Group inverse for two classes of 2×2 block matrices over rings

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Abstract Let R be an associative ring with unity 1. In this paper, existence of the group inverse of two classes of 2×2 block matrices is investigated. We obtain the sufficient and necessary conditions. And further, the representations of the group inverse of two classes are given. (I) $M = \begin{pmatrix} AX + YB & A \\ B & 0 \end{pmatrix}$, where $A^\#$ exists, $XA = AX$ and X is invertible; (ii) $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $CA = C$, $AB = B$ and $(D - CB)^\#$ exists. The results extend the early works of Cao and Li (Appl Math Comput 217:10271–10277, 2011) and Liu and Yang (Appl Math Comput 218:8978–8986, 2012). Some examples are given to illustrate our results.

Keywords Ring · Group inverse · Block matrix

Mathematics Subject Classification (2010) 15A09 · 16U30

1 Introduction

A square matrix A is said to be group invertible if there exists a matrix X such that

$$AXA = A, \quad XAX = X \quad \text{and} \quad AX = XA.$$

We then write $X = A^\#$. It is well known that if $A^\#$ exists, it is unique. The group inverse, which is a particular case of the Drazin inverse, has been studied by many researchers and many applications of such inverse have been found in many literatures. The group inverse

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has applications in singular differential and difference equations, Markov chains and iterative methods. For example, [Heinig \(1997\)](#) investigated the group inverse of Sylvester transformation. [Wei and Diao \(2005\)](#) studied the representation of the group inverse of a real singular Toeplitz matrix which arises in scientific computing and engineering. [Catal et al. \(2008\)](#) studied the existence of $A^\sharp$. More details and its applications about group (Drazin) inverse can be found in the literature ([Soares and Latouche 2002](#); [Bu and Zhang 2011](#); [Golub and Greif 2003](#); [Wei 1998](#); [Ben-Israel and Greville 2003](#)).

In the past few decades, group (Drazin) inverse has been studied extensively. Recently, more and more authors pay close attention to them and a large body of works can be found in literature (see, for example [Cao and Zhao 2012](#); [Bu et al. 2009](#); [Liu et al. 2012](#); [Bu et al. 2010](#); [Liu et al. 2011](#); [Bu et al. 2012](#).) However, most of the relative results were obtained over skew fields and complex fields. Recently, [Zhang and Bu \(2012\)](#) and [Ge et al. \(2012\)](#) investigated the group inverse of some 2×2 block matrices over right Ore domains. It is well known that a ring is called a right Ore domain if it possesses no zero divisors and every two elements of the ring have a right common multiple, see [Zhang and Bu \(2012\)](#). Skew field and complex field are in the right Ore domain. It is very important to study the group inverse in more general rings. In fact, the concept of the Drazin inverse (group inverse) is based on associative rings, see [Drazin \(1958\)](#). Motivated by the excellent results mentioned above, the aim of this paper is to extend some classical results to even more general setting, i.e., associative rings. The results in this paper improve two following works:

[Cao and Li \(2011\)](#) gave the group inverse of the block matrix $\begin{pmatrix} AX + YB & A \\ B & 0 \end{pmatrix}$ under some conditions over skew fields. We extend the relative results to associative rings. We should pointed here that the method used here is quite different from that of [Cao and Li \(2011\)](#).

[Liu and Yang \(2012\)](#) studied the group inverse of the block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under the conditions $D = 0$, $AB = B$, $CA = C$ and $A = A^2$ over complex fields. In this paper, we improve the results to associative rings under more simplify conditions.

Next, we give some explanations on notations used in this paper.

Let $R, \mathfrak{R}, K$ be an associative ring with unity 1, right Ore domain with unity 1 and skew field, respectively. $R^{m \times n}$, $\mathfrak{R}^{m \times n}$ and $K^{m \times n}$ denote the set of all $m \times n$ matrices over $R, \mathfrak{R}$ and K , respectively.

$R^{(m)}(R^m)$ denotes the set of all m -dimensional row (column) vectors over R . Let $A \in \mathfrak{R}^{m \times n}$, denote the range and the row range of a matrix A by $R(A)$ and $R_r(A)$, where $R(A) = \{Ax | x \in \mathfrak{R}^n\}$ and $R_r(A) = \{xA | x \in \mathfrak{R}^{(m)}\}$.

We denote $I_n - AA^\sharp$ by A^π . A matrix A is regular if there exists a matrix X such that $AXA = A$. Define $A\{1\} = \{X : AXA = A\}$, and if one needs to pick an element of $A\{1\}$, simply we write $B \in A\{1\}$ or $A^{(1)} \in A\{1\}$. If $A \in \mathfrak{R}^{m \times n}$, $r(A)$ denotes the rank of A , see [Zhang and Bu \(2012\)](#).

2 Some Lemmas

Lemma 1 *Let $A \in R^{n \times n}$, the followings are equivalent:*

- (i) $A^\sharp$ exists;
- (ii) $A = A^2X$ and $A = Y A^2$ for some $X, Y \in R^{n \times n}$. In this case, $A^\sharp = YAX = AX^2 = Y^2A$;
- (iii) $R(A) = R(A^2)$, $R_r(A) = R_r(A^2)$.

Proof It follows from Proposition 8.22 of Bhaskara Rao (2002) that (i) $\Leftrightarrow$ (ii) holds. Next, we prove the relation of (ii) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (iii) :

It is obviously that $R(A) \supset R(A^2)$, $R_r(A) \supset R_r(A^2)$ hold. From $A = A^2X$ and $A = YA^2$, we can deduce that $R(A) \subset R(A^2)$ and $R_r(A) \subset R_r(A^2)$, respectively.

(iii) $\Rightarrow$ (ii) :

Using the fact that the right Ore domain has unity 1, we can easily obtain that $R(A) = R(A^2)$ and $R_r(A) = R_r(A^2)$ imply $A = A^2X$ and $A = YA^2$, respectively. $\square$

Lemma 2 Ge et al. (2012) Let $A \in \mathfrak{R}^{n \times n}$, then the followings are equivalent:

- (i) A is group invertible;
- (ii) $A^2X = A$ for some $X \in \mathfrak{R}^{n \times n}$. In this case, $A^\sharp = AX^2$;
- (iii) $YA^2 = A$ for some $Y \in \mathfrak{R}^{n \times n}$. In this case, $A^\sharp = Y^2A$;
- (iv) $R(A) = R(A^2)$;
- (v) $R_r(A) = R_r(A^2)$.

Lemma 3 Ge et al. (2012) Let $A \in \mathfrak{R}^{n \times m}$, $B \in \mathfrak{R}^{m \times n}$, then the followings are equivalent:

- (i) $R(BAB) = R(B)$;
- (ii) $(BA)^\sharp$ exists and $(BA)^\pi B = 0$;
- (iii) $R_r(BAB) = R_r(B)$;
- (iv) $(AB)^\sharp$ exists and $B(AB)^\pi = 0$.

Lemma 4 Let $A \in R^{m \times n}$, $B \in R^{n \times m}$, $R(BAB) = R(B)$ and $R_r(BAB) = R_r(B)$, then $(AB)^\sharp$ and $(BA)^\sharp$ exist.

Proof By similar arguments of Lemmas 1 and 3, we can easily obtain the results. $\square$

Lemma 5 Suppose that $A \in R^{m \times n}$, $B \in R^{n \times m}$. If $(AB)^\sharp$ and $(BA)^\sharp$ exist, then the following equalities hold.

- (i) $(AB)^\sharp = A[(BA)^\sharp]^2B$;
- (ii) $(BA)^\sharp = B[(AB)^\sharp]^2A$;
- (iii) $(AB)^\sharp A = A(BA)^\sharp$;
- (iv) $(BA)^\sharp B = B(AB)^\sharp$.

Proof The proof of this Lemma is similar to Lemma 3 in Ge et al. (2012), so we omit it here. $\square$

3 Main results

Theorem 1 Let $M = \begin{pmatrix} AX + YB & A \\ B & 0 \end{pmatrix}$, where $A, B, X, Y \in R^{n \times n}$, $A^\sharp$ exists, $XA = AX$ and X is invertible, then we have

- (i) $M^\sharp$ exists if and only if $R(B) = R(BTB)$, $R_r(B) = R_r(BTB)$, where $T = A^\pi Y - AA^\sharp X^{-1}$;
- (ii) If $M^\sharp$ exists, then $M^\sharp = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, where

$$M_1 = (TB)^\sharp A^\pi + (TB)^\pi A^\sharp X^{-1} - (TB)^\pi A^\sharp ZB(TB)^\sharp A^\pi,$$

$$M_2 = -(TB)^\sharp AA^\sharp X^{-1} + (TB)^\pi A^\sharp X^{-2} + (TB)^\pi A^\sharp ZB(TB)^\sharp AA^\sharp X^{-1},$$

$$\begin{aligned} M_3 &= -B(TB)^\sharp A^\sharp X^{-1} + B[(TB)^\sharp]^2 A^\pi + B(TB)^\sharp A^\sharp ZB(TB)^\sharp A^\pi, \\ M_4 &= -B(TB)^\sharp A^\sharp X^{-2} - B[(TB)^\sharp]^2 AA^\sharp X^{-1} - B(TB)^\sharp A^\sharp ZB(TB)^\sharp AA^\sharp X^{-1}, \text{ and} \\ Z &= X^{-2} + X^{-1}Y. \end{aligned}$$

Proof (i) The “only if” part.

Note that

$$M = \begin{pmatrix} YB & A \\ B & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}, \quad (1)$$

and

$$M^2 = \begin{pmatrix} YBTB & XA^2 + YBA \\ BTB & BA \end{pmatrix} \begin{pmatrix} I & 0 \\ A^\sharp YB & I \end{pmatrix} \begin{pmatrix} I & 0 \\ A^\sharp X^{-1}B & I \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \quad (2)$$

By Lemma 1, there exist matrices $\bar{X}$ and $\bar{Y}$ over R such that $M = M^2\bar{X}$ and $\bar{Y}M^2 = M$.

Let

$$\bar{X} = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^\sharp X^{-1}B & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^\sharp YB & I \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \quad (3)$$

It follows from Eqs. (1)–(3) and $M = M^2\bar{X}$, we can obtain following equations

$$YBTBX_1 + (XA^2 + YBA)X_3 = YB, \quad (4)$$

$$YBTBX_2 + (XA^2 + YBA)X_4 = A, \quad (5)$$

$$BTBX_1 + BAX_3 = B, \quad (6)$$

$$BTBX_2 + BAX_4 = 0. \quad (7)$$

On the other hand, we have

$$M^2 = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} XAYB + AB & XA^2 \\ BYB & BA \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \quad (8)$$

Let

$$\bar{Y} = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix}. \quad (9)$$

It follows from (1), (8), (9) and $\bar{Y}M^2 = M$, we have

$$Y_1(XAYB + AB) + Y_2BYB = YB, \quad (10)$$

$$Y_1XA^2 + Y_2BA = A, \quad (11)$$

$$Y_3(XAYB + AB) + Y_4BYB = B, \quad (12)$$

$$Y_3XA^2 + Y_4BA = 0. \quad (13)$$

From (6), we have

$$YBTBX_1 + YBAX_3 = YB.$$

Substituting it into (4). Obviously, we have

$$XA^2X_3 = 0.$$

Recall that X is invertible and A is group invertible. It follows that

$$XA^2X_3 = 0 \implies AX_3 = 0.$$

Substituting $AX_3 = 0$ into (6), we get $BTBX_1 = B$, i.e., $R(BTB) = R(B)$.

From (13), we can get

$$-Y_4BAA^\sharp YB = Y_3XA^2A^\sharp YB = Y_3XAYB, \quad (14)$$

and

$$-Y_4BAA^\sharp X^{-1}B = Y_3XA^2A^\sharp X^{-1}B = Y_3AB. \quad (15)$$

Substitute (14) into (12), we have

$$-Y_4BAA^\sharp YB + Y_3AB + Y_4BYB = B. \quad (16)$$

Substitute (15) into (16), we have

$$-Y_4BAA^\sharp YB - Y_4BAA^\sharp X^{-1}B + Y_4BYB = B$$

that is $Y_4BTB = B$, which implies $R_r(BTB) = R_r(B)$. This completes proof of “only if” part.

In what follows, we give the proof of “if” part.

It follows from Lemma 4 that $R(BTB) = R(B)$ and $R_r(BTB) = R_r(B)$ imply $(TB)^\sharp$ and $(BT)^\sharp$ exist. Let

$$X_1 = (TB)^\sharp, \quad X_2 = -(TB)^\sharp AA^\sharp X^{-1}, \quad X_3 = 0, \quad X_4 = A^\sharp X^{-1}$$

and

$$Y_1 = (TB)^\pi A^\sharp X^{-1}, \quad Y_2 = (TB)^\sharp T, \quad Y_3 = -B(TB)^\sharp A^\sharp X^{-1}, \quad Y_4 = (BT)^\sharp.$$

From Lemma 3, we can easily obtain that $R(BTB) = R(B)$ implies $B(TB)^\pi = 0$, i.e., $BTB(TB)^\sharp = B$. Further, $XA = AX$ implies $X^{-1}A = AX^{-1}$.

We claim that X_1, X_2, X_3 and X_4 satisfy the Eqs. (4)–(7). Next, we verify the claim by computation separately.

- (i) $YBTBX_1 + (XA^2 + YBA)X_3 = YBTB(TB)^\sharp = YB$;
- (ii) $YBTBX_2 + (XA^2 + YBA)X_4$
 $= -YBTB(TB)^\sharp AA^\sharp X^{-1} + (XA^2 + YBA)A^\sharp X^{-1}$
 $= -YBAA^\sharp X^{-1} + XAX^{-1} + YBAA^\sharp X^{-1} = A$;
- (iii) $BTBX_1 + BAX_3 = BTB(TB)^\sharp = B$;
- (iv) $BTBX_2 + BAX_4 = -BTB(TB)^\sharp AA^\sharp X^{-1} + BAA^\sharp X^{-1}$
 $= -BAA^\sharp X^{-1} + BAA^\sharp X^{-1} = 0$.

From $T = A^\pi Y - AA^\sharp X^{-1}$, we have $TB - YB = -AA^\sharp YB - AA^\sharp X^{-1}B$. By Lemma 5, we know $B(TB)^\sharp = (BT)^\sharp B$.

By computation, we next verify Y_1, Y_2, Y_3 and Y_4 satisfy the Eqs. (10)–(13). We give the proof separately.

- (v) $Y_1(XAYB + AB) + Y_2BYB$
 $= (TB)^\pi A^\sharp X^{-1}(XAYB + AB) + (TB)^\sharp BYB$
 $= (TB)^\pi (A^\sharp AYB + A^\sharp AX^{-1}B) + (TB)^\sharp BYB$
 $= (TB)^\pi (YB - TB) + (TB)^\sharp BYB = YB$
- (vi) $Y_1XA^2 + Y_2BA = (TB)^\pi A^\sharp X^{-1}XA^2 + (TB)^\sharp TBA$
 $= (TB)^\pi A + (TB)^\sharp TBA = A$

$$\begin{aligned}
\text{(vii)} \quad & Y_3(XAYB + AB) + Y_4BYB \\
& = -B(TB)^\sharp A^\sharp X^{-1}(XAYB + AB) + (BT)^\sharp BYB \\
& = -B(TB)^\sharp (A^\sharp AYB + AA^\sharp X^{-1}B - YB) = -B(TB)^\sharp (-TB) = B \\
\text{(viii)} \quad & Y_3XA^2 + Y_4BA = -B(TB)^\sharp A^\sharp X^{-1}XA^2 + (BT)^\sharp BA \\
& = -B(TB)^\sharp A + (BT)^\sharp BA = -(BT)^\sharp BA + (BT)^\sharp BA = 0.
\end{aligned}$$

(ii) By Lemmas 1, 4 and 5, the expression of $M^\sharp$ can be obtained from $M^\sharp = \overline{Y}M\overline{X}$. $\square$

The following corollaries follow from Theorem 1.

Corollary 1 Patrício and Hartwig (2010) Let $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$, where $A, B \in R^{n \times n}$, $A^\sharp$ exists, then the following hold.

- (i) $M^\sharp$ exists if and only if $R(B) = R(BTB)$, $R_r(B) = R_r(BTB)$, where $T = -AA^\sharp$;
- (ii) If $M^\sharp$ exists, then $M^\sharp = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, where

$$\begin{aligned}
M_1 &= (TB)^\sharp A^\pi + (TB)^\pi A^\sharp - (TB)^\pi A^\sharp B(TB)^\sharp A^\pi, \\
M_2 &= -(TB)^\sharp AA^\sharp + (TB)^\pi A^\sharp + (TB)^\pi A^\sharp B(TB)^\sharp AA^\sharp, \\
M_3 &= -B(TB)^\sharp A^\sharp + B[(TB)^\sharp]^2 A^\pi + B(TB)^\sharp A^\sharp B(TB)^\sharp A^\pi \text{ and} \\
M_4 &= -B(TB)^\sharp A^\sharp - B[(TB)^\sharp]^2 AA^\sharp - B(TB)^\sharp A^\sharp B(TB)^\sharp AA^\sharp.
\end{aligned}$$

Proof The proof immediately follows from Theorem 1 if we define $X = I, Y = 0$. The calculations and conclusions are analogous, and we omit them. $\square$

Corollary 2 Let $M = \begin{pmatrix} AX + YB & A \\ B & 0 \end{pmatrix}$, where $A, B, X, Y \in \mathfrak{N}^{n \times n}$, $A^\sharp$ exists, $XA = AX$ and X is invertible, then

- (i) $M^\sharp$ exists if and only if $R(B) = R(BTB)$ if and only if $R_r(B) = R_r(BTB)$, where $T = A^\pi Y - AA^\sharp X^{-1}$;
- (ii) If $M^\sharp$ exists, then the expression of $M^\sharp$ is the same as in Theorem 1.

Proof The result immediately follows from Lemma 2 and 3. $\square$

Corollary 3 Cao and Li (2011) Let $M = \begin{pmatrix} AX + YB & A \\ B & 0 \end{pmatrix}$, where $A, B, X, Y \in K^{n \times n}$, $A^\sharp$ exists, $XA = AX$ and X is invertible, then

- (i) $M^\sharp$ exists if and only if $r(B) = r(BTB)$, where $T = A^\pi Y - AA^\sharp X^{-1}$.
- (ii) If $M^\sharp$ exists, then the representation of $M^\sharp$ is the same as in Theorem 1.

We now give examples of existence and representation of $M^\sharp$ for which the required hypotheses are satisfied.

Example for Theorem 1: Let $R = \mathbb{Z}/[12]$, $M = \begin{pmatrix} AX + YB & A \\ B & 0 \end{pmatrix}$, where

$$A = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 4 \\ -4 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \quad Y = \begin{pmatrix} 4 & -4 \\ 4 & 4 \end{pmatrix}.$$

It is easy to verify $AX = XA$ and X is invertible. By a direct computation, we know that $A^\sharp$ and $(TB)^\sharp$ exist.

Further, we have

$$A^\sharp = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, \quad A^\pi = \begin{pmatrix} -3 & -4 \\ -4 & -3 \end{pmatrix}, \quad TB = \begin{pmatrix} 0 & -4 \\ 4 & 4 \end{pmatrix},$$

$$(TB)^\sharp = \begin{pmatrix} 4 & 4 \\ -4 & 0 \end{pmatrix}, \quad (TB)^\pi = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}.$$

Clearly, $R(B) = R(BTB)$ and $R_r(B) = R_r(BTB)$. According to Theorem 1, we know that $M^\sharp$ exists.

The representation of $M^\sharp$ immediately follows from (ii) of Theorem 1,

$$M^\sharp = \begin{pmatrix} -4 & -4 & 4 & 4 \\ 0 & 4 & 4 & 4 \\ 0 & 4 & -4 & -4 \\ -4 & 4 & 0 & 0 \end{pmatrix}.$$

Example for Corollary 2: Let $\mathfrak{R} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z}\}$ be the quaternion integral domain.

Let $M = \begin{pmatrix} AX + YB & A \\ B & 0 \end{pmatrix}$, where

$$A = \begin{pmatrix} 1 & j \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -i - j + 1 & 0 \\ j & 0 \end{pmatrix}, \quad X = \begin{pmatrix} i & k + 1 \\ 0 & j \end{pmatrix} \text{ and } Y = \begin{pmatrix} i & j \\ 0 & -k \end{pmatrix}.$$

It is easy to verify $AX = XA$ and X is invertible. By a direct computation, we know that $A^\sharp$ and $(TB)^\sharp$ exist.

We can easily obtain

$$A^\sharp = \begin{pmatrix} 1 & j \\ 0 & 0 \end{pmatrix}, \quad A^\pi = \begin{pmatrix} 0 & -j \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} i & i + k \\ 0 & -k \end{pmatrix},$$

$$TB = (TB)^\sharp = \begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix} \text{ and } (TB)^\pi = \begin{pmatrix} 0 & 0 \\ -i & 1 \end{pmatrix}.$$

Clearly, $R(B) = R(BTB)$. The representation of $M^\sharp$ immediately follows from Corollary 2,

$$M^\sharp = \begin{pmatrix} 0 & -j & i & k \\ -1 & -i - 2k & -j - 2 & -2j + 1 \\ i + k + 1 & i - j + 2k - 3 & i + j + 3k + 2 & -3i + 2j + k - 1 \\ -k & -j + 2 & i - 2k & 2i + k \end{pmatrix}.$$

Theorem 2 Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{m \times m}$, $CA = C$, $AB = B$, $S = D - CB$, if $S^\sharp$ exists, then

- (i) $M^\sharp$ exists if and only if $F = A^2 + BS^\pi C$ is regular and $R(F) = R(A)$, $R_r(F) = R_r(A)$;
(ii) If $M^\sharp$ exists, then $M^\sharp = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, where
- $$\begin{aligned} M_1 &= AF^{(1)}(I + BS^\sharp C)AF^{(1)}A, \\ M_2 &= AF^{(1)}(I + BS^\sharp C)AF^{(1)}BS^\pi - AF^{(1)}BS^\sharp, \\ M_3 &= S^\pi CF^{(1)}(I + BS^\sharp C)AF^{(1)}A - S^\sharp CF^{(1)}A, \\ M_4 &= S^\pi CF^{(1)}AF^{(1)}BS^\pi + (S^\pi CF^{(1)}B - I)S^\sharp(CF^{(1)}BS^\pi - I). \end{aligned}$$

Proof The “only if” part.

Obviously,

$$M = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \quad (17)$$

and

$$M^2 = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & BS^\sharp \\ 0 & I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & S^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ S^\sharp C & I \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}. \quad (18)$$

By Lemma 1, $M^\sharp$ exists if and only if there exist matrices X, Y over R such that $M^2X = M$ and $YM^2 = M$.

If we let

$$X = \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -S^\sharp C & I \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \quad (19)$$

and

$$Y = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \begin{pmatrix} I & -BS^\sharp \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}. \quad (20)$$

It follows from (17), (18), (19) and $M^2X = M$, we can obtain

$$\begin{pmatrix} F & 0 \\ 0 & S^2 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} A & -BS^\sharp C \\ 0 & S \end{pmatrix}. \quad (21)$$

Similarly, it follows from (17), (18), (20) and $YM^2 = M$, we can obtain

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & S^2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ -SS^\sharp C & S \end{pmatrix}. \quad (22)$$

It follows from (21) and (22) that

$$FX_1 = A \quad \text{and} \quad Y_1F = A,$$

i.e., $R(A) \subset R(F)$ and $R_r(A) \subset R_r(F)$.

On the other hand, it follows from $F = A^2 + BS^\pi C$, $AB = B$ and $CA = C$ that we have

$$F = A(A + BS^\pi C) \quad \text{and} \quad F = (A + BS^\pi C)A,$$

i.e., $R(F) \subset R(A)$ and $R_r(F) \subset R_r(A)$, which imply $R(F) = R(A)$ and $R_r(F) = R_r(A)$.

It is easy to see that $F = A(A + BS^\pi C) = A(I + BS^\pi C)A = FX_1(I + BS^\pi C)Y_1F$, which implies that F is regular.

The “if” part.

Since F is regular, we can take $F^{(1)} \in F\{1\}$. Let

$$X_1 = F^{(1)}A, \quad X_2 = -F^{(1)}BS^\sharp, \quad X_3 = 0, \quad X_4 = S^\sharp$$

and

$$Y_1 = AF^{(1)}, Y_2 = 0, Y_3 = -SS^\sharp CF^{(1)}, Y_4 = S^\sharp.$$

Next we need to prove $M = M^2X$ and $M = YM^2$ hold, i.e., (21) and (22) hold. In fact, from $X_3 = 0, X_4 = S^\sharp$, it is easy to verify $S^2X_3 = 0$ and $S^2X_4 = S$.

Note that $R(A) = R(F)$, there exists $V \in R^{n \times n}$ such that $FV = A$. Hence, we have

$$A = FV = FF^{(1)}FV = FF^{(1)}A = FX_1.$$

and

$$\begin{aligned} FX_2 &= -SS^\sharp CAF^{(1)}F = -FF^{(1)}ABSS^\sharp = -FF^{(1)}FVBSS^\sharp \\ &= -ABSS^\sharp = -BSS^\sharp. \end{aligned}$$

This implies that (21) holds, i.e., $M = M^2X$.

Similarly, $R_r(A) = R_r(F)$ implies $WF = A$, where $W \in R^{n \times n}$. Hence we have

$$A = WF = WFF^{(1)}F = AF^{(1)}F = Y_1F.$$

and

$$\begin{aligned} Y_3F &= -SS^\sharp CF^{(1)}F = -SS^\sharp CAF^{(1)}F = -SS^\sharp CWWFF^{(1)}F = -SS^\sharp CWF \\ &= -SS^\sharp CA = -SS^\sharp C. \end{aligned}$$

It follows (22) holds, i.e., $M = YM^2$. By Lemma 1, we know $M^\sharp$ exists.

(ii) By Lemma 1, the representation of $M^\sharp$ can be obtained from $M^\sharp = YMX$. Next we calculate the $M^\sharp$,

$$\begin{aligned} M^\sharp &= \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} AF^{(1)} & 0 \\ -SS^\sharp CF^{(1)} & S^\sharp \end{pmatrix} \begin{pmatrix} I & -BS^\sharp \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} X \\ &= \begin{pmatrix} AF^{(1)} & 0 \\ S^\pi CF^{(1)} & S^\sharp \end{pmatrix} \begin{pmatrix} I & -BS^\sharp \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & S \end{pmatrix} \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -S^\sharp C & I \end{pmatrix} \\ &\quad \times \begin{pmatrix} F^{(1)}A & -F^{(1)}BSS^\sharp \\ 0 & S^\sharp \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} AF^{(1)} & 0 \\ S^\pi CF^{(1)} & S^\sharp \end{pmatrix} \begin{pmatrix} A & BS^\pi \\ 0 & S \end{pmatrix} \begin{pmatrix} I + BS^\sharp C & -B \\ -S^\sharp C & I \end{pmatrix} \begin{pmatrix} F^{(1)}A & F^{(1)}BS^\pi \\ 0 & S^\sharp \end{pmatrix} \\ &= \begin{pmatrix} AF^{(1)}(I + BS^\sharp C)A & -AF^{(1)}BSS^\sharp \\ S^\pi CF^{(1)}(I + BS^\sharp C)A - S^\sharp C & -S^\pi CF^{(1)}BSS^\sharp + S^\sharp \end{pmatrix} \\ &\quad \times \begin{pmatrix} F^{(1)}A & F^{(1)}BS^\pi \\ 0 & S^\sharp \end{pmatrix} \\ &= \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}. \end{aligned}$$

This complete the proof of Theorem 2. $\square$

In particular, we now give Corollaries and Example of Theorem 2.

Corollary 4 Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathfrak{R}^{n \times n}$, $D \in \mathfrak{R}^{m \times m}$, $CA = C$, $AB = B$, $S = D - CB$, if $S^\sharp$ exists, then

- (i) $M^\sharp$ exists if and only if $F = A^2 + BS^\pi C$ is regular and $r(F) = r(A)$;
(ii) If $M^\sharp$ exists, then the expression of $M^\sharp$ is the same as in Theorem 2.

Corollary 5 Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in K^{n \times n}$, $D \in K^{m \times m}$, $CA = C$, $AB = B$, $S = D - CB$, then

- (i) $M^\sharp$ exists if and only if $r(F) = r(A)$;
(ii) If $M^\sharp$ exists, then the expression of $M^\sharp$ is the same as in Theorem 2.

Example for Theorem 2: Let Z be the integer ring, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be the matrix over $Z/[8Z]$, where

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ 6 & -4 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 & -2 & 0 \\ 4 & 4 & 0 \end{pmatrix}, D = \begin{pmatrix} 3 & 6 \\ 4 & 0 \end{pmatrix}.$$

Obviously, $AB = B$, $CA = C$. We calculate

$$S = S^\sharp = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, S^\pi = \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}, F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 2, we have

$$M^\sharp = \begin{pmatrix} 7 & -2 & 0 & -3 & -2 \\ 4 & -3 & 0 & -6 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 1 & 2 \\ 4 & -4 & 0 & -4 & 0 \end{pmatrix}.$$

Remark 1 In fact, Theorem 1 improves the result of Theorem 1.1 in Zhao et al. (2012) to the rings, and Theorem 2 reduces the restrict condition of Corollary 2.2 in Liu and Yang (2012) and improves the relative results to even more general rings.

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