Computational and Applied Mathematics (2020) 39:211
https://doi.org/10.1007/s40314-020-01249-w

A 3D digital Jordan–Brouwer separation theorem

Josef Šlapal[1]

Received: 26 November 2019 / Revised: 2 July 2020 / Accepted: 5 July 2020 / Published online: 14 July 2020

Abstract
We introduce and discuss a concept of connectedness induced by an n-ary relation ($n > 1$ an integer). In particular, for every integer $n > 1$, we define an n-ary relation R_n on the digital line $\mathbb{Z}$ and equip the digital space $\mathbb{Z}^3$ with the n-ary relation R_n^3 obtained as a special product of three copies of R_n. For $n = 2$, the connectedness induced by R_n^3 coincides with the connectedness given by the Khalimsky topology on $\mathbb{Z}^3$ and we show that, for every integer $n > 2$, it allows for a digital analog of the Jordan–Brouwer separation theorem for three-dimensional spaces. An advantage of the connectedness induced by R_n^3 ($n > 2$) over that given by the Khalimsky topology is shown, too.

Keywords n-ary relation · Connectedness · Digital space · Digital surface · Jordan–Brouwer separation theorem

Mathematics Subject Classification 68U05 · 68R99 · 52C99

1 Introduction

In imaging applications, three-dimensional (3D for short) images play an increasingly important role. One of the basic tasks in the study and processing of these images is to define digital surfaces. It is desirable that the surfaces satisfy digital analogs of certain properties of real surfaces (i.e., surfaces in the three-dimensional Euclidean space $\mathbb{R}^3$). In particular, we require that they satisfy a digital analog of the Jordan–Brouwer separation theorem (cf. Greenberg 1967) for 3D spaces, also known as the Jordan surface theorem (because it is a three-dimensional extension of the Jordan curve theorem, which states that every simple closed curve in the Euclidean plane divides the plane into exactly two connected components).

Communicated by Carlos Hoppen.

This work was supported by Ministry of Education, Youth and Sports of the Czech Republic from the National Programme of Sustainability (NPU II) project "IT4 Innovations Excellence in Science-LQ1602".

✉ Josef Šlapal
slapal@fme.vutbr.cz

[1] IT4Innovations Centre of Excellence, Brno University of Technology, Brno, Czech Republic

As with the two-dimensional case (cf. Kong and Rosenfeld 1989; Rosenfeld 1979), the classical approach to the problem in the three-dimensional digital space $\mathbb{Z}^3$ is based on employing adjacency relations (6- and 26-adjacency) to define connectedness. This approach was used, for instance, in Morgenthaler and Rosenfeld (1981) where digital Jordan surfaces, i.e., surfaces satisfying a digital Jordan–Brouwer theorem, were introduced and studied (see also Brimkov and Klette 2008; Kong and Roscoe 1985; Reed and Rosenfeld 1982). A disadvantage of the classical approach is that two kinds of connectedness have to be employed (6- and 26-connectedness), one for the surface and the other for its complement. To overcome this disadvantage, a new, topological approach was proposed in Khalimsky et al. (1990) using just one connectedness, namely that provided by the Khalimsky topology. A digital Jordan-Brouwer separation theorem for the Khalimsky topology on $\mathbb{Z}^3$ was proved in Kopperman et al. (1991) and digital Jordan surfaces with respect to the topology were studied in Melin (2007).

In this note, we propose a new approach to defining digital Jordan surfaces, i.e., the surfaces in $\mathbb{Z}^3$ that satisfy a digital analog of the Jordan–Brouwer separation theorem for 3D spaces. While the classical approach is based on using binary (adjacency) relations to introduce a connectedness in $\mathbb{Z}^3$, we employ n-ary ones. We show that every n-ary relation ($n > 1$ an integer) on a set X induces a connectedness on X. Such a connectedness is discussed. In particular, for every integer $n > 1$, we introduce an n-ary relation R_n on $\mathbb{Z}$ and study the connectedness induced by the n-ary relation R_n^3 on $\mathbb{Z}^3$ obtained as a special product of three copies of the relation R_n. The connectedness induced by R_2^3 coincides with that given by the Khalimsky topology on $\mathbb{Z}^3$. As the main result, we prove a digital analog of the Jordan–Brouwer separation theorem for the digital space $\mathbb{Z}^3$ equipped with the (connectedness induced by the) relation R_n^3 where $n > 2$. This extends the results in Šlapal (2018) where a Jordan curve theorem was proved for the digital plane $\mathbb{Z}^2$ equipped with the n-ary relation $R_n^2, n > 2$. We show an advantage of the digital Jordan surfaces with respect to the connectedness induced by R_n^3 ($n > 2$) over those with respect to the Khalimsky topology on $\mathbb{Z}^3$. Our approach is somewhat similar to that used in Artzy et al. (2008) where, like in the present paper, the digital surfaces introduced consist of faces of certain polyhedra.

2 Preliminaries

The present paper uses some techniques developed in Šlapal (2018). To make it self-contained, we repeat the relevant material from Šlapal (2018) without proofs.

We will work with finite sequences, i.e., n-tuples (n a positive integer). Their terms will always be indexed by non-negative integers. Thus, for instance, both $(x_i \mid i < n)$ and $(x_i \mid i \leq n-1)$ will denote the sequence $(x_0, x_1, \ldots, x_{n-1})$. If X is a set, then the sequences $(x_i \mid i < n)$ with $x_i \in X$ for every $i < n$ are exactly the elements of the Cartesian product $X^n = \underbrace{X \times X \times \cdots \times X}_{n-times}$. Hence, the sequences correspond to the maps $\varphi : \{0, 1, \ldots, n-1\} \to X$ where $\varphi(i) = x_i$ for all $i < n$. A sequence $(x_i \mid i < n)$ is said to be *constant* if the corresponding map is constant, i.e., if $x_i = x_j$ for all $i, j < n$.

Let $n > 1$ be an integer and X be a set. Recall that an *n-ary relation* R on X is a subset $R \subseteq X^n$. The pair (X, R) is then called an *n-ary relational system*. We denote by $\Delta_X^{(n)}$ the n-ary *diagonal* on X, i.e., the n-ary relation $\Delta_X^{(n)} = \{(x_i \mid i < n) \in X^n;\ (x_i \mid i < n) \text{ is constant}\}$. For every n-ary relation R on X, we put $\bar{R} = R \cup \Delta_X^{(n)}$. $\bar{R}$ is called the *reflexive hull* of R.

Let R_j be an n-ary relation on a set X_j for every $j = 1, 2, \ldots, m$ ($m > 1$ an integer). Recall that the *Cartesian product* of the relations R_j, $j = 1, 2, \ldots, m$, is the n-ary relation $\prod_{j=1}^{m} R_j$ on the Cartesian product $\prod_{j=1}^{m} X_j$ of the sets X_j, $j = 1, 2, \ldots, m$, given by $\prod_{j=1}^{m} R_j = \{((x_i^1, x_i^2, ..., x_i^m) |\, i < n);\ (x_i^j |\, i < n) \in R_j$ for every $j = 1, 2, \ldots, m\}$. We put $\bigotimes_{j=1}^{m} R_j = \prod_{j=1}^{m} \bar{R}_j - \Delta_Y^{(n)}$ where $Y = \prod_{j=1}^{m} X_j$ and call the n-ary relation $\bigotimes_{j=1}^{m} R_j$ on $\prod_{j=1}^{m} X_j$ the *strong product* of R_j, $j = 1, 2, \ldots, m$.

If $X_j = X$ and $R_j = R$ for every $j = 1, 2, \ldots, m$, we write R^m instead of $\bigotimes_{j=1}^{m} R_j$.

Remark 1 If R_j is an n-ary relation on a set X_j for every $j = 1, 2, \ldots, m$, then we clearly have $\bigotimes_{j=1}^{m} R_j = \{((x_i^1, x_i^2, \ldots, x_i^m) |\, i < n)$; there is a nonempty subset $J \subseteq \{1, 2, \ldots, m\}$ such that $(x_i^j |\, i < n) \in R_j$ for every $j \in J$ and $(x_i^j |\, i < n)$ is a constant sequence for every $j \in \{1, 2, \ldots, m\} - J\}$.

Given an n-ary relation R on a set X, we put
$R^* = \{(x_i |\, i \leq m);\ (x_i |\, i \leq m) \in X^{m+1},\ 0 < m < n$, and there exists $(y_i |\, i < n) \in R$ such that $x_i = y_i$ for every $i \leq m$ or $x_i = y_{m-i}$ for every $i \leq m\}$.
The elements of R^* will be called R-*initial segments*. Thus, the R-initial segments are initial parts of the n-tuples belonging to R directed according to the n-tuples or conversely.

Definition 1 Let R be an n-ary relation on a set X. A sequence $C = (x_i |\, i \leq r)$, $r > 0$ an integer, of elements of X is called an R-*walk* if there is an increasing sequence $(i_k |\, k \leq p)$ of non-negative integers with $i_0 = 0$ and $i_p = r$ such that $i_k - i_{k-1} < n$ and $(x_i |\, i_{k-1} \leq i \leq i_k) \in R^*$ for every k with $0 < k \leq p$. An R-walk $C = (x_i |\, i \leq r)$ is called an R-*path* if its members are pairwise different; it is called an R-*circle* if, for every pair of different non-negative integers $i_0, i_1 \leq r$, $x_{i_0} = x_{i_1}$ is equivalent to $\{i_0, i_1\} = \{0, r\}$.

Definition 2 Let R be an n-ary relation on a set X. A set $Y \subseteq X$ is said to be R-*connected* if any two different elements $x, y \in Y$ can be joined by an R-*walk* contained in Y (i.e., there is an R-walk $(x_i |\, i \leq r)$ with $\{x_i |\, i \leq r\} \subseteq Y$ such that $x_0 = x$ and $x_r = y$). A maximal (with respect to set inclusion) R-connected set is called an R-*component*.

Note that, given an n-ary relation R on a set X, every R-initial segment is R-connected. Of course, the union of a finite sequence of nonempty R-connected sets is R-connected if the intersection of every consecutive pair of sets in the sequence is nonempty. In particular, every R-walk is R-connected.

If R is an n-ary relation on a set X and $Y \subseteq X$, then there is an n-ary relation on Y induced by R, namely $R \cap Y^n$. The relational system $(Y, R \cap Y^n)$ is then called an *induced relational subsystem* of (X, R) and is denoted by Y for short. If a subset $A \subseteq Y$ is $R \cap Y^n$-connected or is an $R \cap Y^n$-component, then we briefly say that it is R-connected or an R-component, respectively. And we say that Y *separates* X into exactly two R-components if the subset $X - Y$ of X has exactly two R-components.

We will need the following statement (Theorem 3.5) proved in Šlapal (2018).

Proposition 1 *Let R_j be an n-ary relation on a set X_j and $Y_j \subseteq X_j$ be a subset for every $j = 1, 2, \ldots, m$ ($m > 0$ an integer). If Y_j is R_j-connected for every $j = 1, 2, \ldots, m$, then $\bigotimes_{j=1}^{m} Y_j$ is $\prod_{j=1}^{m} R_j$-connected.*

As usual, two n-ary relational systems (X, R) and (Y, S) are said to be *isomorphic* if there exists a bijection $f : X \to Y$ such that, for every $(x_i |\, i < n) \in X^n$, $(x_i |\, i < n) \in R \Leftrightarrow (f(x_i) |\, i < n) \in S$.

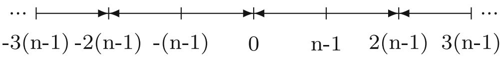

Fig. 1 A portion of R_n

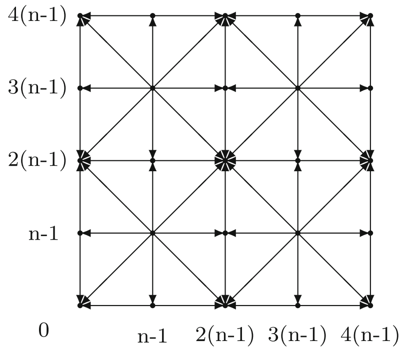

Fig. 2 A portion of R_n^2

3 Relation-induced connectedness in digital spaces

In the sequel, $n > 1$ will be an integer and R_n will denote the n-ary relation on $\mathbb{Z}$ given as follows:
$R_n = \{(x_i|\, i < n);$ there exists an odd integer k such that $x_i = 2k + i$ for all $i < n$ or $x_i = 2k - i$ for all $i < n\}$.
The n-ary relation R_n is demonstrated in Fig. 1 where the n-tuples belonging to R_n are represented by line segments directed from the first to the last members of the n-tuples.

Since $\mathbb{Z}$ is evidently R_n-connected, Proposition 1 implies:

Theorem 1 *$\mathbb{Z}^m$ is R_n^m-connected for every positive integer m.*

The relation R_n^2 is demonstrated in Fig. 2 where, as in Fig. 1, the n-tuples belonging to R_n^2 are represented by line segments directed from the first to the last members of the n-tuples. But, of the n-tuples $((x_i, l)|\, i < n) \in R_n^2$ such that $l \in \mathbb{Z}$ and there exists an odd integer k satisfying $y_i = 2k+i$ for all $i < n$ or $y_i = 2k-i$ for all $i < n$, only those with $l = m(n-1)$ for some $m \in \mathbb{Z}$ are displayed. Similarly, of the n-tuples $((k, y_i)|\, i < n) \in R_n^2$ such that $k \in \mathbb{Z}$ and there exists an odd integer l satisfying $x_i = 2l + i$ for all $i < n$ or $x_i = 2l - i$ for all $i < n$, only those with $k = m(n-1)$ for some $m \in \mathbb{Z}$ are displayed in Fig. 2. In other words, between any two neighboring parallel horizontal or vertical directed line segments (with the same direction), there are $n-2$ more line segments parallel to them and having the same direction, which are not displayed.

The R_2^m-connectedness in $\mathbb{Z}^m$ coincides with the connectedness with respect to the Khalimsky topology on $\mathbb{Z}^m$ for every integer $m > 0$. A digital Jordan curve theorem for the Khalimsky topology on $\mathbb{Z}^2$ and a Jordan–Brouwer separation theorem for the Khalimsky topology on $\mathbb{Z}^3$ were proved in Khalimsky et al. (1990) and Kopperman et al. (1991), respectively. A Jordan curve theorem for the relational systems $(\mathbb{Z}^2, R_n^2)$, $n > 2$ an integer, was proved in Šlapal (2018). In the sequel, we will focus on proving a digital Jordan–Brouwer separation theorem for the n-ary relational systems $(\mathbb{Z}^3, R_n^3)$ where $n > 2$. Therefore, from now on, n will denote an integer with $n > 2$.

Definition 3 Each of the following subsets of $\mathbb{Z}^2$ will be called an *R_n^2-fundamental triangle*:

1. $\{(x, y) \in \mathbb{Z}^2;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le x + (2l-2k)(n-1)\}$, $k, l \in \mathbb{Z}$,

2. $\{(x, y) \in \mathbb{Z}^2;\ 2k(n-1) \le x \le (2k+2)(n-1),\ x + (2l-2k)(n-1) \le y \le (2l+2)(n-1)\}$, $k, l \in \mathbb{Z}$,
3. $\{(x, y) \in \mathbb{Z}^2;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le (2k+2l+2)(n-1)-x\}$, $k, l \in \mathbb{Z}$,
4. $\{(x, y) \in \mathbb{Z}^2;\ 2k(n-1) \le x \le (2k+2)(n-1),\ (2k+2l+2)(n-1) - x \le y \le (2l+2)(n-1)\}$, $k, l \in \mathbb{Z}$.

Every R_n^2-fundamental triangle has the form of a digital rectangular triangle having $2n^2-n$ points. If we define *R_n^2-fundamental squares* to be the sets $\{(x, y) \in \mathbb{Z}^2;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le (2l+2)(n-1)\}$, $k, l \in \mathbb{Z}$, then every R_n^2-fundamental triangle is one of the two (digital) triangles obtained by splitting an R_n^2-fundamental square along one of its two diagonals—cf. Fig. 2.

The following statement follows from (the proof of) Theorem 4.7 in Šlapal (2018).

Lemma 1 *Every R_n^2-fundamental triangle is R_n^2-connected and so is every set obtained from an R_n^2-fundamental triangle by deleting some of its sides.*

Since every R_n^2-fundamental square is the union of a pair of fundamental triangles having a common hypotenuse (which is a diagonal of the square), Lemma 1 is valid for R_n^2-fundamental squares as well.

Now, we will switch from $\mathbb{Z}^2$ to $\mathbb{Z}^3$. Clearly, the coordinate planes xy, xz, and zy, when regarded as the induced relational subsystems of $(\mathbb{Z}^3, R_n^3)$, are isomorphic to $(\mathbb{Z}^2, R_n^2)$. The same is true for the digital planes $\{(x, y, 2k(n-1));\ (x, y) \in \mathbb{Z}^2\}$, $\{(x, 2k(n-1), z);\ (x, y) \in \mathbb{Z}^2\}$, and $\{(2k(n-1), y, z);\ (x, y) \in \mathbb{Z}^2\}$ where $k \in \mathbb{Z}$ (these planes may be obtained by shifting the coordinate planes along the coordinate axes perpendicular to them).

Definition 4 A subset $B \subseteq \mathbb{Z}^3$ will be called an *R_n^3-fundamental triangle* if:

1. There is an R_n^2-fundamental triangle A in the coordinate plane xy and an integer $k \in \mathbb{Z}$ such that $B = \{(x, y, z) \in \mathbb{Z}^3;\ (x, y) \in A \text{ and } z = 2k(n-1)\}$ or
2. There is an R_n^2-fundamental triangle A in the coordinate plane xz and an integer $k \in \mathbb{Z}$ such that $B = \{(x, y, z) \in \mathbb{Z}^3;\ (x, z) \in A \text{ and } y = 2k(n-1)\}$ or
3. There is an R_n^2-fundamental triangle A in the coordinate plane yz and an integer $k \in \mathbb{Z}$ such that $B = \{(x, y, z) \in \mathbb{Z}^3;\ (y, z) \in A \text{ and } x = 2k(n-1)\}$.

Proposition 2 *Every R_n^3-fundamental triangle is R_n^3-connected and so is every set obtained from an R_n^3-fundamental triangle by deleting some of its sides.*

Proof Clearly, every R_n^3-fundamental triangle T is obtained by shifting an R_n^3-fundamental triangle T_0 lying in a coordinate plane along the coordinate axis perpendicular to the plane such that the coordinates of all points of the shifted triangle with respect to the axis equal $2k(n-1)$, $k \in \mathbb{Z}$. Thus, the triangles T and T_0 (regarded as the induced relational subsystems of $(\mathbb{Z}^3, R_n^3)$) are isomorphic. Obviously, T_0 is also isomorphic to an R_n^2-fundamental triangle (obtained by deleting, from the points of T_0, the (0-valued) coordinates with respect to the axis perpendicular to the plane containing T_0), which is R_n^2-connected by Lemma 1. Hence, T_0 is R_n^3-connected and, therefore, T is R_n^3-connected, too. □

It is clear which points are vertices and which sets of points (digital line segments) are sides of an R_n^3-fundamental triangle.

For our convenience, we define the concept of an *R_n^3-fundamental square* to be any of the following subsets of $\mathbb{Z}^3$:

1. $\{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le (2l+2)(n-1),\ z = 2m(n-1)\}, k, l, m \in \mathbb{Z}$,
2. $\{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ y = 2l(n-1),\ 2m(n-1) \le z \le (2m+2)(n-1)\}, k, l, m \in \mathbb{Z}$,
3. $\{(x, y, z) \in \mathbb{Z}^3;\ x = 2k(n-1),\ 2l(n-1) \le y \le (2l+2)(n-1),\ 2m(n-1) \le z \le (2m+2)(n-1)\}, k, l, m \in \mathbb{Z}$.

Remark 2 Clearly, R^3_n-fundamental squares are digital squares with $(2n-1)^2$ points. They are exactly the unions of pairs of R^3_n-fundamental triangles having a common hypotenuse. Therefore, by Proposition 2, every R^3_n-fundamental square is R^3_n-connected and so is every set obtained from an R^3_n-fundamental square by deleting some of its sides.

Definition 5 Each of the following subsets of $\mathbb{Z}^3$ will be called an R^3_n-*fundamental rectangle*:

1. $\{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ y = x+(2l-2k)(n-1),\ 2m(n-1) \le z \le (2m+2)(n-1)\}, k, l, m \in \mathbb{Z}$,
2. $\{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ y = (2k+2l+1)-x,\ 2m(n-1) \le z \le (2m+2)(n-1)\}, k, l, m \in \mathbb{Z}$,
3. $\{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le (2l+2)(n-1),\ z = x + (2m-2k)(n-1)\}, k, l, m \in \mathbb{Z}$,
4. $\{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le (2l+2)(n-1),\ z = (2k+2m+1)-x\}, k, l, m \in \mathbb{Z}$,
5. $\{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le (2l+2)(n-1), z = y + (2m-2l)(n-1)\}, k, l, m \in \mathbb{Z}$,
6. $\{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le (2l+2)(n-1),\ z = (2l+2m+1)-y\}, k, l, m \in \mathbb{Z}$.

Every R^3_n-fundamental rectangle D has $(2n-1)^2$ points and has the form of a digital rectangle perpendicular to a coordinate plane such that the angle between D and each of the other two coordinate planes is $\frac{\pi}{4}$. If we define R^3_n-*fundamental cubes* to be the sets $\{(x, y, z) \in \mathcal{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le (2l+2)(n-1),\ 2m(n-1) \le z \le (2m+2)(n-1)\}, k, l, m \in \mathbb{Z}$, then every R^3_n-fundamental rectangle is obtained as the intersection of an R^3_n-fundamental cube and the (digital) plane that is perpendicular to a face of the cube and contains a (digital) diagonal of the face. Thus, it is clear which points are vertices and which sets of points (digital line segments) are sides of an R^3_n-fundamental rectangle. Note also that the R^3_n-fundamental squares are nothing but the faces of R^3_n-fundamental cubes.

An R^3_n-fundamental cube and an R^3_n-fundamental rectangle are demonstrated in Fig. 3 where only the points on the edges of the cube and those on the sides of the rectangle are visualized (the latter are ringed). The faces of the cube represent six R^3_n-fundamental squares and, on each of the two horizontal faces, a pair of R^3_n-fundamental triangles is demonstrated by dividing the face by one of its two diagonals.

Proposition 3 *Every R^3_n-fundamental rectangle is R^3_n-connected and so is every set obtained from an R^3_n-fundamental rectangle by deleting some of its sides.*

Proof Consider an R^3_n-fundamental rectangle satisfying condition (1) in Definition 5 and denote it by D. Clearly, putting $f(x, y, z) = (0, y, z)$ and $g(x, y, z) = (x, 0, z)$, we get bijections of D onto the R^3_n-fundamental squares $H_{yz} = \{(0, y, z) \in \mathbb{Z}^3;\ 2l(n-1) \le y \le (2l+2)(n-1),\ 2m(n-1) \le z \le (2m+2)(n-1)\}$ and $H_{yz} = \{(x, 0, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2m(n-1) \le z \le (2m+2)(n-1)\}$, which are contained in the

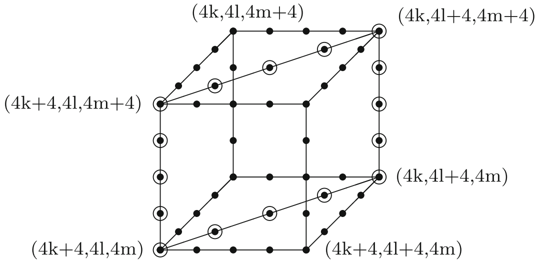

Fig. 3 An R^3_3-fundamental cube and an R^3_3-fundamental rectangle

coordinate planes yz and xz, respectively. Evidently, for every n-tuple $((x_i, y_i, z_i)|\ i < n)$ with $(x_i, y_i, z_i) \in \mathbb{Z}^3$ whenever $i < n$, we have $((x_i, y_i, z_i)|\ i < n) \in R^3_n$ if and only if both $((0, y_i, z_i)|\ i < n) \in R^3_n$ and $((x_i, 0, z_i)|\ i < n) \in R^3_n$ (where $(0, y_i, z_i)$ belongs to the coordinate plane yz for every $i < n$ and $(x_i, 0, z_i)$ belongs to the coordinate plane xz for every $i < n$). In particular, for an n-tuple $((x_i, y_i, z_i)|\ i < n)$ with $(x_i, y_i, z_i) \in D$ for every $i < n$, we have $((x_i, y_i, z_i)|\ i < n) \in R^3_n$ if and only if both $(f(x_i, y_i, z_i)|\ i < n) \in R^3_n$ and $(g(x_i, y_i, z_i)|\ i < n) \in R^3_n$ (note that $f(x_i, y_i, z_i) \in H_{yz}$ for every $i < n$ and $g(x_i, y_i, z_i) \in H_{xz}$ for every $i < n$). Since $g \circ f^{-1} : H_{yz} \to H_{xz}$ and $f \circ g^{-1} : H_{xz} \to H_{yz}$ are isomorphisms (inverse to each other), for every n-tuple $((x_i, y_i, z_i)|\ i < n)$ with $(x_i, y_i, z_i) \in D$ whenever $i < n$, we have $(f(x_i, y_i, z_i)|\ i < n) \in R^3_n$ if and only if $(g(x_i, y_i, z_i)|\ i < n) \in R^3_n$. Therefore, both $f : D \to H_{yz}$ and $g : D \to H_{xz}$ are isomorphisms. Since, by Remark 2, each of the R^3_n-fundamental squares H_{yz} and H_{xz} is R^3_n-connected and so is every set obtained from the R^3_n-fundamental square by deleting some of its sides, the same is true for the R^3_n-fundamental rectangle D. For an R^3_n-fundamental rectangle satisfying any of the conditions (2)-(6) in Definition 5, the proof is analogous. □

Definition 6 A subset $P \subseteq \mathbb{Z}^3$ will be called an R^3_n-*fundamental prism* if:

1. There is an R^2_n-fundamental triangle A in the coordinate plane xy and an integer $k \in \mathbb{Z}$ such that $P = \{(x, y, z) \in \mathbb{Z}^3;\ (x, y) \in A \text{ and } 2k(n-1) \le z \le (2k+2)(n-1)\}$ or
2. There is an R^2_n-fundamental triangle A in the coordinate plane xz and an integer $k \in \mathbb{Z}$ such that $P = \{(x, y, z) \in \mathbb{Z}^3;\ (x, z) \in A \text{ and } 2k(n-1) \le y \le (2k+2)(n-1)\}$ or
3. There is an R^2_n-fundamental triangle A in the coordinate plane yz and an integer $k \in \mathbb{Z}$ such that $P = \{(x, y, z) \in \mathbb{Z}^3;\ (y, z) \in A \text{ and } 2k(n-1) \le x \le (2k+2)(n-1)\}$.

Clearly, every R^3_n-fundamental prism has the form of a digital triangular prism with $n(2n-1)^2$ points such that two of its faces are R^3_n-fundamental triangles, another two are R^3_n-fundamental squares and one face is an R^3_n-fundamental rectangle. In other words, each R^3_n-fundamental prism is one of the two (digital) prisms obtained by splitting an R^3_n-fundamental cube along a plane perpendicular to a face of the cube and containing a diagonal of the face. Thus, every R^3_n-fundamental cube gives rise to 12 different R^3_n-fundamental prisms and the cube is the union of any pair of them having a common face (which is an R^3_n-fundamental rectangle).

Proposition 4 *Every R^3_n-fundamental prism is R^3_n-connected and so is every subset of $\mathbb{Z}^3$ obtained from an R^3_n-fundamental prism by removing some of its faces.*

Proof Consider the fundamental prism $P = \{(x, y, z) \in \mathbb{Z}^3;\ 2k(n-1) \le x \le (2k+2)(n-1),\ 2l(n-1) \le y \le x + (2l-2k)(n-1),\ 2m(n-1) \le z \le (2m+2)(n-1)\}$ where $k = l = m = 0$. Then, P is the union of the $2n-1$ digital triangles obtained as the intersection of P

with the digital plane (parallel to the coordinate plane xy) $\{(x, y, z) \in \mathbb{Z}^3;\ z = k\}$, where k is one of the integers $0, 1, \dots, 2n-2$. Each of the $2n-1$ triangles is isomorphic to (is obtained by shifting along the axis z of) the R_n^2-fundamental triangle $(0,0)(2n-2,0)(2n-2,2n-2)$ in the coordinate plane xy, which is R_n^2-connected by Lemma 1. Therefore, each of the $2n-1$ triangles is R_n^3-connected. The same is true for every set obtained from any of the triangles by removing some of its sides. Put $C = ((2n-3,1,0), (2n-3,1,1), (2n-3,1,2), \dots, (2n-3,1,2n-2))$. Since both $((2n-3,1,n-1), (2n-3,1,n-2), \dots, (2n-3,1,0))$ and $((2n-3,1,n-1), (2n-3,1,n), \dots, (2n-3,1,2n-2))$ belong to R_n^3, C is an R_n^3-path. Thus, C is R_n^3-connected. Clearly, C meets each of the $2n-1$ triangles and is contained in their union and the same is true even if some of the sides of the triangles are removed. Consequently, the fundamental prism P (which is the union of the $2n-1$ triangles) is connected and the same is true if some of the faces of the prism are removed. If some of the integers k, l, m differ from 0, the proof is much the same. Thus, the statement is valid for the R_n^3-fundamental prisms satisfying condition (1) in Definition 6 where the R_n^2-fundamental triangle A satisfies condition (1) in Definition 4. For all the other cases of R_n^3-fundamental prisms, the proofs may be done along similar lines. □

Definition 7 By an R_n^3-*Jordan surface*, we understand every R_n^3-connected set $S = \bigcup \mathcal{F}$, where $\mathcal{F}$ is a finite nonempty set such that every element of $\mathcal{F}$ is an R_n^3-fundamental triangle or an R_n^3-fundamental rectangle and the following two conditions are satisfied:

1. For every pair $F_1, F_2 \in \mathcal{F}$ of distinct elements, $F_1 \cap F_2$ is an empty set or a singleton containing a common vertex of F_1 and F_2 or a common side of of F_1 and F_2.
2. For every $F_1 \in \mathcal{F}$, if E is a side of F_1, then there exists exactly one $F_2 \in \mathcal{F}$, $F_1 \neq F_2$, such that E is a side of F_2.

Thus, every R_n^3-Jordan surface consists of R_n^3-fundamental rectangles, triangles and squares (the latter being unions of pairs of R_n^3-fundamental triangles having a common hypotenuse).

Theorem 2 (3D Digital Jordan–Brouwer Separation Theorem) *Every R_n^3-Jordan surface S separates $\mathbb{Z}^3$ into exactly two R_n^3-components and the union of S with each of them is R_n^3-connected.*

Proof Let $S = \bigcup \mathcal{F}$ be an R_n^3-Jordan surface. Then, it is evident that S is a polyhedral surface, i.e., the union of all faces of a polyhedron $T \subseteq \mathbb{Z}^3$. By Definition 7, if $D \subseteq T$ is an R_n^3-fundamental rectangle, then T contains (as a subset) at least one of the two R_n^3-fundamental prisms having D as a side. Consequently, T may be expressed as the union of a (finite) sequence of pairwise different R_n^3-fundamental prisms such that any two of them are disjoint or meet in just one face in common and every R_n^3-fundamental prism in the sequence, except for the first one, has a face in common with at least one of its predecessors. However, the set $U = (\mathbb{Z}^3 - T) \cup S$, too, may be written as the union of such an (infinite) sequence of R_n^3-fundamental prisms. By Proposition 4, T, $T - S$, U, and $U - S$ are R_n^3-connected.

It is easy to see that every R_n^3-walk $C = (z_i \mid i \leq k)$, $k > 0$ an integer, joining a point of $T - S$ with a point of $U - S$ meets S (i.e., meets an R_n^3-fundamental prism face contained in S). Thus, the set $\mathbb{Z}^3 - S = (T - S) \cup (U - S)$ is not R_n^3-connected. Hence, $T - S$ and $U - S$ are R_n^3-components of the subspace $\mathbb{Z}^3 - S$ of $(\mathbb{Z}^3, R_n^3)$, $T - S$ finite and $U - S$ infinite, with T and U R_n^3-connected. □

By the proof of Theorem 2, every R_n^3-Jordan surface is the surface of the polyhedron $\mathcal{P}$ that is the union of a finite set $\mathcal{F}$ of R_n^3-fundamental prisms such that any two of them are disjoint or intersect at a single vertex, in a single edge or a single face. Since the union of

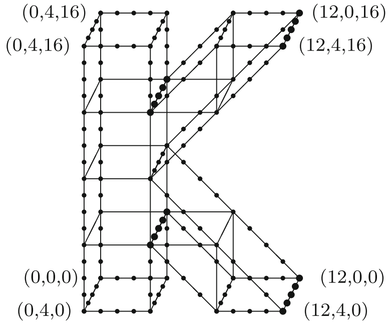

Fig. 4 An R_3^3-Jordan surface

a pair of R_n^3-fundamental prisms with a common face that is an R_n^3-fundamental rectangle is an R_n^3-fundamental cube, $\mathcal{P}$ may be regarded as the union of R_n^3-fundamental prisms and R_n^3-fundamental cubes (if any).

4 Conclusion

We have found a connectedness structure for the digital space $\mathbb{Z}^3$, namely the relation R_n^3 ($n > 2$ an integer), which can be used to obtain a 3D digital Jordan–Brouwer separation theorem (Theorem 2). An advantage of the R_n^3-Jordan surfaces introduced over the Jordan surfaces with respect to the Khalimsky topology on $\mathbb{Z}^3$ proposed in Kopperman et al. (1991) is that pairs of plane segments (R_n^3-fundamental squares and rectangles) of an R_n^3-Jordan surface may meet at the acute angle $\frac{\pi}{4}$. The advantage is demonstrated in Fig. 4 by an R_3^3-Jordan surface in $\mathbb{Z}^3$ that is a digital image of the letter K. In the figure, only the points on the edges of the surface are visualized. The R_n^3-fundamental squares, triangles and rectangles of which the surface consists are emphasized by their sides represented as line segments. These line segments are the edges of the R_n^3-fundamental cubes and prisms whose union is the polyhedron with the surface being the R_3^3-Jordan surface demonstrated. This surface is not a Jordan surface with respect to the Khalimsky topology on $\mathbb{Z}^3$ in the sense of Kopperman et al. (1991), because there are four pairs of plane segments of the surface meeting at the angle $\frac{\pi}{4}$—the four meeting digital line segments are denoted by the bold dots in the figure. Thus, the relation R_n^3 provides a connectedness structure convenient for the study of three-dimensional digital images that allows for a richer variety of digital Jordan surfaces than the Khalimsky topology does.

References

Artzy E, Frieder G, Herman G (2008) The theory, design, implementation, and evaluation of a three-dimensional surface-detection algorithm. Comput Gr Image Process 15:1–24

Brimkov VE, Klette R (2008) Border and surface tracing - theoretical foundations. IEEE Trans. Patt. Anal. Machine. Intell 30:577–590

Čech E (1966) Topological spaces (Revised by Z.Frolík and M.Katětov). Academia, Prague

Čech E (1968) Topological spaces. In: Topological Papers of Eduard Čech. Academia, Prague, ch. 28, pp 436–472

Engelking R (1977) Gen Topol. Państwowe Wydawnictwo Naukowe, Warszawa

Greenberg M (1967) Lectures on algebraic topology. Benjamin, New York

Khalimsky ED, Kopperman R, Meyer PR (1990) Computer graphics and connected topologies on finite ordered sets. Topol Appl 36:1–17
Kong TY, Roscoe W (1985) Continuous analogs of axiomatized digital surfaces. Comput Vis Gr Image Process 29:60–86
Kong TY, Rosenfeld A (1989) Digital topology: Introduction and survey. Comput. Vision Graphics Image Process. 48:357–393
Kopperman R, Meyer PR, Wilson RG (1991) A Jordan surface theorem for three-dimensional digital spaces. Discrete Comput Geom 6:155–161
Melin E (2007) Digital surfaces and boundaries in Khalimsky spaces. J. Math. Imaging and Vision 28:169–177
Morgenthaler DG, Rosenfeld A (1981) Surfaces in three dimensional digital images. Inf Control 28:227–247
Reed M, Rosenfeld A (1982) Recognition of surfaces in three dimensional digital images. Inf Control 53:108–120
Rosenfeld A (1979) Picture Languages. Academic Press, New York
Šlapal J (2018) Relation-induced connectedness in the digital plane. Aequat Math 92:75–90