Optimization Methods for Mixed Integer Weakly Concave Programming Problems

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Received: 8 March 2014 / Revised: 8 May 2014 / Accepted: 13 May 2014 /

Published online: 27 June 2014

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Abstract In this paper, we consider a class of mixed integer weakly concave programming problems (MIWCPP) consisting of minimizing a difference of a quadratic function and a convex function. A new necessary global optimality conditions for MIWCPP is presented in this paper. A new local optimization method for MIWCPP is designed based on the necessary global optimality conditions, which is different from the traditional local optimization method. A global optimization method is proposed by combining some auxiliary functions and the new local optimization method. Furthermore, numerical examples are also presented to show that the proposed global optimization method for MIWCPP is efficient.

Keywords Global optimality conditions · Local optimization method · Global optimization method · Mixed integer weakly concave programming problems

Mathematics Subject Classification 41A65 · 41A29 · 90C30

This research was partly supported by Natural Science Foundation of Chongqing (Nos. cstc2013jjB00001 and cstc2011jjA00010).

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1 Introduction

It is well known that the sum of the quadratic and convex function is called to be weakly convex function (see [1]), which has been investigated in many literatures, such as [1, 2]. In this paper, we consider the class of mixed integer weakly concave programming problems (MIWCPP) consisting of minimizing a difference of a quadratic function and a convex function.

Models of MIWCPP cover a large spectrum of global optimization problems including mixed integer quadratic optimization problems and mixed integer concave minimization problems. Some problems in the layout design of integrated electronic circuits can be modeled as a quadratic program [3], and the $\{0, 1\}$ quadratic problems also arise in combinatorial optimization [4], financial analysis problems [5], computer-aided design problems [6], and message traffic management [7]. For concave minimization problems, one can refer to [8–12].

As far as we know, there are many applications for mixed integer programming problems in many areas, such as engineering design, computational chemistry, computational biology, communications and finance, reliability networks, and optimization of core reload patterns for nuclear reactors, as well as the planning of the production activities. Floudas [13] gives an overview of many applications, including process synthesis, process design, process synthesis and design under uncertainty, molecular design, interaction of design, synthesis and control, process operations, facility location and allocation, facility planning, and scheduling and topology of transportation networks. For applications of mixed integer programming models in reliability optimization, chemical engineering, and core reload patterns for nuclear reactor can be found in [14–16]. More applications can be found in [17–23]. Recently, the $\{0, 1\}$ mixed integer programming problems are also discussed by many researchers (see [24–27]).

Most approaches for the mixed integer nonlinear programming problems are branch-and-bound, decomposition, and outer approximation method (see [21, 24, 28–35]). It is very difficult to solve the mixed integer programming problems due to the nonlinearity and the mixture of discrete and continuous variables. The existing algorithms are not polynomial, especially when the size of the objective function is very large; these algorithms manifest their inadequate, also we even do not know whether the results we find are the global solutions or not. Thus, the global optimality conditions and other novel algorithms for mixed integer programming become the focus of many researches in recent years. Some global optimality conditions characterizing global minimizer of quadratic minimization problem has been discussed in [36–39]. [40] discussed some global optimality conditions for mixed integer quadratic programming problems. [41] discussed some global optimality conditions for MIWCPP with only box or binary constraints. [42] discussed some global optimality conditions for MIWCPP with mixed integer constraints.

In this paper, we will further discuss some new necessary global optimality conditions for MIWCPP, which extend the results given in [41] and [42]. Then, we will use the given necessary global optimality conditions to develop a new local optimization method (LOM) for MIWCPP. Finally, a global optimization method for MIWCPP is also proposed in this paper, which is designed based on the new

local optimization method and some special auxiliary functions. The methods proposed in this paper are very different from the existed methods. First, the LOM given in this paper is based on the necessary global optimality conditions of MIWCPP. Second, the global optimization method is combined the new local optimization method and some auxiliary functions for MIWCPP. Some numerical examples are also presented to show that the proposed novel optimization methods for MIWCPP are efficient.

The layout of the paper is as follows. Some new necessary global optimality conditions for MIWCPP are proposed in Sect. 2. A new LOM based on the necessary global optimality conditions is developed in Sect. 3. A global optimization method based on auxiliary functions and the proposed new LOM is followed in Sect. 4. Section 5 provides several examples to illustrate the efficiency of the given methods. Section 6 gives the conclusion of this paper

2 Necessary Global Optimality Conditions for MIWCPP

Throughout this paper, the real line is denoted by $\mathbb{R}$ and the n -dimensional Euclidean space is denoted by $\mathbb{R}^n$. For vectors $x, y \in \mathbb{R}^n$, $x \geq y$ means that $x_i \geq y_i$, for $i = 1, \dots, n$. The notation $A \succeq B$ means $A - B$ is positive semidefinite and $A \preceq 0$ means $-A \succeq 0$. A diagonal matrix with diagonal elements $\alpha_1, \dots, \alpha_n$ is denoted by $\text{diag}(\alpha)$ or $\text{diag}(\alpha_1, \dots, \alpha_n)$, where $\alpha = (\alpha_1, \dots, \alpha_n)^T$.

In this paper, we consider the following MIWCPP:

$$\begin{aligned} \text{MIWCPP} \quad & \min \quad f(x) = \frac{1}{2}x^T Ax - g(x) \\ & \text{s.t.} \quad x \in S, \end{aligned} \quad (2.1)$$

where $S \subset \mathbb{R}^n$ and $S = \prod_{i=1}^n S_i$, $S_i = \{0, 1\}$, $i = 1, \dots, k$ and $S_i = [0, 1]$, $i = k + 1, \dots, n$, $A = (a_{ij})_{n \times n} \in S^n$ and S^n is the set of all symmetric $n \times n$ matrices, $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable convex on $\mathbb{R}^n$. For $\bar{x} \in S$, let

$$\begin{aligned} y^{(i)} &:= \bar{x} + (1 - 2\bar{x}_i)e_i, i = 1, \dots, n, \\ \tilde{\bar{x}}_i &:= \begin{cases} -1, & \text{if } \bar{x}_i = 0, \\ 1, & \text{if } \bar{x}_i = 1, \\ \text{sign}(A\bar{x} - \nabla g(\bar{x}))_i, & \text{if } 0 < \bar{x}_i < 1, \end{cases} \quad i = 1, \dots, n, \end{aligned} \quad (2.2)$$

$$\tilde{\bar{X}} := \text{diag}(\tilde{\bar{x}}_1, \dots, \tilde{\bar{x}}_n), \quad (2.3)$$

$$b_{\bar{x}_i} := \begin{cases} \tilde{\bar{x}}_i(A\bar{x})_i + (g(y^{(i)}) - g(\bar{x})), & \bar{x}_i \in \{0, 1\}, i = 1, \dots, n, \\ \tilde{\bar{x}}_i(A\bar{x} - \nabla g(\bar{x}))_i, & 0 < \bar{x}_i < 1, \end{cases} \quad (2.4)$$

$$b_{\bar{x}} := (b_{\bar{x}_1}, \dots, b_{\bar{x}_n})^T, \quad (2.5)$$

$$d_{\bar{x}_i} := \tilde{x}_i(A\bar{x} - \nabla g(\bar{x}))_i, k+1 \leq i \leq n, \quad (2.6)$$

$$d_{\bar{x}} := (d_{\bar{x}_{k+1}}, \dots, d_{\bar{x}_n})^T, \quad (2.7)$$

where e_i is a i -th unit vector in $\mathbb{R}^n$, and

$$\text{sign}(A\bar{x} - \nabla g(\bar{x}))_i = \begin{cases} -1, & (A\bar{x} - \nabla g(\bar{x}))_i < 0, \\ 0, & (A\bar{x} - \nabla g(\bar{x}))_i = 0, \\ 1, & (A\bar{x} - \nabla g(\bar{x}))_i > 0, \end{cases}$$

Theorem 2.1 (Necessary Global Optimality Conditions for MIWCPP) *Let $\bar{x} \in S$ and let $\text{diag}(A) = \text{diag}(a_{11}, \dots, a_{nn})$. If $\bar{x}$ is a global minimizer of MIWCPP, then*

$$[\text{NC1}] \quad \begin{cases} \text{diag}(b_{\bar{x}}) \preceq \frac{1}{2} \text{diag}(A), \\ d_{\bar{x}} \leq 0. \end{cases}$$

Proof Let $\bar{x}$ be a global minimizer of MIWCPP. Then

$$\frac{1}{2}x^T Ax - \frac{1}{2}\bar{x}^T A\bar{x} - g(x) + g(\bar{x}) \geq 0, \text{ for any } x \in S.$$

Let $x := (\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n)^T$, where $x_i \in S_i, i = 1, \dots, n$. Then

$$\begin{aligned} \frac{1}{2}x^T Ax - \frac{1}{2}\bar{x}^T A\bar{x} - g(x) + g(\bar{x}) &\geq 0 \\ \Leftrightarrow h_i(x_i) &:= \frac{1}{2}(x_i - \bar{x}_i)^2 a_{ii} + (x_i - \bar{x}_i)(A\bar{x})_i - g(x) + g(\bar{x}) \geq 0, \\ \text{for any } x_i &\in S_i, i \in \{1, \dots, n\}. \end{aligned} \quad (2.8)$$

For any $i = 1, \dots, n$, if $\bar{x}_i \in \{0, 1\}$, let $x_i := 1 - \bar{x}_i$, then $x_i \in S_i$ and $y^{(i)} = \bar{x} + (1 - 2\bar{x}_i)e_i \in S$. It follows from (2.8) that

$$\begin{aligned} \frac{1}{2}(1 - 2\bar{x}_i)^2 a_{ii} + (1 - 2\bar{x}_i)(A\bar{x})_i - g(y^{(i)}) + g(\bar{x}) &\geq 0, \text{ for any } i \in \{1, \dots, n\} \\ \Leftrightarrow b_{\bar{x}_i} &= \tilde{x}_i(A\bar{x})_i + (g(y^{(i)}) - g(\bar{x})) \leq \frac{a_{ii}}{2}. \end{aligned}$$

If $\bar{x}_i \in (0, 1)$, then (2.8) implies that $\nabla h_i(\bar{x}_i) = (A\bar{x} - \nabla g(\bar{x}))_i = 0$ since $\bar{x}_i$ is a global minimizer of $h_i(x_i)$ on S_i . Therefore, $b_{\bar{x}_i} = 0$ and it follows from (2.8) that

$$\begin{aligned} \frac{1}{2}(x_i - \bar{x}_i)^2 a_{ii} + (x_i - \bar{x}_i)(\nabla g(\bar{x}))_i - g(x) + g(\bar{x}) &\geq 0 \\ \Rightarrow \frac{1}{2}(x_i - \bar{x}_i)^2 a_{ii} &\geq 0 \text{ (since } g(x) \text{ is a convex function)} \\ \Rightarrow a_{ii} &\geq 0. \end{aligned}$$

Hence, we have that $b_{\bar{x}_i} \leq \frac{a_{ii}}{2}$ for any $i = 1, \dots, n$.

Moreover, for the case of $k+1 \leq i \leq n$, we break the discussion into three different cases:

1° if $\bar{x}_i = 0$, then it follows from (2.8) that

$$\begin{aligned} & \frac{1}{2}a_{ii}(x_i - \bar{x}_i) + (A\bar{x} - \nabla g(\bar{x}))_i \geq 0, \text{ for any } x_i \in (0, 1] \\ & \Rightarrow (A\bar{x} - \nabla g(\bar{x}))_i \geq 0 \quad (\text{let } x_i \rightarrow \bar{x}_i) \\ & \Leftrightarrow \tilde{x}_i(A\bar{x} - \nabla g(\bar{x}))_i \leq 0, \text{ i.e., } d_{\bar{x}_i} \leq 0. \end{aligned}$$

2° If $\bar{x}_i = 1$, then it follows from (2.8) that

$$\begin{aligned} & \frac{1}{2}a_{ii}(x_i - \bar{x}_i) + (A\bar{x} - \nabla g(\bar{x}))_i \leq 0, \text{ for any } x_i \in [0, 1) \\ & \Rightarrow (A\bar{x} - \nabla g(\bar{x}))_i \leq 0 \quad (\text{let } x_i \rightarrow \bar{x}_i) \\ & \Leftrightarrow \tilde{x}_i(A\bar{x} - \nabla g(\bar{x}))_i \leq 0, \text{ i.e., } d_{\bar{x}_i} \leq 0. \end{aligned}$$

3° If $0 < \bar{x}_i < 1$, then it follows from (2.8) that

$$\begin{aligned} & \begin{cases} \frac{1}{2}a_{ii}(x_i - \bar{x}_i) + (A\bar{x} - \nabla g(\bar{x}))_i \leq 0, & \text{for any } x_i \in [0, \bar{x}_i), \\ \frac{1}{2}a_{ii}(x_i - \bar{x}_i) + (A\bar{x} - \nabla g(\bar{x}))_i \geq 0, & \text{for any } x_i \in (\bar{x}_i, 1] \end{cases} \\ & \Rightarrow (A\bar{x} - \nabla g(\bar{x}))_i = 0 \text{ and } a_{ii} \geq 0 \\ & \Leftrightarrow \tilde{x}_i(A\bar{x} - \nabla g(\bar{x}))_i \leq 0, \text{ i.e., } d_{\bar{x}_i} \leq 0. \end{aligned}$$

By the above discussion, we know that [NC1] holds.

Remark 2.2 By the convexity of $g(x)$, we have that

$$g(y^{(i)}) - g(\bar{x}) \geq \nabla g(\bar{x})^T(y^{(i)} - \bar{x}).$$

If $\bar{x}_i \in \{0, 1\}$, then $\nabla g(\bar{x})^T(y^{(i)} - \bar{x}) = -\tilde{x}_i(\nabla g(\bar{x}))_i$. So

$$[\text{NC1}] \Rightarrow [\text{NC1}]^* : \begin{cases} \tilde{x}_i(A\bar{x} - \nabla g(\bar{x}))_i \leq \frac{a_{ii}}{2}, \\ d_{\bar{x}} \leq 0, \end{cases}$$

where $[\text{NC1}]^*$ is just the necessary global optimality conditions given in [42] for MIWCPP with $u_i = p_i = 0$ and $v_i = q_i = 1$ and the necessary condition given by Theorem 2.1 in [41] for MIWCPP with box constraints. So [NC1] extends the condition $[\text{NC1}]^*$ given in [42] for MIWCPP with $u_i = p_i = 0$ and $v_i = q_i = 1$ and extends the conditions given in [41] for MIWCPP with box constraints.

Corollary 2.3 Let $\bar{x} \in S$, $A \in S^n$. If $k = n$, then [NC1] is equivalent to

$$[\text{NC2}] \quad \text{diag}(b_{\bar{x}}) \preceq \frac{1}{2} \text{diag}(A).$$

Proof Using the same proof in Theorem 2.1, we can prove it.

Remark 2.4 By the convexity of $g(x)$ and by $k = n$, we have that

$$g(y^{(i)}) - g(\bar{x}) \geq \nabla g(\bar{x})^T (y^{(i)} - \bar{x}) = -\tilde{x}_i (\nabla g(\bar{x}))_i.$$

So

$$[\text{NC2}] \Rightarrow [\text{NC2}]^* : \tilde{x}_i (A\bar{x} - \nabla g(\bar{x}))_i \leq \frac{a_{ii}}{2}.$$

Note that the necessary global optimality conditions [BNC1] for MIWCPP with binary constraints given by Theorem 4.1 in [41] is

$$[\text{BNC1}] \quad \frac{a_{ii}}{2} + \tilde{x}_i (A\bar{x} - (\bar{x}))_i \leq 0,$$

which should be changed into $[\text{NC2}]^*$ from the proof of Theorem 4.1 in [41]. So $[\text{NC2}]$ extends the condition $[\text{NC2}]^*$ given in [41] for MIWCPP with binary constraints.

Corollary 2.5 Let $\bar{x} \in S$, $A \in S^n$. If $g(x) = a^T x$, then [NC1] is equivalent to

$$[\text{NC3}] \quad \text{diag}(\tilde{X}(A\bar{x} - a)) \preceq \frac{1}{2} \text{diag}(\tilde{A}),$$

where $\text{diag}(\tilde{A}) = \text{diag}(\tilde{a}_{11}, \dots, \tilde{a}_{nn})$ and $\tilde{a}_{ii} := \begin{cases} a_{ii}, & \text{if } i = 1, \dots, k, \\ \min\{0, a_{ii}\}, & \text{if } i = k+1, \dots, n. \end{cases}$

Moreover, if A is a diagonal matrix, then $\bar{x}$ is a global minimizer of MIWCPP if and only if [NC3] holds.

Proof If $g(x) = a^T x$, then $b_{\bar{x}_i} = \tilde{x}_i (A\bar{x} - a)_i$, for any $i = 1, \dots, n$ and $d_{\bar{x}_i} = \tilde{x}_i (A\bar{x} - a)_i$, for any $i = k+1, \dots, n$. Hence, we can easily check that [NC1] is equivalent to [NC3].

Assume that A is a diagonal matrix and [NC3] holds, then $a_{kt} = 0$, for any $t = 1, 2, \dots, n$, $k \neq t$. Hence,

$$f(x) - f(\bar{x}) = \frac{1}{2} \sum_{k=1}^n a_{kk} (x_k - \bar{x}_k)^2 + \sum_{k=1}^n (A\bar{x} - a)_k (x_k - \bar{x}_k) \geq 0$$

if and only if

$$\frac{1}{2} a_{kk} (x_k - \bar{x}_k)^2 + (A\bar{x} - a)_k (x_k - \bar{x}_k) \geq 0, \text{ for any } k = 1, 2, \dots, n. \quad (2.9)$$

As the same proof in Theorem 1, we can prove that (2.9) holds if and only if [NC3] holds. Hence, $\bar{x}$ is a global minimizer of MIWCPP if and only if [NC3] holds.

Note that the condition [NC3] is just the necessary condition given by Theorem 3.7 in [40].

In the following section, we will discuss a new local optimization method using the given necessary global optimality condition [NC1].

3 A New Local Optimization Method

First, we consider the following general problem (P):

$$(P) \quad \begin{aligned} & \min f(x) \\ & \text{s.t. } x \in S, \end{aligned}$$

where $S = \prod_{i=1}^n S_i$ and $S_i = \{0, 1\}$ for $i \in I$, $S_i = [0, 1]$ for $i \in \bar{I}$, f is a differentiable function on $\prod_{i=1}^n [0, 1]$, where

$$I = \{1, \dots, k\} \text{ and } \bar{I} = \{k+1, \dots, n\}.$$

For any $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)^T \in S$, let

$$I_{\bar{x}} := \{i \mid \bar{x}_i(1 - \bar{x}_i) = 0, i \in I \cup \bar{I}\}, \quad (3.1)$$

$$N_i(\bar{x}) := \{y^{(i)} = \bar{x} + (1 - 2\bar{x}_i)e_i\}, \text{ for any } i \in I_{\bar{x}}. \quad (3.2)$$

For $\delta > 0$, let

$$N_{i,\delta}(\bar{x}) := \left\{ \bar{x} + \alpha e_i \mid \begin{array}{ll} \alpha \in (0, \delta), & \text{if } \bar{x}_i = 0 \\ \alpha \in (-\delta, 0), & \text{if } \bar{x}_i = 1 \\ \alpha \in (-\delta, \delta), & \text{if } \bar{x}_i \in (0, 1) \end{array} \right\}, \text{ for any } i \in \bar{I} \quad (3.3)$$

and let

$$N_{\delta}(\bar{x}) := \cup_{i \in I_{\bar{x}}} N_i(\bar{x}) \cup \cup_{i \in \bar{I}} N_{i,\delta}(\bar{x}). \quad (3.4)$$

Let

$$\delta(\bar{x}) := \begin{cases} \min\{\bar{x}_i, 1 - \bar{x}_i, i \in (I \cup \bar{I}) \setminus I_{\bar{x}}\}, & \text{if } (I \cup \bar{I}) \setminus I_{\bar{x}} \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Obviously, $|N_i(\bar{x})| = 1$, for any $i \in I_{\bar{x}}$ and $N_{\delta}(\bar{x}) \subset S$ if $\delta < \delta(\bar{x})$, where $|N_i(\bar{x})|$ means the number of the element in $N_i(\bar{x})$.

Definition 3.1 Let $\bar{x} \in S$. For $\delta > 0$ such that $\delta \leq \delta(\bar{x})$, $N_{\delta}(\bar{x})$ is said to be a neighborhood of $\bar{x}$ with respect to S .

Note that if $I_{\bar{x}} \neq \emptyset$, then $\bar{x} \notin N_{\delta}(\bar{x})$.

Definition 3.2 Let $\bar{x} \in S$. If there exists a $0 < \delta \leq \delta(\bar{x})$ such that $f(\bar{x}) \leq f(x)$ ($f(\bar{x}) \geq f(x)$) for any $x \in N_{\delta}(\bar{x})$, then $\bar{x}$ is said to be a local minimizer (maximizer)

of problem (P); furthermore, if $f(\bar{x}) < f(x)$ ($f(\bar{x}) > f(x)$) for any $x \in N_\delta(\bar{x}) \setminus \{\bar{x}\}$, then $\bar{x}$ is said to be a strict local minimizer (maximizer) of problem (P).

Let

$$S_{\bar{I}} := \prod_{i \in \bar{I}} S_i = \prod_{i \in \bar{I}} [0, 1], \quad (3.5)$$

$$S(\bar{x}) := \{x = (\bar{x}_1, \dots, \bar{x}_k, z_1, \dots, z_{n-k})^T \mid (z_1, \dots, z_{n-k})^T \in S_{\bar{I}}\}, \quad (3.6)$$

$$h(z) := f(x), \text{ where } z = (z_1, \dots, z_{n-k})^T \text{ and } x = (\bar{x}_1, \dots, \bar{x}_k, z_1, \dots, z_{n-k})^T \in S(\bar{x}). \quad (3.7)$$

From [43], we have the following KKT problem $\min_{z \in S_{\bar{I}}} h(z)$.

Lemma 3.3 [43] *Let $\bar{z} \in S_{\bar{I}}$. $\bar{z}$ is an MIWCPP point of $h(z)$ on $S_{\bar{I}}$ if and only if*

$$\nabla h(\bar{z})^T (z - \bar{z}) \geq 0, \text{ for any } z \in S_{\bar{I}}. \quad (3.8)$$

In the following, we will define the KKT point for mixed programming problem (P).

Definition 3.4 Let $\bar{x} \in S$ and let $\bar{z} := (\bar{x}_{k+1}, \dots, \bar{x}_n)^T$. $\bar{x}$ is said to be a KKT point of problem (P) if

$$[\text{KKT}] \quad \begin{cases} f(\bar{x}) \leq \min\{f(x) \mid x \in \cup_{i \in I_{\bar{x}}} N_i(\bar{x})\}, \\ \bar{z} \text{ is a KKT point of } h(z) \text{ on } S_{\bar{I}}. \end{cases} \quad (3.9)$$

By Lemma 3.3 and Definition 3.4, we can obtain the following KKT condition for mixed problem (P).

Proposition 3.5 *Let $\bar{x} \in S$. Then, $\bar{x}$ is a KKT point of problem (P) if and only if the following condition [KKTC] holds:*

$$[\text{KKTC}] \quad \begin{cases} f(\bar{x}) \leq \min\{f(x) \mid x \in \cup_{i \in I_{\bar{x}}} N_i(\bar{x})\}, \\ \tilde{\bar{x}}_i(\bar{x})_i \leq 0, \text{ for any } i \in \bar{I}, \end{cases}. \quad (3.10)$$

where $\tilde{\bar{x}}_i$ is defined by (2.2).

Proof By the definition of KKT point, here we just need to prove that $\bar{z}$ is a KKT point of $h(z)$ on $S_{\bar{I}}$ if and only if $\tilde{\bar{x}}_i(\nabla f(\bar{x}))_i \leq 0$, for any $i \in \bar{I}$. By Lemma 3.3, we know that $\bar{z}$ is a KKT point of $h(z)$ on $S_{\bar{I}}$ if and only if

$$\nabla h(\bar{z})^T (z - \bar{z}) \geq 0, \text{ for any } z \in S_{\bar{I}}. \quad (3.11)$$

We can easily verify that (3.8) is equivalent to $\tilde{x}_i(\nabla f(\bar{x}))_i \leq 0$, for any $i \in \bar{I}$.

Proposition 3.6 *Let $\bar{x} \in S$. If $\bar{x}$ is a local minimizer of problem (P), then $\bar{x}$ is a KKT point of problem (P).*

Proof By Definition 3.2, we know that $\bar{x}$ is a local minimizer of problem (P) if and only if there exists a $0 < \delta \leq \delta(\bar{x})$ such that $f(\bar{x}) \leq f(x)$ for any $x \in N_\delta(\bar{x})$. By $\cup_{i \in I_x} N_i(\bar{x}) \subset N_\delta(\bar{x})$, we know that

$$f(\bar{x}) \leq \min\{f(x) \mid x \in \cup_{i \in I_x} N_i(\bar{x})\}. \quad (3.12)$$

Moreover, $f(\bar{x}) \leq f(x)$ for any $x \in N_\delta(\bar{x})$ implies that

$$\begin{aligned} f(x) &\geq f(\bar{x}), \text{ for any } x \in \cup_{i \in \bar{I}} N_{i,\delta}(\bar{x}), \text{ where } \bar{x} := (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)^T \\ &\Rightarrow h(\bar{z}) \geq h(\bar{z}), \text{ for any } \bar{z} \in o(\bar{z}, \delta) \cap S_{\bar{I}}, \text{ where } \bar{z} := (\bar{x}_{k+1}, \dots, \bar{x}_n)^T, \\ o(\bar{z}, \delta) &= \{z \in \mathbb{R}^{n-k} \mid \|z - \bar{z}\| \leq \delta\}, h(z) \text{ is defined by (16)} \\ &\Leftrightarrow \bar{z} \text{ is a local minimizer of } h(z) \text{ on } S_{\bar{I}} \\ &\Rightarrow \bar{z} \text{ is a KKT point of } h(z) \text{ on } S_{\bar{I}}. \end{aligned} \quad (3.13)$$

(3.12) and (3.13) imply that $\bar{x}$ is a KKT point of problem (P).

Proposition 3.7 *Let $\bar{x} \in S$. Then, the following conditions hold:*

- (i) *If $\bar{x}$ is a local minimizer of MIWCPP, then [NC1] holds;*
- (ii) *If [NC1] holds, then $\bar{x}$ is a KKT point of MIWCPP.*

Moreover, if for any $i \in \{1, \dots, n\} \setminus I_{\bar{x}}$, $a_{ii} \geq 0$, then $\bar{x}$ is a KKT point of MIWCPP if and only if [NC1] holds.

Proof

- (i) If $\bar{x}$ is a local minimizer of MIWCPP, then $f(\bar{x}) \leq \min\{f(x) \mid x \in \cup_{i \in I_x} N_i(\bar{x})\}$. We can easily verify that

$$f(\bar{x}) \leq \min\{f(x) \mid x \in \cup_{i \in I_x} N_i(\bar{x})\} \Leftrightarrow b_{\bar{x}_i} \leq \frac{a_{ii}}{2}, \text{ for any } i \in I_{\bar{x}}. \quad (3.14)$$

Moreover, if $\bar{x}$ is a local minimizer of MIWCPP, then there exists a $0 < \delta \leq \delta(\bar{x})$ such that

$$\begin{aligned} f(x) - f(\bar{x}) &= \frac{1}{2}(x - \bar{x})^T A(x - \bar{x}) + (A\bar{x})^T(x - \bar{x}) - g(x) + g(\bar{x}) \geq 0, \\ &\text{for any } x \in \cup_{i \in \bar{I}} N_{i,\delta}(\bar{x}), \end{aligned}$$

which implies that

$$\begin{cases} (A\bar{x} - \nabla g(\bar{x}))_i = 0, a_{ii} \geq 0 \text{ for any } i \in \{1, \dots, n\} \setminus I_{\bar{x}}, \\ d_{\bar{x}_i} \leq 0 \text{ for any } i \in I_{\bar{x}}. \end{cases} \quad (3.15)$$

(3.14) and (3.15) imply that [NC1] holds.

(ii) If [NC1] holds, then

$$b_{\bar{x}_i} \leq \frac{a_{ii}}{2}, \text{ for any } i = 1, \dots, n.$$

By (3.14), we know that

$$f(\bar{x}) \leq \min\{f(x) \mid x \in \cup_{i \in I_{\bar{x}}} N_i(\bar{x})\}.$$

By Proposition 3.5, we know that $\bar{x}$ is a KKT point of MIWCPP.

Moreover, if for any $i \in \{1, \dots, n\} \setminus I_{\bar{x}}$, $a_{ii} \geq 0$, and if $\bar{x}$ is a KKT point of MIWCPP, then $b_{\bar{x}_i} \leq \frac{a_{ii}}{2}$ for any $i \in \{1, \dots, n\} \setminus I_{\bar{x}}$ since $b_{\bar{x}_i} = 0$, for any $i \in \{1, \dots, n\} \setminus I_{\bar{x}}$ if $\bar{x}$ is a KKT point of MIWCPP. Hence, by (3.14) and Proposition 3.5, we know that condition [NC1] holds. Therefore, if for any $i \in \{1, \dots, n\} \setminus I_{\bar{x}}$, $a_{ii} \geq 0$, then $\bar{x}$ is a KKT point of MIWCPP if and only if [NC1] holds.

By the condition [KKTC] for problem (P), we can obtain the following LOM for problem (P).

Algorithm 3.1 *Local Optimization Method (LOM)*

Step 1. Take an initial point $x_0 = (x_0^1, \dots, x_0^n)^T \in S$. Let $\bar{x} := x_0, k := 1$.

Step 2. Let $I_{\bar{x}} := \{i \mid \bar{x}_i(1 - \bar{x}_i) = 0\}$ and check whether the following condition holds.

$$[\text{KKTC}]_1 : f(\bar{x}) \leq f(x), \text{ for any } x \in \cup_{i \in I_{\bar{x}}} N_i(\bar{x}).$$

If $[\text{KKTC}]_1$ does not hold, goto *Step 3*; otherwise, check whether the following condition holds.

$$[\text{KKTC}]_2 : \tilde{x}_i(\nabla f(\bar{x}))_i \leq 0, \text{ for any } i \in \bar{I}.$$

If $[\text{KKTC}]_2$ holds, goto *Step 5*, else goto *Step 4*.

Step 3. Let $x^* = (x_1^*, \dots, x_n^*)^T := \text{augmin} \{f(x) \mid x \in \cup_{i \in I_{\bar{x}}} N_i(\bar{x})\}$ and let $\bar{x} := x^*$, goto *Step 2*.

Step 4. Let $h(z) := f((\bar{x}_1, \dots, \bar{x}_k, z_1, \dots, z_{n-k}))$ and let $z^* := (z_1^*, \dots, z_{n-k}^*)^T$ be a local minimizer or a KKT point of $h(z)$ on $S_{\bar{I}}$ starting from $(\bar{x}_{k+1}, \dots, \bar{x}_n)^T$. Let $\bar{x} := (\bar{x}_1, \dots, \bar{x}_k, z_1^*, \dots, z_{n-k}^*)^T$, goto *Step 2*.

Step 5. Stop. $\bar{x}$ is a KKT point or a local minimizer of problem (P).

Remark 3.1 1° If $f(x) = \frac{1}{2}x^T Ax - g(x)$, where $g(x)$ is a differentiable convex function, i.e., if the problem (P) is MIWCPP, then the condition $[\text{KKTC}]_1$ can be replaced by the following condition $[\text{NC1}]_1$.

$$[\text{NC1}]_1 : b_{\bar{x}_i} \leq \frac{a_{ii}}{2}, \text{ for any } i \in I_{\bar{x}}.$$

2° In *Step 4*, we can use any local optimization methods to obtain a local or KKT point for smooth programming problem: $\min_{z \in S_f} h(z)$ with only continuous variables since any local minimizer of problem $\min_{z \in S_f} h(z)$ must be a KKT point. In Sect. 5, in the numerical examples, we use the following three local optimization methods: Quasi-Newton, DFP and BFGS to solve problem $\min_{z \in S_f} h(z)$.

Theorem 3.2 *For a given initial point $x_0 \in S$, we can obtain a local minimizer or a KKT point $\bar{x}$ for problem (P) in finite steps by the given LOM.*

Proof Here, we just need to prove that the Algorithm (LOM) needs only finite steps from *Step 2* to *Step 5* since a point $\bar{x}$ can go to *Step 5* from *Step 2* only when it satisfies the conditions $[\text{KKT}C]_1$ and $[\text{KKT}C]_2$, i.e., $\bar{x}$ is already a KKT point of problem (P). Without loss of generality, suppose that $f(x)$ is not a constant function on $\{0, 1\}^n$. Let $\eta := \min\{|f(x) - f(y)| \mid x, y \in \{0, 1\}^n \text{ and } f(x) \neq f(y)\}$, $M := \max\{f(x) \mid x \in \{0, 1\}^n\}$ and $m := \min\{f(x) \mid x \in \{0, 1\}^n\}$. Then, $\eta > 0$ since the number of set $\{0, 1\}^n$ is finite. Then, we can prove that a local minimizer or a KKT point $\bar{x}$ for problem (P) starting from a given point x_0 can be obtained in at most $\frac{2(M-m)}{\eta} + 1$ steps by the given LOM. If $f(x)$ is a constant function on $\{0, 1\}^n$, then a local minimizer or a KKT point $\bar{x}$ for problem (P) starting from a given point x_0 can be obtained in at most 1 step by the given LOM.

In fact, from *Step 2* to *Step 3*, we have that $f(x^*) < f(\bar{x})$ since it can happen only when the condition $[\text{KKT}C]_1$ does not hold, i.e., there must exist $i_0 \in I_{\bar{x}}$ and $y_{i_0} = \bar{x} + (1 - 2\bar{x}_{i_0})e_{i_0}$ such that $f(y_{i_0}) < f(\bar{x})$ otherwise *Step 2* will not goto *Step 3*, and since $f(x^*) \leq f(y_{i_0})$. Hence, there are at most $\frac{M-m}{\eta}$ steps from *Step 2* to *Step 3*.

Moreover, *Step 2* goes to *Step 4* only when condition $[\text{KKT}C]_1$ holds, but condition $[\text{KKT}C]_2$ does not hold. Since z^* obtained in *Step 4* is the local point or KKT point of programming problem $\min_{z \in S_f} h(z)$, we have that $\tilde{x}_i(\nabla f(\bar{x}))_i \leq 0$, for any $i \in \bar{I}$, where $\bar{x} = (\bar{x}_1, \dots, \bar{x}_k, z_1^*, \dots, z_{n-k}^*)^T$, i.e., the condition $[\text{KKT}C]_2$ holds for the point $\bar{x}$, which means that if condition $[\text{KKT}C]_1$ also holds at this point $\bar{x}$, then *Step 2* goes to *Step 5*, i.e., this point $\bar{x}$ is the obtained local minimizer or KKT point of problem (P) and the algorithm stops; if condition $[\text{KKT}C]_1$ does not hold at this point $\bar{x}$, then *Step 2* must go to *Step 3*. Therefore, the steps from *Step 2* to *Step 4* are less than or equal to the steps from *Step 2* to *Step 3* plus 1 when condition $[\text{KKT}C]_1$ holds and $[\text{KKT}C]_2$ does not hold for given point $\bar{x}$, and the steps from *Step 2* to *Step 4* are less than or equal to the steps from *Step 2* to *Step 3* when condition $[\text{KKT}C]_1$ does not hold for the given point $\bar{x}$.

Hence, the total steps from *Step 2* to *Step 5* are at most $\frac{2(M-m)}{\eta} + 1$.

If $f(x)$ is a constant function on $\{0, 1\}^n$, then the condition $[\text{KKT}C]_1$ always holds. So we just need check whether the condition $[\text{KKT}C]_2$ holds. If the condition $[\text{KKT}C]_2$ also holds, then $\bar{x}$ is already a local minimizer or a KKT point $\bar{x}$ for problem (P). If the condition $[\text{KKT}C]_2$ does not hold, then we just need one step from *Step 2* to *Step 4*, and then we can sure that both of the

conditions $[\text{KKTC}]_2$ and $[\text{KKTC}]_1$ hold. Then, the total steps from *Step 2* to *Step 5* are at most 1.

Here, we have given an LOM for problem (P) (or MIWCPP) using the KKT condition $[\text{KKTC}]$ (or by the necessary condition $[\text{NC1}]$). In the following, we will introduce a global optimization method using some auxiliary functions and the given LOM for problem (P) (or MIWCPP).

4 Global Optimization Method for MIWCPP

To introduce the global optimization method, first we will introduce the following auxiliary functions.

For any $r > 0$, let

$$g_r(t) = \begin{cases} 1, & t > 0, \\ -\frac{2}{r^3}t^3 - \frac{3}{r^2}t^2 + 1, & -r < t \leq 0, \\ 0, & t \leq -r, \end{cases} \quad (4.1)$$

$$f_r(t) = \begin{cases} t + r, & t \leq -r, \\ \frac{r-2}{r^3}t^3 + \frac{r-3}{r^2}t^2 + 1, & -r < t \leq 0, \\ 1, & t > 0 \end{cases} \quad (4.2)$$

and let

$$F_{r,\bar{x}}(x) = \frac{1}{\|x - \bar{x}\|^2 + 1} g_r(f(x) - f(\bar{x})) + f_r(f(x) - f(\bar{x})), \quad (4.3)$$

where $r > 0$ is a parameter, and $\bar{x}$ is the current local minimum. Please notice that functions f_r and g_r are just the functions given by (3.4) and (3.5) in [44]. Even though function $F_{r,\bar{x}}(x)$ is also similar to the function $H_{q,e,x^*}(x)$ given in [44], but since the MIWCPP is mixed integer program problem and since the definitions for local minimizer and KKT point given here are different from others, here we think that it is necessary to introduce the properties of function $F_{r,\bar{x}}(x)$, which is as follows:

Theorem 4.1 *Suppose that $\bar{x}$ is a local minimizer of MIWCPP, then $\bar{x}$ is a strictly local maximizer of $F_{r,\bar{x}}(x)$ on S for any $r > 0$.*

Proof Since $\bar{x}$ is a local minimizer of MIWCPP, there exists a $\delta > 0$ such that $N_\delta(\bar{x}) \subset S$ and

$$f(x) \geq f(\bar{x}), \text{ for any } x \in N_\delta(\bar{x}).$$

Hence, we have that

$$F_{r,\bar{x}}(x) = \frac{1}{\|x - \bar{x}\|^2 + 1} + 1 < 1 + 1 = F_{r,\bar{x}}(\bar{x}), \text{ for any } x \in N_\delta(\bar{x}) \setminus \{\bar{x}\}.$$

Hence, $\bar{x}$ is a strictly local maximizer of $F_{r,\bar{x}}(x)$ on S for any $r > 0$.

Let x^* be the global minimizer of MIWCPP and let

$$\beta = f(\bar{x}) - f(x^*). \quad (4.4)$$

Then, we have the following results:

Theorem 4.2 *If $\bar{x}$ is not a global minimizer of MIWCPP, i.e., $\beta > 0$, then x^* is a local minimizer of $F_{r,x^*}(x)$ on S when $r \leq \beta$.*

Proof When $r \leq \beta$, we have that $f(x^*) - f(\bar{x}) \leq -r$. Hence, $F_{r,\bar{x}}(x^*) = f(x^*) - f(\bar{x}) + r \leq 0$ and for any $x \in S$,

$$F_{r,\bar{x}}(x) = \begin{cases} f(x) - f(\bar{x}) + r \geq f(x^*) - f(\bar{x}) + r, & \text{if } f(x) - f(\bar{x}) \leq -r, \\ \frac{1}{\|x - \bar{x}\|^2 + 1} g_r(f(x) - f(\bar{x})) + f_r(f(x) - f(\bar{x})) \geq 0, & \text{if } -r < f(x) - f(\bar{x}) < 0, \\ \frac{1}{\|x - \bar{x}\|^2 + 1} + 1, & \text{if } f(x) - f(\bar{x}) \geq 0. \end{cases}$$

Hence, $F_{r,x^*}(x^*) \leq F_{r,\bar{x}}(x)$, for all $x \in S$, i.e., x^* is a global minimizer of $F_{r,\bar{x}}(x)$ on S . Of course, x^* is a local minimizer of $F_{r,x^*}(x)$ on S when $r \leq \beta$.

Theorem 4.3 *Any KKT point $\hat{x}$ of $F_{r,\bar{x}}(x)$ on S satisfies one of the following conditions:*

- 1° $f(\hat{x}) < f(\bar{x})$;
- 2° $\hat{x} := (\hat{x}_1, \dots, \hat{x}_n)$, where $\hat{x}_i = \begin{cases} 1 - \bar{x}_i, & \text{if } i \in I_{\bar{x}}, \\ 0 \text{ or } 1 \text{ or } \bar{x}_i, & \text{if } i \in (I \cup \bar{I}) \setminus I_{\bar{x}}. \end{cases}$

Proof First, we can prove that if $\hat{x}$ is a KKT point of $F_{r,\bar{x}}(x)$ on S and $f(\hat{x}) \geq f(\bar{x})$, then $\hat{x}_i \in \{0, 1\}$ when $i \in I$ and $\hat{x}_i \in \{0, 1, \bar{x}_i\}$ when $i \in \bar{I}$. In fact, if $i \in I$, then $\hat{x}_i \in S_i = \{0, 1\}$. If $i \in \bar{I}$, and if $\hat{x}_i \notin \{0, 1, \bar{x}_i\}$, then we have that $\hat{x}_i \in (0, 1)$ and $\hat{x}_i \neq \bar{x}_i$. By $\hat{x}_i \in (0, 1)$ and by the condition [KKTC], we know that $\frac{\partial F_{r,\bar{x}}(\hat{x})}{\partial x_i} = 0$. But if $f(\hat{x}) \geq f(\bar{x})$, by $\hat{x}_i \neq \bar{x}_i$, we have that $\frac{\partial F_{r,\bar{x}}(\hat{x})}{\partial x_i} = -\frac{\hat{x}_i - \bar{x}_i}{(\|x - \bar{x}\|^2 + 1)^2} \neq 0$, which contradicts with $\frac{\partial F_{r,\bar{x}}(\hat{x})}{\partial x_i} = 0$.

Second, we can prove that if $\hat{x}$ is a KKT point of $F_{r,\bar{x}}(x)$ on S and if $f(\hat{x}) \geq f(\bar{x})$, then $\hat{x}_i \neq \bar{x}_i$ for $i \in I_{\bar{x}}$. In fact, if there exists an $i_0 \in I_{\bar{x}}$ such that $\hat{x}_{i_0} = \bar{x}_{i_0}$, then $\tilde{x} \in N_{i_0}(\hat{x})$, where $\tilde{x} := \hat{x} + (1 - 2\hat{x}_{i_0})e_{i_0}$. By the definition of KKT point, we should have that $F_{r,\bar{x}}(\tilde{x}) \geq F_{r,\bar{x}}(\hat{x})$, which is contradicts with that

$$\begin{aligned}
F_{r,\bar{x}}(\tilde{x}) &\leq \frac{1}{\sum_{i=1}^n |\tilde{x}_i - \bar{x}_i|^2 + 1} + 1 \\
&= \frac{1}{\sum_{i \neq i_0} |\hat{x}_i - \bar{x}_i|^2 + |\hat{x}_{i_0} - \bar{x}_{i_0} + 1 - 2\bar{x}_{i_0}|^2 + 1} + 1 \\
&= \frac{1}{\sum_{i \neq i_0} |\hat{x}_i - \bar{x}_i|^2 + |1 - 2\bar{x}_{i_0}|^2 + 1} + 1 \\
&= \frac{1}{\sum_{i=1}^n |\hat{x}_i - \bar{x}_i|^2 + 2} + 1 \\
&< \frac{1}{\sum_{i=1}^n |\hat{x}_i - \bar{x}_i|^2 + 1} + 1 \\
&= F_{r,\bar{x}}(\hat{x}).
\end{aligned}$$

Hence, for any $i \in I_{\bar{x}}$, we have that $\hat{x}_i = 1 - \bar{x}_i$.

Corollary 4.4 Any local minimizer $\hat{x}$ of $F_{r,\bar{x}}(x)$ on S satisfies one of the following conditions:

- 1° $f(\hat{x}) < f(\bar{x})$;
 2° $\hat{x} := (\hat{x}_1, \dots, \hat{x}_n)$, where $\hat{x}_i = \begin{cases} 1 - \bar{x}_i, & \text{if } i \in I_{\bar{x}}, \\ 0 \text{ or } 1, & \text{if } i \in (I \cup \bar{I}) \setminus I_{\bar{x}}. \end{cases}$

Proof Since $\hat{x}$ is a local minimizer of $F_{r,\bar{x}}(x)$ on S , there exists a $0 < \delta \leq \delta(\hat{x})$ such that $F_{r,\bar{x}}(x) \geq F_{r,\bar{x}}(\hat{x})$ for any $x \in N_\delta(\hat{x})$.

By Proposition 3.6 and by Theorem 4.3, we know that if $\hat{x}$ is a local minimizer of $F_{r,\bar{x}}(x)$ on S , then one of the following conditions holds:

- 1° $f(\hat{x}) < f(\bar{x})$;
 2° $\hat{x} := (\hat{x}_1, \dots, \hat{x}_n)$, where $\hat{x}_i = \begin{cases} 1 - \bar{x}_i, & \text{if } i \in I_{\bar{x}}, \\ 0 \text{ or } 1 \text{ or } \bar{x}_i, & \text{if } i \in (I \cup \bar{I}) \setminus I_{\bar{x}}. \end{cases}$

So, here we just need to prove that if $i \in (I \cup \bar{I}) \setminus I_{\bar{x}}$, then $\hat{x}_i \neq \bar{x}_i$. In fact, if there exists an $i_0 \in (I \cup \bar{I}) \setminus I_{\bar{x}}$ such that $\hat{x}_{i_0} = \bar{x}_{i_0}$, then let $\alpha_{i_0} := \frac{\delta}{2}$ and let $\tilde{x} := \hat{x} + \alpha_{i_0} e_{i_0}$. Then, $\tilde{x} \in N_\delta(\hat{x})$ and

$$\begin{aligned}
F_{r,\bar{x}}(\tilde{x}) &\leq \frac{1}{\sum_{i=1}^n |\tilde{x}_i - \bar{x}_i|^2 + 1} + 1 \\
&= \frac{1}{\sum_{i \neq i_0} |\hat{x}_i - \bar{x}_i|^2 + |\hat{x}_{i_0} - \bar{x}_{i_0} + \alpha_{i_0}|^2 + 1} + 1 \\
&= \frac{1}{\sum_{i \neq i_0} |\hat{x}_i - \bar{x}_i|^2 + 1 + |\alpha_{i_0}|^2} + 1 \\
&< \frac{1}{\sum_{i=1}^n |\tilde{x}_i - \bar{x}_i|^2 + 1} + 1 \\
&= F_{r,\bar{x}}(\hat{x}),
\end{aligned}$$

which contradicts $F_{r,\bar{x}}(x) \geq F_{r,\bar{x}}(\hat{x})$, for any $x \in N_\delta(\hat{x})$.

In the following, we will introduce a global optimization method for MIWCPP. This method combines the local optimization method for MIWCPP and (P) and the auxiliary function $F_{r,\bar{x}}(x)$. The auxiliary function is used to escape the current local minimizer and to find a better feasible point of MIWCPP.

Algorithm 4.1 *Global Optimization Method (GOM)*

Step 0. Take an initial point $x_1 \in S$, a sufficiently small positive number μ , and an initial $r_1 > 0$ ($r_1 > \mu$, in the numerical examples in the next section, we choose $\mu = 10^{-10}$ and $r_1 = 1$). Set $k := 1$ and $r := r_1$.

Step 1. Use the LOM to solve MIWCPP starting from x_k . Let x_k^* be the obtained local minimizer or KKT point of MIWCPP. If $k \geq 2$, then goto *Step 2*; otherwise goto *Step 3*.

Step 2. If $f(x_k^*) < f(x_{k-1}^*)$, then let $r := r_1$ and let $\bar{x} := x_k^*$, goto *Step 3*, otherwise let $\bar{x} := x_{k-1}^*$, goto *Step 5*.

Step 3. Construct the following auxiliary function:

$$F_{r,\bar{x}}(x) = \frac{1}{\|x - \bar{x}\|^2 + 1} g_r(f(x) - f(\bar{x})) + f_r(f(x) - f(\bar{x})).$$

Consider the following problem:

$$\begin{aligned} \min \quad & F_{r,\bar{x}}(x) \\ \text{s.t.} \quad & x \in S, \end{aligned} \tag{4.5}$$

goto *Step 4*.

Step 4. Use the LOM to solve problem (4.5) starting from $\bar{x}$. Let $\bar{x}_k^*$ be the local minimizer of problem (4.5) and let $x_{k+1} := \bar{x}_k^*$, $k := k + 1$, goto *Step 1*.

Step 5. If $r \geq \mu$, decrease r (such as, in the numerical examples in the next section, we let $r := r/10$), go to *Step 3*; otherwise, go to *Step 6*.

Step 6. Stop. $\bar{x}$ is the obtained global minimizer.

Remark 4.5 In Algorithm 4.1, we use auxiliary function $F_{r,\bar{x}}(x)$ to improve the current best local minimizer $\bar{x}$ by adjusting the parameter r . From the properties given by Theorems 4.1–4.3, we know that if the current best local minimizer $\bar{x}$ is not a global minimizer, then we can try to find a better one by local searching the auxiliary problem: $\min_{x \in S} F_{r,\bar{x}}(x)$. When r is small enough, we also cannot find any better point, then we will stop and the current best local minimizer is the best point which we can find (we regarded it as the global minimizer). The numerical examples given in the following section illustrate that the global minimization method Algorithm 4.1 is very efficient and stable.

5 Example

In this section, we apply Algorithm 4.1 to the following test examples, where

(k) is the iterative number in finding the k -th local minimizer or KKT point of MIWCPP;

$x_k, k \geq 1$ is the k -th initial point;

$f(x_k), k \geq 1$ is the function value of k -th initial point;

$x_k^*, k \geq 1$ is the k -th KKT point or local minimizer of MIWCPP starting from x_k ;

$f(x_k^*), k \geq 1$ is the function value of f at x_k^* ;

$\bar{x}_k^*, k \geq 1$ is the k -th KKT point or local minimizer of problem (4.5) starting from x_k^* ;

$f(\bar{x}_k^*), k \geq 1$ is the function value of f at $\bar{x}_k^*$.

Example 5.1 Consider the problem

$$\begin{aligned} \min \quad & f(x) := \frac{1}{2}x^T Ax - \|Bx - b\|_2^2 \\ \text{s.t.} \quad & x_1, x_2 \in \{0, 1\}, \\ & x_3, x_4 \in [0, 1]. \end{aligned}$$

Here $A = \begin{pmatrix} 1 & -2 & -3 & -4 \\ -2 & -2 & 5 & 6 \\ -3 & 5 & 3 & 7 \\ -4 & 6 & 7 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 26 & 14 & -3 \\ 6 & 0 & -5 & 11 \\ 4 & -4 & 7 & 8 \\ -7 & 11 & 4 & 9 \end{pmatrix},$

$b = (1, 2, 3, 4)^T$. The optimal solution is $\bar{x} = (1, 1, 1, 1)^T$ and the optimal value $f(\bar{x}) = -1774.000$. Table 1 records the numerical results of solving Example 5.1 by Algorithm 4.1.

From Table 1, we see that the first local minimizer x_1^* of $f(x)$ from the initial point x_1 is not the global minimizer of Example 5.1, the auxiliary function is used to find a new initial point x_2 , which is the local minimizer of the auxiliary function problem (4.5) from x_1^* . Then, the global minimizer x_2^* of Example 5.1 is obtained starting from the new initial point x_2 .

Example 5.2 Consider the problem

$$\begin{aligned} \min \quad & f(x) := \frac{1}{2}x^T Ax - \log(\exp(x_1) + \exp(x_2) + \exp(x_3) + \exp(x_4) + \exp(x_5)) \\ \text{s.t.} \quad & x_1, x_2, x_3 \in \{0, 1\}, \\ & x_4, x_5 \in [0, 1]. \end{aligned}$$

Here $A = \begin{pmatrix} 1 & 26 & 14 & -3 & 5 \\ 26 & 0 & -5 & 11 & 2 \\ 14 & -5 & 7 & 4 & -10 \\ -3 & 11 & 4 & 9 & 19 \\ 5 & 2 & -10 & 19 & -21 \end{pmatrix}$. The optimal solution is $\bar{x} = (0, 1, 1,$

$0, 1)^T$ and the optimal value $f(\bar{x}) = -22.31795$. Table 2 records the numerical results of solving Example 5.2 by Algorithm 4.1.

Table 1 Numerical results for Example 5.1

(k)	x_k	$f(x_k)$	k -th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k -th KKT or local minimizer $\bar{x}_k^*$ of $F_{r,\bar{x}}(x)$	$f(\bar{x}_k^*)$
(1)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	-30.000	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	-728.000	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$	-1 685.500
(2)	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$	-1 685.500	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 759.500	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 774.000
(1)	$\begin{pmatrix} 0 \\ 0 \\ 0.1 \\ 0.9 \end{pmatrix}$	-105.285	$\begin{pmatrix} 0 \\ 1 \\ 0.1 \\ 0.9 \end{pmatrix}$	-853.585	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.9 \end{pmatrix}$	-1 727.170
(2)	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.9 \end{pmatrix}$	-1 727.170	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 759.500	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 774.000
(1)	$\begin{pmatrix} 0 \\ 0 \\ 0.8 \\ 0.3 \end{pmatrix}$	-199.930	$\begin{pmatrix} 0 \\ 1 \\ 0.8 \\ 0.3 \end{pmatrix}$	-1 413.530	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.3 \end{pmatrix}$	-1 649.530
(2)	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.3 \end{pmatrix}$	-1 649.530	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 759.500	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 774.000
(1)	$\begin{pmatrix} 1 \\ 0 \\ 0.5 \\ 0.4 \end{pmatrix}$	-158.045	$\begin{pmatrix} 1 \\ 1 \\ 0.5 \\ 0.4 \end{pmatrix}$	-1 090.345	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0.4 \end{pmatrix}$	-1 622.320
(2)	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0.4 \end{pmatrix}$	-1 622.320	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 774.000	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 774.000
(1)	$\begin{pmatrix} 0 \\ 1 \\ 0.2 \\ 0.6 \end{pmatrix}$	-860.060	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.6 \end{pmatrix}$	-1 663.420	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 759.500
(2)	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 759.500	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 774.000	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	-1 774.000

Table 2 Numerical results for Example 5.2

(k)	x_k	$f(x_k)$	k -th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k -th KKT or local minimizer $\bar{x}_k^*$ of $F_{r,\bar{x}}(x)$	$f(\bar{x}_k^*)$
(1)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.00000 \\ 0.00000 \end{pmatrix}$	-1.60944	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-12.40483	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-19.13258
(2)	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-19.13258	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.31795	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.31795
(1)	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	58.39056	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	16.02572	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.31795
(1)	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.20000 \\ 0.90000 \end{pmatrix}$	-12.91927	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 0.90000 \end{pmatrix}$	-19.49715	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.31795
(1)	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0.60000 \\ 0.30000 \end{pmatrix}$	9.22937	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0.00000 \\ 0.30000 \end{pmatrix}$	-2.30060	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-10.63258
(2)	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-10.63258	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.31795	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.31795
(1)	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0.10000 \\ 0.80000 \end{pmatrix}$	-3.04055	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.10000 \\ 0.80000 \end{pmatrix}$	-7.00041	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.10000 \\ 0.80000 \end{pmatrix}$	-11.34055
(2)	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.10000 \\ 0.80000 \end{pmatrix}$	-11.34055	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.10000 \\ 1.00000 \end{pmatrix}$	-11.79996	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-19.13258
(3)	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-19.13258	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.31795	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.31795

Table 2 continued

(k)	x_k	$f(x_k)$	k-th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k-th KKT or local minimizer $\bar{x}_k^*$ of $F_{r,\bar{x}}(x)$	$f(\bar{x}_k^*)$
(1)	$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0.00000 \\ 0.00000 \end{pmatrix}$	15.867 42	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0.00000 \\ 0.00000 \end{pmatrix}$	-1.404 83	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-7.132 58
(2)	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-7.132 58	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-12.404 83	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-19.132 58
(3)	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-19.132 58	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.317 95	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 1.00000 \end{pmatrix}$	-22.317 95

Example 5.3 Consider the problem

$$\begin{aligned} \min \quad & f(x) := \frac{1}{2}x^T A x - \exp^{u(x)} \\ \text{s.t.} \quad & x_1, x_2, x_3, x_4 \in \{0, 1\}, \\ & x_5, x_6, x_7 \in [0, 1]. \end{aligned}$$

$$\text{Here } A = \begin{pmatrix} 1567 & -16 & 14 & -13 & 12 & -11 & 9 \\ -16 & -14 & -5 & 11 & 2 & 11 & 3 \\ 14 & -5 & 745 & 4 & -10 & -6 & -7 \\ -13 & 11 & 4 & 234 & 19 & 20 & -1 \\ 12 & 2 & -10 & 19 & -35 & 4 & 8 \\ -11 & 11 & -6 & 20 & 4 & -123 & -3 \\ 9 & 3 & -7 & -1 & 8 & -3 & 578 \end{pmatrix}, \quad u(x) = \sum_{i=1}^7 x_i^2.$$

The optimal solution is $\bar{x} = (0, 1, 0, 0, 1, 1, 0)^T$ and the optimal value $f(\bar{x}) = -89.085\,533$. Table 3 records the numerical results of solving Example 5.3 by Algorithm 4.1.

Example 5.4 Consider the problem

$$\begin{aligned} \min \quad & f(x) := \frac{1}{2}x^T A x - \sum_{i=1}^7 a(i) \exp^{b(i)x} \\ \text{s.t.} \quad & x_1, x_2, x_3 \in \{0, 1\}, \\ & x_4, x_5, x_6, x_7 \in [0, 1]. \end{aligned}$$

Table 3 Numerical results for Example 5.3

(k)	x_k	$f(x_k)$	k -th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k -th KKT or local minimizer x_k^* of $F_{\gamma,\bar{x}}(x)$	$f(x_k^*)$
(1)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix}$	-1.00000	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	-64.21828	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	-82.38905
(2)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	-82.38905	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	-89.08553	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	-89.08553
(1)	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	424.36685	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	339.071197	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	245.58683
(2)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	245.58683	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	43.40185	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.00000 \\ 1.00000 \\ 0.00000 \end{pmatrix}$	-89.08553

Table 3 continued

(k)	x_k	$f(x_k)$	k -th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k -th KKT or local minimizer x_k^* of $F_{\tau, \bar{x}}(x)$	$f(x_k^*)$
(1)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5000 \\ 0.5000 \\ 0.5000 \end{pmatrix}$	52.63299	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5000 \\ 0.5000 \\ 0.0000 \end{pmatrix}$	-20.39872	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5000 \\ 1.0000 \\ 0.0000 \end{pmatrix}$	-67.36534
(2)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5000 \\ 1.0000 \\ 0.0000 \end{pmatrix}$	-67.36534	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.0000 \\ 1.0000 \\ 0.0000 \end{pmatrix}$	-82.38905	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.0000 \\ 1.0000 \\ 0.0000 \end{pmatrix}$	-89.08553
(1)	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0.2000 \\ 0.9000 \\ 0.3400 \end{pmatrix}$	117.61306	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0.20531 \\ 0.97066 \\ 0.25508 \end{pmatrix}$	-42.31357	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.0000 \\ 0.97066 \\ 0.25508 \end{pmatrix}$	-65.24771
(2)	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.0000 \\ 0.97066 \\ 0.25508 \end{pmatrix}$	-65.24771	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.0000 \\ 0.97066 \\ 0.0000 \end{pmatrix}$	-84.84122	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.0000 \\ 1.0000 \\ 0.0000 \end{pmatrix}$	-89.08553

Table 3 continued

(k)	x_k	$f(x_k)$	k -th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k -th KKT or local minimizer x_k^* of $F_{\pi,\bar{\pi}}(x)$	$f(x_k^*)$
(1)	$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0.123\,00 \\ 0.234\,00 \\ 0.345\,00 \end{pmatrix}$	1189.01086	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0.124\,20 \\ 0.258\,54 \\ 0.315\,44 \end{pmatrix}$	388.81058	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3.111\,55E-006 \\ 0.281\,84 \\ 0.175\,98 \end{pmatrix}$	2.79952
(2)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3.111\,55E-006 \\ 0.281\,84 \\ 0.175\,98 \end{pmatrix}$	2.79952	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3.111\,55E-006 \\ 1.000\,00 \\ 0.175\,98 \end{pmatrix}$	-55.88119	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.000\,00 \\ 1.000\,00 \\ 0.175\,98 \end{pmatrix}$	-72.79102
(3)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.000\,00 \\ 1.000\,00 \\ 0.175\,98 \end{pmatrix}$	-72.79102	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.000\,00 \\ 1.000\,00 \\ 0.000\,00 \end{pmatrix}$	-82.38905	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1.000\,00 \\ 1.000\,00 \\ 0.000\,00 \end{pmatrix}$	-89.08553

Table 4 Numerical results for Example 5.4

(k)	x_k	$f(x_k)$	k -th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k -th KKT or local minimizer $\bar{x}_k^*$ of $F_{r,\bar{x}}(x)$	$f(\bar{x}_k^*)$
(1)	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	53.837 6	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1.00000 \\ 0.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	27.179 2	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1.00000 \\ 0.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	12.339 2
(2)	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1.00000 \\ 0.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	12.339 2	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0.00000 \\ 0.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	-3.833 6	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 0.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	-11.092 6
(1)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.50000 \\ 0.50000 \\ 0.50000 \\ 0.50000 \end{pmatrix}$	9.584 02	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.50000 \\ 0.00000 \\ 0.50000 \\ 0.50000 \end{pmatrix}$	2.452 29	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.00000 \\ 0.00000 \\ 0.50000 \\ 0.50000 \end{pmatrix}$	-0.424 13
(2)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.00000 \\ 0.00000 \\ 0.50000 \\ 0.50000 \end{pmatrix}$	-0.424 13	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 0.00000 \\ 0.50000 \\ 0.50000 \end{pmatrix}$	-3.412 39	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.00000 \\ 0.00000 \\ 1.00000 \\ 0.50000 \end{pmatrix}$	7.376 67

Table 4 continued

(k)	x_k	$f(x_k)$	k -th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k -th KKT or local minimizer $\bar{x}_k^*$ of $F_{r,\bar{x}}(x)$	$f(\bar{x}_k^*)$
(3)	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.0000 \\ 0.0000 \\ 1.0000 \\ 0.5000 \end{pmatrix}$	-7.37667	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.0000 \\ 0.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$	-11.09261		
(1)	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0.3000 \\ 0.4000 \\ 0.5000 \\ 0.6000 \end{pmatrix}$	28.10599	$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0.29728 \\ 0.35736 \\ 0.47769 \\ 0.57532 \end{pmatrix}$	16.64041	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.301867 \\ 0.30917 \\ 0.45425 \\ 0.56014 \end{pmatrix}$	3.99959
(2)	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.301867 \\ 0.30917 \\ 0.45425 \\ 0.56014 \end{pmatrix}$	3.99959	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0.301867 \\ 0.30917 \\ 0.45425 \\ 0.56014 \end{pmatrix}$	-0.562045	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.301867 \\ 0.30917 \\ 1.0000 \\ 0.56014 \end{pmatrix}$	-4.67494
(3)	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.301867 \\ 0.30917 \\ 1.0000 \\ 0.56014 \end{pmatrix}$	-4.67494	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.0000 \\ 0.30917 \\ 1.0000 \\ 0.56014 \end{pmatrix}$	-7.73742	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.0000 \\ 0.30917 \\ 1.0000 \\ 1.0000 \end{pmatrix}$	-9.78586

Table 4 continued

(k)	x_k	$f(x_k)$	k -th KKT or local minimizer x_k^* of $f(x)$	$f(x_k^*)$	k -th KKT or local minimizer $\bar{x}_k^*$ of $F_{r,\bar{x}}(x)$	$f(\bar{x}_k^*)$
(4)	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.000\,00 \\ 0.309\,17 \\ 1.000\,00 \\ 1.000\,00 \end{pmatrix}$	-9.78586	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0.000\,00 \\ 0.000\,00 \\ 1.000\,00 \\ 1.000\,00 \end{pmatrix}$	-11.09261		

Here
$$A = \begin{pmatrix} 20 & -6 & 4 & -3 & 5 & 2 & 9 \\ -6 & 16 & -5 & 1 & 3 & 1 & 3 \\ 4 & -5 & 19 & 4 & -10 & -16 & -7 \\ -3 & 1 & 4 & 24 & 9 & 0 & -1 \\ 5 & 3 & -10 & 9 & 17 & 4 & 8 \\ 2 & 1 & -16 & 0 & 4 & 16 & -8 \\ 9 & 3 & -7 & -1 & 8 & -8 & 11 \end{pmatrix}, \quad a(i) = \frac{1}{1+i},$$

$b(1) = (1.2, 0, 0, 0, 0, 0, 0)$, $b(2) = (0, -1.5, 0, 0, 0, 0, 0)$, $b(3) = (0, 0, 1.6, 0, 0, 0, 0)$, $b(4) = (0, 0, 0, -2.0, 0, 0, 0)$, $b(5) = (0, 0, 0, 0, 1.8, 0, 0)$, $b(6) = (0, 0, 0, 0, 0, -1.4, 0)$, $b(7) = (0, 0, 0, 0, 0, 0, 1.6)$. The optimal solution is $\bar{x} = (0, 0, 1, 0.0000, 0.0000, 1.0000, 0)^T$ and the optimal value $f(\bar{x}) = -11.0926$. Table 4 records the numerical results of solving Example 5.4 by Algorithm 4.1.

6 Conclusion

In this paper, we developed some new necessary global optimality conditions for MIWCPP, which extend the results given in [41] and [42]. A new LOM for MIWCPP is presented according to the given necessary global optimality conditions, which is different from the traditional one. A global optimization method for MIWCPP is also proposed in this paper, which is designed by combining the new LOM and an auxiliary function. Some numerical examples are also presented to show that the proposed novel optimization methods for MIWCPP are efficient.

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