Acta Math Vietnam (2014) 39:121–131
DOI 10.1007/s40306-014-0054-3

GLOBAL EXISTENCE AND DECAY RATES OF SOLUTIONS OF GENERALIZED BENJAMIN–BONA–MAHONY EQUATIONS IN MULTIPLE DIMENSIONS

Hongmei Xu · Guyu Xin

Received: 3 December 2012 / Accepted: 4 January 2013 / Published online: 13 March 2014

Abstract We study the global existence and decay rates of the Cauchy problem for the generalized Benjamin–Bona–Mahony equations in multi-dimensional spaces. By using Fourier analysis, frequency decomposition, pseudo-differential operators and the energy method, we obtain global existence and optimal L_2 convergence rates of the solution.

Keywords Cauchy problem · Generalized Benjamin–Bona–Mahony equation · Multiple dimensions · Global existence · Optimal L_2 decay estimate

Mathematics Subject Classification (2010) 35L05

1 Introduction

In this paper, we study the global existence and decay rates of the smooth solution $u(x,t)$ to the scalar multi-dimensional generalized Benjamin–Bona–Mahony (GBBM) equations of the form

$$\begin{cases} \partial_t u - \Delta\partial_t u - \eta\Delta u + (\beta\cdot\nabla)u + \operatorname{div} f(u) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Here η is a positive constant, β is a real constant vector, $f(u) = (u^2, u^2, \dots, u^2)$, $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian, ∇ is a gradient operator, $n \geq 2$ is the spatial dimension.

The well known Benjamin–Bona–Mahony (BBM) equation

$$u_t - u_{xxt} + u_x + uu_x = 0, \quad -\infty < x < \infty,\ t > 0$$

H. Xu (✉)
College of Science, Hohai University, Nanjing 210098, Jiangsu Province, China
e-mail: xxu_hongmei@163.com

G. Xin
China Electronics Technology Group Corporation No. 28 Research Institute, Nanjing 210007, Jiangsu Province, China

and its counterpart, the Korteweg–de Vries (KdV) equation

$$u_t - u_{xxx} + u_x + uu_x = 0, \qquad -\infty < x < \infty,\ t > 0,$$

were both suggested as model equations for long waves in nonlinear dispersive media. The BBM equation was advocated by Benjamin, Bona and Mahony [3] in 1972. Since then, the periodic boundary value problems, the initial value problems and the initial boundary value problems, for various generalized BBM equations have been studied. The existence and uniqueness of solutions for GBBM have been proved by many authors [2, 3, 7, 8]. The decays of solutions were also studied in [1, 4–6, 10]. In [1, 4, 5, 10], the authors studied the equation in low spatial dimension. In [6], the equation in high spatial dimension was studied, but the authors just got a global solution $u(x,t) \in H^1$. In this paper, the goal is to get a global smoother solution and give its L_2 decay rates in high spatial dimension.

Throughout this paper, we denote the generic constants by C and write $D^\alpha f = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Let $W^{s,p}(\mathbb{R}^n)$, $s \in Z_+$, $p \in [1,\infty]$, be the usual Sobolev space with the norm

$$\|f\|_{W^{s,p}} := \sum_{|\alpha|=0}^{s} \|D^\alpha f\|_{L_p}.$$

In particular, $W^{s,2} = H^s$. The Fourier transformation with respect to the variable $x \in \mathbb{R}^n$ is

$$\hat{f}(\xi,t) = \int_{\mathbb{R}^n} f(x,t) e^{-\sqrt{-1}x\cdot\xi}\, dx$$

and the inverse Fourier transformation with respect to the variable ξ is

$$f(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi,t) e^{\sqrt{-1}x\cdot\xi}\, d\xi.$$

In this paper, all convolutions are only with respect to the spatial variable x.

Our main result is the following:

Theorem 1.1 *If $E = (\|u_0\|_{H^l} + \|\nabla u_0\|_{H^l})$ is small enough, $l \geq 1 + [\frac{n}{2}]$, then there exists a global solution $u(x,t)$ of (1.1) such that*

$$u \in L_\infty\big(0,\infty; H^l\big(\mathbb{R}^n\big)\big).$$

Moreover, we have $\|D_x^\alpha u\|_{L_2} \leq C(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}}$ for $|\alpha| \leq l$.

Remark The decay rate is the same as that of the heat equation, so our estimate is optimal.

The rest of the paper is arranged as follows. In Sect. 2, we get the local existence of the solution directly by constructing a Cauchy sequence and using energy estimation. In Sect. 3, by means of the Green function, we obtain a bound of the solution, then we extend the local solution to the global one. In Sect. 4, we get an L_2 decay estimate of the solution.

2 Local existence

In this section, we will construct a convergent sequence $\{u^{(m)}(x,t)\}$ to get the local solution, where $u^{(m)}(x,t)$ satisfy the following linear problem

$$\begin{cases} \partial_t u^{(m)} - \Delta \partial_t u^{(m)} - \eta \Delta u^{(m)} + (\beta \cdot \nabla) u^{(m)} = -\operatorname{div} f(u^{(m-1)}), \\ u^{(m)}(0,x) = u_0(x), \quad u^{(0)}(t,x) = 0. \end{cases} \tag{2.1}$$

We will construct a Banach space and prove that the sequence is convergent in this space, so the limit is the solution of (1.1).

First, we define a function space

$$\mathbf{X}_{T_0} = \Big\{ u(x,t) | \|u\|_{\mathbf{X}_{T_0}} = \sup_{0 \le t \le T_0} \|u\|_{H^l} \le C_0 E \Big\}.$$

Here $E = (\|u_0\|_{H^l} + \|\nabla u_0\|_{H^l})$ is small enough, $C_0 > \sqrt{l}$, $l \ge 1 + [\frac{n}{2}]$ and $n \ge 2$. The metric in $\mathbf{X}_{T_0}$ is induced by the norm $\|u\|_{\mathbf{X}_{T_0}}$:

$$\rho(u,v) = \|u - v\|_{\mathbf{X}_{T_0}} \quad \forall u, v \in \mathbf{X}_{T_0}.$$

It is obvious that $\mathbf{X}_{T_0}$ is a nonempty complete space.

Lemma 2.1 *There exists some constant T_1 such that $\{u^{(m)}(x,t)\}$ belongs to $\boldsymbol{X}_{T_1}$.*

Proof We will prove this lemma by induction on m. When $m = 1$, we have

$$\begin{cases} u_t^{(1)} - \Delta u_t^{(1)} - \eta \Delta u^{(1)} + (\beta \cdot \nabla) u^{(1)} = 0, \\ u^{(1)}(0,x) = u_0(x). \end{cases} \tag{2.2}$$

Multiplying the first equation by $u^{(1)}$ and integrating with respect to the variable x in $\mathbb{R}^n$, we get

$$\frac{1}{2} \frac{\partial}{\partial t} \big\| u^{(1)} \big\|_{L_2}^2 + \frac{1}{2} \frac{\partial}{\partial t} \big\| \nabla u^{(1)} \big\|_{L_2}^2 + \eta \big\| \nabla u^{(1)} \big\|_{L_2}^2 = 0,$$

here we use the fact that $\int (\beta \cdot \nabla u^{(1)}) \cdot u^{(1)} dx = 0$. Then

$$\big\| u^{(1)} \big\|_{L_2}^2 + \big\| \nabla u^{(1)} \big\|_{L_2}^2 \le \|u_0\|_{L_2}^2 + \|\nabla u_0\|_{L_2}^2.$$

From (2.2) we have

$$D_x^\alpha u_t^{(1)} - D_x^\alpha \Delta u_t^{(1)} - \eta D_x^\alpha \Delta u^{(1)} + D_x^\alpha (\beta \cdot \nabla) u^{(1)} = -D_x^\alpha \operatorname{div} f\big(u^{(0)}\big) = 0. \tag{2.3}$$

Multiplying the two sides in (2.3) by $D_x^\alpha u^{(1)}$ and integrating with respect to the variable x in $\mathbb{R}^n$, we also get

$$\frac{1}{2} \frac{\partial}{\partial t} \big\| D_x^\alpha u^{(1)} \big\|_{L_2}^2 + \frac{1}{2} \frac{\partial}{\partial t} \big\| D_x^\alpha \nabla u^{(1)} \big\|_{L_2}^2 + \eta \big\| \nabla D_x^\alpha u^{(1)} \big\|_{L_2}^2 = 0. \tag{2.4}$$

From (2.4) we have

$$\big\| D_x^\alpha u^{(1)} \big\|_{L_2}^2 (t) + \big\| D_x^\alpha \nabla u^{(1)} \big\|_{L_2}^2 (t) \le \big\| D_x^\alpha u_0 \big\|_{L_2}^2 + \big\| D_x^\alpha \nabla u_0 \big\|_{L_2}^2.$$

Thus, for every T_1, we have $u^{(1)} \in \mathbf{X}_{T_1}$. Supposing that $u^{(m)}(x,t) \in \mathbf{X}_{T_1}$, we next prove that $u^{(m+1)}(x,t) \in \mathbf{X}_{T_1}$. By definition, $u^{(m+1)}(x,t)$ satisfies the following equation:

$$\begin{cases} \partial_t u^{(m+1)} - \Delta\partial_t u^{(m+1)} - \eta\Delta u^{(m+1)} + (\beta\cdot\nabla)u^{(m+1)} = -\operatorname{div} f(u^{(m)}), \\ u^{(m+1)}(0,x) = u_0(x). \end{cases}$$

Then

$$D_x^\alpha \partial_t u^{(m+1)} - D_x^\alpha \Delta\partial_t u^{(m+1)} - \eta D_x^\alpha \Delta u^{(m+1)} + D_x^\alpha(\beta\cdot\nabla)u^{(m+1)} = -D_x^\alpha \operatorname{div} f\big(u^{(m)}\big),$$

so, we get

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\big\|D_x^\alpha u^{(m+1)}\big\|_{L_2}^2 + \frac{1}{2}\frac{\partial}{\partial t}\big\|D_x^\alpha \nabla u^{(m+1)}\big\|_{L_2}^2 + \eta\big\|\nabla D_x^\alpha u^{(m+1)}\big\|_{L_2}^2 \\ &\quad = -\int D_x^\alpha \operatorname{div} f\big(u^{(m)}\big) D_x^\alpha u^{(m+1)}\,dx \\ &\quad = \int D_x^\alpha f\big(u^{(m)}\big)\nabla D_x^\alpha u^{(m+1)}\,dx \\ &\quad \le \frac{\eta}{2}\big\|\nabla D_x^\alpha u^{(m+1)}\big\|_{L_2}^2 + C\big\|D_x^\alpha f\big(u^{(m)}\big)\big\|_{L_2}^2. \end{aligned} \tag{2.5}$$

To proceed, we need the following inequality:

$$\big\|D^s(FG)\big\|_{L^r} \le C\big(\big\|D^s F\big\|_{L^p}\|G\|_{L^q} + \|F\|_{L^p}\big\|D^s G\big\|_{L^q}\big), \tag{2.6}$$

where $1 \le r, p, q \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Noticing that $f(u) = (u^2, u^2, \dots, u^2)$, we have

$$\big\|D_x^\alpha f\big(u^{(m)}\big)\big\|_{L_2} \le C\big\|u^{(m)}\big\|_{L_\infty}\big\|D_x^\alpha u^{(m)}\big\|_{L_2}. \tag{2.7}$$

By the Sobolev embedding inequality, we have $\|u\|_{L_\infty} \le C\|u\|_{H^l}$ if $l \ge 1 + [\frac{n}{2}]$. From (2.5), (2.7), we get

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\big\|D_x^\alpha u^{(m+1)}\big\|_{L^2}^2 + \frac{1}{2}\frac{\partial}{\partial t}\big\|D_x^\alpha \nabla u^{(m+1)}\big\|_{L^2}^2 + \frac{\eta}{2}\big\|\nabla D_x^\alpha u^{(m+1)}\big\|_{L^2}^2 \\ &\quad \le C\big\|u^{(m)}\big\|_{H^l}^2\big\|D_x^\alpha u^{(m)}\big\|_{L^2}^2. \end{aligned}$$

Then

$$\begin{aligned} \big\|D_x^\alpha u^{(m+1)}\big\|_{L^2}^2 + \big\|D_x^\alpha \nabla u^{(m+1)}\big\|_{L^2}^2 &\le \big\|D_x^\alpha u_0\big\|_{L^2}^2 + \big\|D_x^\alpha \nabla u_0\big\|_{L^2}^2 + C\int_0^t E^2E^2\,ds \\ &\le E^2 + CE^4 t. \end{aligned} \tag{2.8}$$

When $T_1 \le (C_0^2 - 1)\frac{1}{CE^2}$, from (2.8) we have $\{u^{(m+1)}(x,t)\} \in \mathbf{X}_{T_1}$. By the induction method, our lemma is proved. □

Lemma 2.2 *There exists a positive constant T_0 such that $\{u^{(m)}(x,t)\}$ constructed by* (2.1) *is a Cauchy sequence in $\mathbf{X}_{T_0}$.*

Proof We only need to prove that

$$\|u^{(m+1)} - u^{(m)}\|_{\mathbf{X}_{T_0}} \le \lambda \|u^{(m)} - u^{(m-1)}\|_{\mathbf{X}_{T_0}} \tag{2.9}$$

for some $0 < \lambda < 1$. By (2.1), for every m, $u^{(m+1)} - u^{(m)}$ satisfies the following equation:

$$\begin{cases} \partial_t(u^{(m+1)} - u^{(m)}) - \Delta\partial_t(u^{(m+1)} - u^{(m)}) - \eta\Delta(u^{(m+1)} - u^{(m)}) \\ \quad + (\beta \cdot \nabla)(u^{(m+1)} - u^{(m)}) = -\operatorname{div} f(u^{(m)}) + \operatorname{div} f(u^{(m-1)}), \\ (u^{(m+1)} - u^{(m)})(0, x) = 0. \end{cases} \tag{2.10}$$

Thus we have

$$\begin{aligned} &D_x^\alpha \partial_t\big(u^{(m+1)} - u^{(m)}\big) - D_x^\alpha \Delta\partial_t\big(u^{(m+1)} - u^{(m)}\big) - \eta D_x^\alpha \Delta\big(u^{(m+1)} - u^{(m)}\big) \\ &\quad + D_x^\alpha(\beta \cdot \nabla)\big(u^{(m+1)} - u^{(m)}\big) = -D_x^\alpha \operatorname{div}\big(f\big(u^{(m)}\big) - f\big(u^{(m-1)}\big)\big). \end{aligned} \tag{2.11}$$

Multiplying the two sides of (2.11) by $D_x^\alpha(u^{(m+1)} - u^{(m)})$ and integrating with respect to the variable x in $\mathbb{R}^n$, we get

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\|D_x^\alpha\big(u^{(m+1)} - u^{(m)}\big)\|_{L_2}^2 + \frac{1}{2}\frac{\partial}{\partial t}\|D_x^\alpha\nabla\big(u^{(m+1)} - u^{(m)}\big)\|_{L_2}^2 + \eta\|\nabla D_x^\alpha\big(u^{(m+1)} - u^{(m)}\big)\|_{L_2}^2 \\ &\quad = -\int D_x^\alpha \operatorname{div}\big(f\big(u^{(m)}\big) - f\big(u^{(m-1)}\big)\big) D_x^\alpha\big(u^{(m+1)} - u^{(m)}\big)\,dx \\ &\quad = \int D_x^\alpha\big(f\big(u^{(m)}\big) - f\big(u^{(m-1)}\big)\big)\nabla D_x^\alpha\big(u^{(m+1)} - u^{(m)}\big)\,dx \\ &\quad \le \frac{\eta}{2}\|\nabla D_x^\alpha\big(u^{(m+1)} - u^{(m)}\big)\|_{L_2}^2 + C\big(\|u^{(m)}\|_{H^l}^2 + \|u^{(m-1)}\|_{H^l}^2\big)\|D_x^\alpha\big(u^{(m)} - u^{(m-1)}\big)\|_{L_2}^2. \end{aligned}$$

Thus

$$\frac{\partial}{\partial t}\big(\|D_x^\alpha\big(u^{(m+1)} - u^{(m)}\big)\|_{L_2}^2 \le C\big(\|u^{(m)}\|_{H^l}^2 + \|u^{(m-1)}\|_{H^l}^2\big)\big(\|D_x^\alpha\big(u^{(m)} - u^{(m-1)}\big)\|_{L_2}^2\big)\big).$$

Noticing that $(u^{(m+1)} - u^{(m)})(0, x) = 0$, and choosing $T_0 = \min(T_1, \frac{1}{4CE^2})$, we have

$$\begin{aligned} \|\big(u^{(m+1)} - u^{(m)}\big)\|_{H^l}^2 &\le C\int_0^t \big(\|u^{(m)}\|_{H^l}^2 + \|u^{(m-1)}\|_{H^l}^2\big)\|u^{(m)} - u^{(m-1)}\|_{H^l}^2\,ds \\ &\le 2CE^2T_0\|u^{(m)} - u^{(m-1)}\|_{\mathbf{X}_{T_0}}^2 \\ &\le \frac{1}{2}\|u^{(m)} - u^{(m-1)}\|_{\mathbf{X}_{T_0}}^2. \end{aligned}$$

Thus we can get (2.9). □

Since $\mathbf{X}_{T_0}$ is a complete metric space, by Lemmas 2.1 and 2.2, there exists a $u(x,t) \in \mathbf{X}_{T_0}$ such that

$$\begin{cases} \partial_t u - \Delta\partial_t u - \eta\Delta u + (\beta \cdot \nabla)u + \operatorname{div} f(u) = 0, \\ u(0, x) = u_0(x). \end{cases}$$

Thus we have proved the local existence of (1.1). Next we will prove the global existence.

3 Bounded estimates and global existence

In this section, we want to get $\|u\|_{H^l} \le CE$, then extend the local solution to a global one.

Lemma 3.1 *If $u_0 \in H^1$, we have $u \in L_\infty(0, +\infty; H^1)$.*

Proof Multiplying the first equation by u and integrating with respect to the variable x in $\mathbb{R}^n$, we get

$$\partial_t u - \Delta\partial_t u - \eta\Delta u + (\beta\cdot\nabla)u = -\operatorname{div} f(u),$$

then we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_{L_2}^2 + \frac{1}{2}\frac{\partial}{\partial t}\|\nabla u\|_{L_2}^2 + \eta\|\nabla u\|_{L_2}^2 = -\int u \operatorname{div} f(u)\,dx = \int f(u)\cdot\nabla u\,dx = 0.$$

Thus

$$\|u\|_{L_2}^2 + \|\nabla u\|_{L_2}^2 + \int_0^t \|\nabla u\|_{L_2}^2 ds \le \|u_0\|_{L_2}^2 + \|\nabla u_0\|_{L_2}^2, \tag{3.1}$$

and our lemma is proved. □

In order to get a bound of u, first we want to get an explicit expression of u through the Green function. The linearized system of (1.1) is of the form

$$\partial_t u - \Delta\partial_t u - \eta\Delta u + (\beta\cdot\nabla)u = 0. \tag{3.2}$$

The Green function of (3.2) is defined as

$$\begin{cases}\partial_t G - \Delta\partial_t G - \eta\Delta G + (\beta\cdot\nabla)G = 0,\\ G|_{t=0} = \delta(x).\end{cases}$$

Direct calculation shows that the Fourier transformation of G is

$$\hat{G} = e^{-\frac{\eta|\xi|^2}{1+|\xi|^2}t - \frac{i\beta\cdot\xi t}{1+|\xi|^2}}. \tag{3.3}$$

Set

$$\hat{H} = \frac{1}{1+|\xi|^2}\hat{G}. \tag{3.4}$$

Due to Duhamel's principle, we know that the solution of (1.1) can be expressed by

$$u(x,t) = G * u_0 - \int_0^t H(t-s) * \operatorname{div} f(u)(s)\,ds. \tag{3.5}$$

Next we want to analyze u through estimates of G, H. For this part, we must estimate decay rates of $\hat{G}, \hat{H}$ separately for low and high frequencies. Thus we set

$$\chi_1(\xi) = \begin{cases}1, & |\xi| \le R,\\ 0, & |\xi| > R+1,\end{cases} \qquad \chi_2(\xi) = \begin{cases}1, & |\xi| \ge R+1,\\ 0, & |\xi| < R,\end{cases}$$

where χ_1, χ_2 are smooth cut-off functions and $\chi_1(\xi) + \chi_2(\xi) = 1$.

Set $\hat{G}_i = \chi_i\hat{G}$, $\hat{H}_i = \chi_i\hat{H}$. For low frequencies, we have the following lemma.

Lemma 3.2 *For* $2 \le p \le \infty$, *we have* $\|D_x^\alpha G_1\|_{L_p} \le Ct^{-\frac{|\alpha|}{2}-\frac{n}{2}(1-\frac{1}{p})}$, $\|D_x^\alpha H_1\|_{L_p} \le Ct^{-\frac{|\alpha|}{2}-\frac{n}{2}(1-\frac{1}{p})}$.

Proof By the Hausdorff–Young inequality, if the integers $p \ge 2$ and q satisfy $\frac{1}{p}+\frac{1}{q}=1$, then we have

$$\begin{aligned}\big\|D_x^\alpha G_1\big\|_{L_p} \le C\big\|\xi^\alpha \hat{G}_1\big\|_{L_q} &\le C\bigg(\int_{|\xi|\le R+1} |\xi|^{|\alpha|q} e^{-\frac{\eta q|\xi|^2}{1+|\xi|^2}t}\,d\xi\bigg)^{\frac{1}{q}} \\ &\le C\bigg(\int_{|\xi|\le R+1} |\xi|^{|\alpha|q} e^{-\frac{\eta q|\xi|^2}{1+(R+1)^2}t}\,d\xi\bigg)^{\frac{1}{q}} \\ &\le C\big(t^{-\frac{n}{2}-\frac{|\alpha|q}{2}}\big)^{\frac{1}{q}} \\ &= Ct^{-\frac{|\alpha|}{2}-\frac{n}{2}(1-\frac{1}{p})}\end{aligned}$$

and

$$\big\|D_x^\alpha H_1\big\|_{L_p} \le C\bigg\|\xi^\alpha \frac{1}{1+|\xi|^2}\hat{G}_1\bigg\|_{L_q} \le C\big\|\xi^\alpha \hat{G}_1\big\|_{L_q} \le Ct^{-\frac{|\alpha|}{2}-\frac{n}{2}(1-\frac{1}{p})}.$$

□

Next we analyze the constructions of G, H for high frequencies.

Lemma 3.3 *There exist a positive constant b and functions $f_1(x)$, $f_2(x)$, $f_3(x)$ such that $G_2 \le Ce^{-bt}(\delta(x)+f_1(x))$, $H_2 \le Ce^{-bt}(\delta(x)+f_2(x))$, $\nabla H_2 \le Ce^{-bt}(\delta(x)+f_3(x))$. Here $\|f_1\|_{L_1} < C$, $\|f_2\|_{L_1} < C$, $\|f_3\|_{L_1} < C$.*

Proof We just prove the first inequality; the proofs of the others are similar. By (3.3), if $|\beta| \ge 1$ then we have

$$\big|D_\xi^\beta \hat{G}_2\big| \le C|\xi|^{-|\beta|-1} e^{-\frac{\eta|\xi|^2}{1+|\xi|^2}t} \le C|\xi|^{-|\beta|-1}e^{-bt}.$$

From this and Lemma 3.2 in [9], we get our result. □

Lemma 3.4 $\|u\|_{L_\infty} \le CE$.

Proof By Young's inequality, from (3.5) we have

$$\begin{aligned}\|u\|_{L_\infty} \le \|G_1 * u_0\|_{L_\infty} + \|G_2 * u_0\|_{L_\infty} + \int_0^t \big\|H_1(t-s)\big\|_{L_\infty} \|\operatorname{div} f(u)\|_{L_1}\,ds \\ + \int_0^t \big\|H_2(t-s) * \operatorname{div} f(u)\big\|_{L_\infty}\,ds. \qquad (3.6)\end{aligned}$$

By Lemmas 3.2, 3.3 and the Sobolev embedding inequality, we have

$$\begin{aligned}\|G_1 * u_0\|_{L_\infty} + \|G_2 * u_0\|_{L_\infty} &\le \|G_1 * u_0\|_{H^l} + Ce^{-bt}\big(\|u_0\|_{L_\infty} + \|f_1\|_{L_1}\|u_0\|_{L_\infty}\big)\\ &\le C\|G_1\|_{L_\infty}\|u_0\|_{H^l} + Ce^{-bt}\|u_0\|_{H^l} \\ &\le CE.\end{aligned} \tag{3.7}$$

Setting $M(t) = \sup_{0\le s\le t, |\alpha|\le 1} \|D_x^\alpha u\|_{L_\infty}$, we have

$$\big\|D_x^\alpha u\big\|_{L_\infty} \le M, \quad |\alpha| \le 1. \tag{3.8}$$

Next we will prove that $M \le CE$. From (3.1) we have

$$\|u\|_{L_2} \le CE, \qquad \int_0^t \|\nabla u\|_{L_2}^2 ds \le CE^2.$$

By Lemma 3.2, (3.1) and Hölder inequality, we have

$$\begin{aligned}\int_0^t \big\|H_1(t-s)\big\|_{L_\infty}\big\|\operatorname{div} f(u)\big\|_{L_1} ds &\le C\int_0^t (1+t-s)^{-\frac{n}{2}}\|\nabla u\|_{L_2}\|u\|_{L_2}\, ds\\ &\le C\bigg(\int_0^t (1+t-s)^{-n}\|u\|_{L_2}^2 ds\bigg)^{\frac12}\bigg(\int_0^t \|\nabla u\|_{L_2}^2\, ds\bigg)^{\frac12}\\ &\le CE\end{aligned} \tag{3.9}$$

By Lemma 3.3 and (3.8),

$$\begin{aligned}\int_0^t \big\|H_2(t-s) * \operatorname{div} f(u)\big\|_{L_\infty} ds &\le \int_0^t e^{-b(t-s)}\big(\big\|\operatorname{div} f(u)\big\|_{L_\infty} + \|f_2\|_{L_1}\big\|\operatorname{div} f(u)\big\|_{L_\infty}\big)\\ &\le \int_0^t e^{-b(t-s)}\|u\|_{L_\infty}\|\nabla u\|_{L_\infty}\, ds\\ &\le C\int_0^t e^{-b(t-s)} M^2\, ds\\ &\le CM^2.\end{aligned} \tag{3.10}$$

By (3.6), (3.7), (3.9), and (3.10), we have

$$\|u\|_{L_\infty} \le CE + CE + CM^2.$$

Thus $M(t) \le CE + CM^2(t)$. Because $M(0) = \sup_{|\alpha|\le 1}\|D_x^\alpha u_0\|_{L_\infty} \le \|u_0\|_{H^l} + \|\nabla u_0\|_{H^l} \le CE$, this implies that $M(t) \le CE$; thus our result is proved. □

Lemma 3.5 $\|u\|_{H^l} \le CE$. $\|\nabla u\|_{H^l} \le CE$.

Proof By (2.6) and the formula $f(u) = (u^2, u^2, \ldots, u^2)$, we can assert that

$$\big\|D_x^\alpha f(u)\big\|_{L_2} \le C\|u\|_{L_\infty}\big\|D_x^\alpha u\big\|_{L_2}. \tag{3.11}$$

From (1.1) we have

$$D_x^\alpha \partial_t u - D_x^\alpha \Delta\partial_t u - \eta D_x^\alpha \Delta u + D_x^\alpha(\beta\cdot\nabla)u = -D_x^\alpha \operatorname{div} f(u). \tag{3.12}$$

By (3.12) and Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left\|D_x^\alpha u\right\|_{L_2}^2+\frac{1}{2}\frac{\partial}{\partial t}\left\|D_x^\alpha \nabla u\right\|_{L_2}^2+\eta\left\|D_x^\alpha \nabla u\right\|_{L_2}^2\\
&\quad=-\int D_x^\alpha u D_x^\alpha \operatorname{div} f(u)dx\\
&\quad=\int D_x^\alpha f(u)\cdot D_x^\alpha \nabla u\, dx\\
&\quad\le \frac{\eta}{2}\left\|D_x^\alpha \nabla u\right\|_{L_2}^2+C\left\|D_x^\alpha f(u)\right\|_{L_2}^2. \qquad (3.13)
\end{aligned}$$

From (3.11), (3.13) and Lemma 3.4, we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\left\|D_x^\alpha u\right\|_{L_2}^2+\frac{1}{2}\frac{\partial}{\partial t}\left\|D_x^\alpha \nabla u\right\|_{L_2}^2+\frac{\eta}{2}\left\|D_x^\alpha \nabla u\right\|_{L_2}^2 &\le C\left\|D_x^\alpha f(u)\right\|_{L_2}^2\\
&\le C\|u\|_{L_\infty}^2\left\|D_x^\alpha u\right\|_{L_2}^2\\
&\le CE^2\left\|D_x^\alpha u\right\|_{L_2}^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\left\|D_x^\alpha u\right\|_{L_2}^2+\left\|D_x^\alpha \nabla u\right\|_{L_2}^2+\frac{\eta}{2}\int_0^t\left\|D_x^\alpha \nabla u\right\|_{L_2}^2\\
&\quad\le CE^2\int_0^t\left\|D_x^\alpha u\right\|_{L_2}^2 ds+\left\|D_x^\alpha u_0\right\|_{L_2}^2+\left\|D_x^\alpha \nabla u_0\right\|_{L_2}^2.
\end{aligned}$$

By (3.1), if $|\alpha|=1$ then

$$\left\|D_x^\alpha u\right\|_{L_2}\le CE,\qquad \int_0^t\left\|D_x^\alpha \nabla u\right\|_{L_2}^2 ds\le CE^2.$$

Using induction, we have

$$\left\|D_x^\alpha u\right\|_{L_2}\le CE,\qquad \left\|D_x^\alpha \nabla u\right\|_{L_2}\le CE,\quad |\alpha|\le l. \qquad (3.14)$$

Thus our lemma is proved. □

According to Lemma 3.5 and using the local solution, by the usual method, we can derive a global solution for (1.1) such that $u\in L_\infty(0,\infty,H^l(R^n))$.

4 Decay estimation

Setting $M(t)=\sup_{|\alpha|\le l,0\le s\le t}\|D_x^\alpha u\|_{L_2}(1+s)^{\frac{n}{4}+\frac{|\alpha|}{2}}$, we have

$$\left\|D_x^\alpha u\right\|_{L_2}\le M(t)(1+s)^{-\frac{n}{4}-\frac{|\alpha|}{2}},\quad |\alpha|\le l. \qquad (4.1)$$

By (4.1),

$$\left\|D_x^\alpha f(u)(s)\right\|_{L_1}\le \sum_{|\alpha_1|+|\alpha_2|=|\alpha|}\left\|D_x^{\alpha_1} u\right\|_{L_2}\left\|D_x^{\alpha_1} u\right\|_{L_2}\le CM^2(t)(1+s)^{-\frac{n}{2}-\frac{|\alpha|}{2}}. \qquad (4.2)$$

From (3.11), Lemma 3.4 and (4.1), it follows that

$$\left\|D_x^\alpha f(u)\right\|_{L_2} \le C\|u\|_{L_\infty}\left\|D_x^\alpha u\right\|_{L_2} \le CEM(t)(1+s)^{-\frac{n}{4}-\frac{|\alpha|}{2}}, \quad |\alpha| \le l. \tag{4.3}$$

By (3.4) and Young's inequality,

$$\begin{aligned}\left\|D_x^\alpha u\right\|_{L_2} \le{}& \left\|D_x^\alpha G_1\right\|_{L_2}\|u_0\|_{L_1} + \left\|G_2 * D_x^\alpha u_0\right\|_{L_2} \\ &+ \int_0^t \left\|D_x^\alpha H_1(t-s) * \operatorname{div} f(u)\right\|_{L_2} ds + \int_0^t \left\|\nabla H_2(t-s) * D_x^\alpha f(u)\right\|_{L_2} ds.\end{aligned} \tag{4.4}$$

From Lemmas 3.2 and 3.3, we have

$$\left\|D_x^\alpha G_1\right\|_{L_2}\|u_0\|_{L_1} + \left\|G_2 * D_x^\alpha u_0\right\|_{L_2} \le CE(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}}, \quad |\alpha| \le l. \tag{4.5}$$

When $|\alpha| \le l$, from Lemma 3.2, (4.2) and (3.1), we get

$$\begin{aligned}&\int_0^t \left\|D_x^\alpha H_1(t-s) * \operatorname{div} f(u)\right\|_{L_2} ds \\ &\le \int_0^{\frac{t}{2}} \left\|D_x^\alpha \nabla H_1(t-s)\right\|_{L_2}\left\|f(u)\right\|_{L_1} ds \\ &\quad + \int_{\frac{t}{2}}^t \left\|\nabla H_1(t-s)\right\|_{L_2}\left\|D_x^\alpha f(u)\right\|_{L_1} ds \\ &\le \int_0^{\frac{t}{2}} (1+t-s)^{-\frac{n}{4}-\frac{|\alpha|}{2}-\frac{1}{2}} M^2 (1+s)^{-\frac{n}{2}} ds \\ &\quad + \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{n}{4}-\frac{1}{2}} M^2(t)(1+s)^{-\frac{n}{2}-\frac{|\alpha|}{2}} ds \\ &\le CM^2(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}} + C(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}} M^2(t).\end{aligned} \tag{4.6}$$

From Lemma 3.3 and (4.3), with $|\alpha| \le l$, we have

$$\int_0^t \left\|\nabla H_2(t-s) * D_x^\alpha f(u)\right\|_{L_2} ds \tag{4.7}$$

$$\begin{aligned}&\le \int_0^t e^{-b(t-s)}\left(\left\|D_x^\alpha f(u)\right\|_{L_2} + \|f_3\|_{L_1}\left\|D_x^\alpha f(u)\right\|_{L_2}\right) ds \\ &\le \int_0^t e^{-b(t-s)} CEM(t)(1+s)^{-\frac{n}{4}-\frac{|\alpha|}{2}} ds \\ &\le CEM(t)(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}}.\end{aligned} \tag{4.8}$$

From (4.4)–(4.7) we obtain

$$\left\|D_x^\alpha u\right\|_{L_2} \le CE(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}} + CM^2(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}} + CEM(t)(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}},$$

thus $M(t) \leq CE + CM^2(t) + CEM(t)$. In addition, since $M(0) \leq CE$, we have $M(t) \leq CE$. Therefore,

$$\left\| D_x^\alpha u \right\|_{L_2} \leq CE(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}}. \tag{4.9}$$

Thus we get Theorem 1.1, our main result in this paper.

Acknowledgements The first author was supported by NNSF 11101121.

References

1. Albert, J.: On the decay of solutions of the generalized Benjamin–Bona–Mahony equations. J. Math. Anal. Appl. **141**, 527–537 (1989)
2. Avrin, J., Goldstein, J.A.: Global existence for the Benjamin–Bona–Mahony equation in arbitrary dimensions. Nonlinear Anal. **9**, 861–865 (1985)
3. Benjamin, T.B., Bona, J.L., Mahony, J.J.: Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. Ser. A, Math. Phys. Sci. **272**, 47–78 (1972)
4. Biler, P.: Long time behavior of solutions of the generalized Benjamin–Bona–Mahony equation in two space dimensions. Differ. Integral Equ. **5**, 891–901 (1992)
5. Fang, S.M., Guo, B.L.: Long time behavior for solution of initial-boundary value problem for one class of system with multidimensional inhomogeneous GBBM equations. Appl. Math. Mech. **26**(6), 659–664 (2005)
6. Fang, S., Guo, B.: The decay rates of solutions of generalized Benjamin–Bona–Mahony equations in multi-dimensions. Nonlinear Anal. **69**, 2230–2235 (2008)
7. Goldstein, J.A., Wichnoski, B.J.: On the Benjamin–Bona–Mahony equation in higher dimensions. Nonlinear Anal. **4**, 665–675 (1980)
8. Guo, B.L.: Initial boundary value problem for one class of system of multidimensional inhomogeneous GBBM equations. Chin. Ann. Math., Ser. B **8**(2), 226–238 (1987)
9. Wang, W., Yang, T. : The pointwise estimate of solutions of Euler equations with damping in multidimensions. J. Differ. Equ. **173**, 410–450 (2001)
10. Zhang, L.H.: Decay of solutions of generalized Benjamin–Bona–Mahony–Burgers equations in n-space dimensions. Nonlinear Anal. **25**, 1343–1369 (1995)