathbf{Z})$. Poincaré is extremely familiar with this group, which plays a role of primary importance in the theory of automorphic functions; for more details, we refer the reader to [28] and [25, chap. 3].

¹⁹ More technically, this manifold is a torus fibration on the circle.

²⁰ Poincaré gives the name *lacet* (loop) to a trivial closed path.

²¹ In today’s terminology, the functions F are not defined on the manifold but on its universal cover.

meeting Poincaré) and in Gottingen [34], defended an ambitious dissertation entitled ‘Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhaeng’ (Preliminary studies towards the topological theory of connectivity of algebraic surfaces) [26, 27], in which he provided an original description of the three-dimensional manifold in terms of certain ‘diagrams’, which still constitute today one of the most effective methods for representing closed and oriented 3-manifolds. Heegaard was very critical of the results obtained by Poincaré in ‘Analysis Situs’:

‘... the theory of Reimann and Betti regarding the connections number has many shortcomings and is difficult to use in the case of manifolds whose dimension is greater than two. Poincaré ... has sought to complete it but we do not believe that he has succeeded’ [26, p. 5; 27, p. 167].

In particular, on the basis of the examples discussed, his criticisms were aimed at the duality theorem: ‘... not only is the theorem not proved; it cannot be correct’ [26, p. 72; 27, p. 220].

By March 1899 Poincaré had already been able to read the *travail très remarquable* of his young, unknown Danish colleague (moreover, he read it in the original Danish, probably aided by his knowledge of German and his own geometric intuition). In an admirable gesture of fair play, he accepted Heegaard’s objections regarding the duality theorem and admitted his errors:

These criticisms are founded in part; the theorem is not true for the Betti numbers *as defined by Betti*; this can be seen in an example cited by Heegaard, and on the other hand can also be seen in an example that I described in my Memoir. However, the theorem is true for the Betti numbers as I have defined them; I have found a proof... and will develop it in a more extended memoir.²²

This proof, based on the assumption that every manifold admits a decomposition into ‘simply connected polyhedra’ (that is, into cells, because here ‘simply connected’ means homeomorphic to a hypersphere) [9, p. 337], is completely different from that given in ‘Analysis situs’ and introduces a rather artificial dis-

tinction between Betti numbers *tout court* and reduced Betti numbers. This distinction is only elucidated in the second supplement to *Analysis situs* [11], a work of great elegance and mastery, in which is defined the key concept of ‘torsion coefficient’. Briefly said, Poincaré became aware that it might happen that the validity of the $\sum_i kv_i \sim 0$ does not imply the validity of the homology $\sum_i v_i \sim 0$ (to give an elementary example, on the projective plane there exists a closed curve C that is not a boundary—and thus $C \sim 0$ does not hold—but when traversed two times divides the projective plane into a disk and a Möbius strip—thus we have $2C \sim 0$, since $2C$ is the boundary of both the disk and the Möbius strip). To calculate the (reduced) Betti numbers taking torsion into account, Poincaré devises a combinatoric method based on *tableaux d’incidence* (tables, or matrices we would call them today, of incidence) T_q , whose elements express the relations between the polyhedra of dimension q and those of dimension $q - 1$.²³

On the last page of the second supplement Poincaré states the first version of his famous conjecture, optimistically observing that it is a theorem ‘the proof of which will require further developments’: Each polyhedron which has all its Betti numbers equal to 1 and all its tables T_q orientable is simply connected, i.e. homeomorphic to a hypersphere [11, p. 308; Eng. trans. p. 134].

Neither Poincaré nor anyone else worried about proving or disproving the truth of this theorem over the course of the 2 years that followed. In 1902 the third [12] and fourth [13] supplements to ‘Analysis situs’ were published, but we won’t dwell on those, as they primarily deal with questions related to algebraic geometry.²⁴ We need only note that Poincaré, in keeping with the ideas also carried forward by his colleague Émile Picard,²⁵ adopted the strategy of describing an algebraic surface as the total space of a family of hyperplane sections (that which today is called a ‘Lefschetz pencil’).²⁶

²³ To be more precise, given a decomposition of the manifold in question into polyhedra, the columns of the *tableau* T_q correspond to the oriented q -polyhedra and the rows to the oriented $(q - 1)$ -polyhedra; at each intersection of a row and column is placed either 0, 1, or -1 , depending whether, respectively, the $(q - 1)$ -polyhedron is not a face of the q -polyhedron, is a face of it and has the same orientation, or is a face of it but has the opposite orientation. A *tableau* is said to be orientable [*bilatère*] if it contains only chains that are null or orientable [11, p. 304]. A clear exposition, in relatively modern language, of the computation of homology groups by means of incidence matrices can be found in the *Lehrbuch der Topologie* (1934) by Herbert Seifert and William Threlfall [42, chap. 3, § 21].

²⁴ For more about Poincaré’s contributions in the field of algebraic geometry, see the article by Ciro Ciliberto, ‘Henri Poincaré and Algebraic Geometry’, in this same issue.

²⁵ See, for example, [36, cap. iv].

²⁶ The observations regarding this in [22, p. 436] are illuminating.

²² Ces critiques sont en partie fondées; le théorème n’est pas vrai des nombres de Betti tels que Betti les définit; c’est ce qui résulte d’un exemple cité par M. Heegaard; c’est ce qui résultait d’ailleurs d’un exemple que j’avais moi-même rencontré dans mon Mémoire. Le théorème est vrai, au contraire, des nombres de Betti tels que je les définis; j’en ai trouvé une démonstration qui est fondée sur la considération des polyèdres à n dimensions et que je développerai prochainement dans un Mémoire plus étendu [8, pp. 629–630].

Poincaré, who possessed a global vision of mathematics, was gifted with a phenomenal ability to transpose—operating by analogy—techniques and concepts from one theoretical context to another, even when the two fields were profoundly different from each other. Thus it comes as no surprise that in the fifth supplement to ‘Analysis situs’ [14] he succeeds in exploiting his research in the theory of algebraic surfaces in setting out a new method for studying the topological structure of real manifolds. The basic idea—in fact, the same idea on which is based what we call today ‘Morse theory’—consists in ‘slicing’ a given manifold V of dimension m immersed in $\mathbf{R}^k$ by means of a one-parameter family of $(k-1)$ -dimensional hypersurfaces, $\varphi(x_1, x_2, \dots, x_k) = t$: in this way we have a ‘system’ $W(t)$ —composed (generally) of “a certain number of $(m-1)$ -dimensional manifolds”—which, ‘when t varies continuously from $-\infty$ to $+\infty$,... varies continuously and generates the manifold V ’ [14, p. 46; Eng. trans. p. 180]. Poincaré associates a point of three-dimensional space to each connected component of $W(t)$ and obtains, as t varies, a ‘sort of network of lines’ that he calls the ‘skeleton’ (*squelette*) of the manifold V :

Under these circumstances, when t varies in a continuous manner, the points representing the p manifolds

$$w_1(t); w_2(t); \dots; w_p(t)$$

generate p continuous lines

$$L_1; L_2; \dots; L_p$$

at least as long as the number p does not change. But this number can change at $t = t_0$, if one of the manifolds decomposes into two, or if, on the contrary, two manifolds merge into one. In the first case one of the lines L bifurcates, in the second case two of the lines L combine into one [14, p. 47; Eng. trans., p. 180].

The fundamental observation, at this point, is the following:

If we follow one of these lines, L_1 for example, described by the point representing $w_1(t)$, we see that this manifold remains homeomorphic to itself ... as long as we do not pass through a value t such that $w_1(t)$ has a singular point [14, p. 47; Eng. trans., p. 180].

It is thus necessary to ‘study these singular points’, and Poincaré does not retreat. He proves the result that is called the ‘Morse lemma’ in today’s textbooks of differential topology²⁷; he proves that ‘if a two-dimensional V is ori-

entable, its skeleton will not have singular points other than *culs-de-sac* and bifurcations’ [14, p. 54; Eng. trans., p. 185]; he exploits this fact (using as well a healthy dose of hyperbolic geometry) to classify closed and orientable surfaces; and he applies the new methods to manifolds of dimension 3.²⁸

The crowning achievement of the fifth supplement is the construction, in the final four pages, of ‘the case ... where the Betti numbers and torsion coefficients are equal to 1 [and where] nevertheless V is not simply connected [14, p. 46; Eng. trans., p. 223], that is,—in Poincaré’s terminology—it is not homeomorphic to the three-dimensional sphere: the ‘theorem’ stated at the end of the second supplement is thus false. The procedure followed by Poincaré consists in considering two 3-manifolds, V' e V'' whose boundary is a surface W of genus 2 (that is, two handlebodies of genus 2) and gluing them together, identifying in a suitable way the boundary surfaces, obtaining a 3-manifold V . Given that ‘every cycle of V is equivalent [homotope] to a cycle of W [14, p. 102; Eng. trans., p. 218], to determine ‘the homologies’ and the fundamental group of V , it is sufficient to consider a base C_1, C_2, C_3, C_4 of the homology of W and to take into account the relations that derive from the gluing homeomorphism. To this end, Poincaré identifies the “fundamental cycles” K'_1 and K'_2 of V' with C_1 and C_2 . Describing surface W as shown in Fig. 2 (Fig. 4 in the fifth supplement),—which is drawn well, but behind which, in all probability, lie hidden dozens and dozens of geometric experiments on figures drawn badly²⁹—cycle C_1 is given by the ‘conjugate circles’ $-A$ and $+A$, while C_2 is given by the ‘conjugate circles’ $-B$ and $+B$; the ‘fundamental cycles’ K''_1 and K''_2 of V'' ‘are represented by the arcs of curve running between points on the perimeter’ of the figure; more precisely, ‘The arcs which represent K''_1 are shown as unbroken lines; those which represent K''_2 are dotted [14, p. 106; Eng. trans., p. 220].

Expressing cycles K''_1 and K''_2 in terms of cycles C_1, C_2, C_3, C_4 , Poincaré arrives at the relations

$$-C_2 + C_4 - C_2 + C_4 \equiv 0; \quad 5C_2 \equiv 0; \quad 3C_4 \equiv 0,$$

which constitute—as he immediately observes—‘the relations of the structure in which the substitutions C_2 and C_4 generate the *icosahedral group*’ [14, p. 110; Eng. trans., p. 224]. It is thus clear that the fundamental group of

²⁷ See, for example, [32, p. 6].

²⁸ For more on this, see [40, pp. 158–164] and [25, pp. 451–466].

²⁹ Cameron McA. Gordon has this to say: ‘It is clear that in order to arrive at his example of a nonsimply-connected homology sphere ... Poincaré must have done a good deal of experimentation with Heegaard diagrams, of genus 2, and presumably of higher genus also’ [23, p. 462].

Fig. 2 The diagram that defines Poincaré's homology sphere [14, p. 106; Eng. trans., p. 221]

V cannot be reduced to identical substitution, since it contains the icosahedral group as a subgroup.³⁰

As Poincaré said in his talk at the fourth International Congress of Mathematicians in Rome in 1908,³¹ 'There are no longer some problems solved and others unsolved, there are only problems more or less solved' [15, p. 34; Eng. trans. p. 404]. In this case as well, 'one question remains to be dealt with: Is it possible for the fundamental group of V to reduce to the identity without V being simply connected?' [14, p. 110; Eng. trans., p. 224].

So it is not a conjecture, but a simple question, one whose answer—negative—would be provided only a hundred years later, by Grigori Perelman.³²

Translated from the Italian by Kim Williams.

³⁰ For the manifold V —Poincaré's homology sphere—it is possible to provide other descriptions that are perhaps less arduous to visualise. It is possible to obtain V as a 'dodecahedral space': consider a dodecahedron (as a solid) and identify the opposite faces, subjecting each face to a clockwise twist of $2\pi/10$; this proves that the quotient manifold is homeomorphic to V . Another construction directly involves the icosahedral group, which is isomorphic to the alternating group A_5 (the group of even permutation of 5 elements): this is the only simple group with 60 elements. Given that the isometries of the icosahedron are (proper) rotations, there is a natural immersion $A_5 \subset \text{SO}(3)$. Let us consider the double cover $\text{SU}(2) \rightarrow \text{SO}(3)$ and indicate with $\Gamma \subset \text{SU}(2)$ the inverse image of A_5 . Now $\text{SU}(2)$ is homeomorphic to the three-dimensional sphere S^3 ; the quotient manifold $\text{SU}(2)/\Gamma$ has a fundamental group isomorphic to Γ . On the other hand, as a direct consequence of the fact that A_5 is simple, $[\Gamma, \Gamma] = \Gamma$ (in other words, Γ is perfect); by the Hurewicz theorem, the first homology group of $\text{SU}(2)/\Gamma$ is zero. It can be shown that Γ is isomorphic to the fundamental group of V and that V is homeomorphic to $\text{SU}(2)/\Gamma$.

³¹ For Poincaré's report on the 1908 congress, see the "Appendix".

³² For a general overview of the history of the attempts to prove the Poincaré conjecture, see [39] and [43].

Appendix: Poincaré at the 1908 International Congress of Mathematicians in Rome

In April 1908, Poincaré went to Rome to take part in the International Congress of Mathematicians. Soon after arriving, however, he began to feel ill, and was forced to follow the events of the Congress from his hotel, where he was confined for the entire time. Even so, by speaking with those who attended, he was able to fulfil his promise to provide an account for the Paris newspaper *Le Temps* ('Compte rendu d'ensemble des travaux du IVe Congrès des Mathématiciens tenu à Rome en 1908', *Le Temps* 48: 2–3). Here is the account he wrote:

Mr Director, I had promised to send you news of the fourth Congress of Mathematicians which has just concluded in Rome. Unfortunately I am able to maintain my promise only in part because I was confined to bed in the hotel due to illness for a large part of the Congress. I was not even able to read the lecture that I had prepared, which was instead read by Darboux. However, I remained constantly informed of events thanks to the numerous participants who were lodged in the same hotel, and it is thanks to this circumstance that today I can keep my word.

The number of participants was higher than the previous Congresses; this is doubtless due to the attractiveness exerted by the Eternal City, but it is not the only reason, because the Congresses have become increasingly more attended and this is proof of their success.

France was brilliantly represented by four members of the Institute, many professors from the Sorbonne and of Universities in the provinces. There were also many distinguished representatives of German science, although unfortunately the professors of the University of Berlin were not among them. Similarly, two of the professors of Göttingen, Klein and Hilbert, who are universally regarded as two of the most important mathematicians of our age, after having given to believe they would be present as well as having announced their lecture, were detained in Germany for various reasons. In any case, no nation was absent: England sent Sir George Darwin, son of the celebrated naturalist and who, if this name were not already illustrious in itself, would have become famous for his works on the tides and the origin of the solar system. America was represented by the illustrious astronomer Newcomb, foreign member of the Institute of France; Sweden, by Mittag-Leffler, already well known to readers of *Le Temps*, and Holland by Lorentz, to whom we owe a new theory of electricity and matter.

It goes without saying that Italy had the greatest and most brilliant representation. For 30 years now the Italian mathematical movement has been very intense, both in Rome as well as in other Universities in the provinces: I could cite a great number of names that will occupy a very honourable place in the history of the sciences, but seeing them gathered together at this Congress, one understands much better what an active life Italian science has had. I hesitate somewhat to mention names, because I fear, indeed I am sure, that I will forget someone important. I cannot however neglect Blaserna, president of the Congress, vice-president of the Senate, a physicist who has worked on the questions of acoustics that concerned Helmholtz; nor Volterra, the celebrated analyst; nor Castelnuovo, Enriques and Severi, who took a decisive step forward in the theory of surfaces; nor above all Guccia, who did fine work in geometry and founded in Palermo an international mathematical society and one of the mathematical journals most widely read in the whole world.

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Author Biography

Claudio Bartocci is associate professor of mathematics at the University of Genoa. His research interests are focused on the following topics: algebraic geometry and mathematical physics, history of the mathematical thought in the nineteenth and twentieth centuries, philosophy of mathematics. His recent books include: *Una piramide di problemi* (Raffaello Cortina, Milan, 2012); *New Trends in Geometry: Their Role in the Natural and Life Sciences*

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