Herscovici's Conjecture on the Product of the Thorn Graphs of the Complete Graphs

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Abstract Given a distribution of pebbles on the vertices of a connected graph G , a pebbling move on G consists of taking two pebbles off one vertex and placing one on an adjacent vertex. The t -pebbling number $f_t(G)$ of a simple connected graph G is the smallest positive integer such that for every distribution of $f_t(G)$ pebbles on the vertices of G , we can move t pebbles to any target vertex by a sequence of pebbling moves. Graham conjectured that for any connected graphs G and H , $f_1(G \times H) \leq f_1(G)f_1(H)$. Herscovici further conjectured that $f_{st}(G \times H) \leq f_s(G)f_t(H)$ for any positive integers s and t . Wang et al. (Discret Math, 309: 3431–3435, 2009) proved that Graham's conjecture holds when G is a thorn graph of a complete graph and H is a graph having the 2-pebbling property. In this paper, we further show that Herscovici's conjecture is true when G is a thorn graph of a complete graph and H is a graph having the $2t$ -pebbling property.

Keywords Thorn graph · t -Pebbling number · Graham's conjecture · Herscovici's conjecture

1 Introduction

Pebbling in graphs was first studied by Chung [1]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Considering a pebble distribution p on the vertices of G , $p(v)$ denotes the number of pebbles on v for any vertex v of G , $p(H) = \sum_{v \in V(H)} p(v)$, where H is the subgraph of G . A *pebbling move* consists of

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removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. The *pebbling number* of a vertex v in a graph G is the smallest positive integer $f(G, v)$ such that every distribution of $f(G, v)$ pebbles on the vertices of G , we can move one pebble to v by a sequence of pebbling moves. The *pebbling number* $f(G)$ of a simple connected graph G is the maximum of $f(G, v)$ over all vertices v . Similarly, the *t -pebbling number* of a vertex v in a graph G is the smallest positive integer $f_t(G, v)$ such that every distribution of $f_t(G, v)$ pebbles on the vertices of G , we can move t pebbles to v by a sequence of pebbling moves. The *t -pebbling number* $f_t(G)$ of a simple connected graph G is the maximum of $f_t(G, v)$ over all vertices v . Evidently, $f_1(G) = f(G)$.

We say a graph G has the *2-pebbling property* if two pebbles can be moved to any specified vertex when the total starting number of pebbles is $2f(G) - q + 1$, where q is the number of vertices with at least one pebble. Similarly, a graph G has the *$2t$ -pebbling property* if $2t$ pebbles can be moved to any specified vertex when the total starting number of pebbles is $2f_t(G) - q + 1$, where q is the number of vertices with at least one pebble. Lourdasamy et al. [8, 9] have proved that the star graph, the n -cube, the complete graph, the complete multi-partite graph, the fan graph, the wheel graph, and the even cycle have the $2t$ -pebbling property.

Conjecture 1.1 (Graham [1]) *For any connected graphs G and H , $f(G \times H) \leq f(G)f(H)$.*

Many articles (see, e.g., [1–3, 5, 10–12]) have given evidences supporting Conjecture 1.1. Recently, Herscovici further extended Conjecture 1.1 as follows:

Conjecture 1.2 (Herscovici [6]) *For any connected graphs G and H , $f_{st}(G \times H) \leq f_s(G)f_t(H)$.*

Gao and Yin [4] proved that if G is a tree, or a complete graph, or an even cycle and H has the $2t$ -pebbling property, then Conjecture 1.2 holds.

Kirlangic [7] defined a thorn graph of the graph G as follows: Let G be a graph with $V(G) = \{v_1, v_2, \dots, v_n\}$ and $p_1, \dots, p_n$ be positive integers with $p_i \geq 1$ for each i . The thorn graph of G , with parameters $p_1, p_2, \dots, p_n$, is obtained by attaching p_i new vertices of degree 1 to vertex v_i of G for $i \in \{1, 2, \dots, n\}$. The thorn graph of G is denoted by G^* or $G^*(p_1, p_2, \dots, p_n)$. When $G = K_n$, Wang et al. [12] proved the following three theorems:

Theorem 1.1 ([12]) *If $n \geq 2$ and $p_i \geq 2$ for each i , then $f(K_n^*) = \sum_{i=1}^n p_i + 2n + 2$.*

Theorem 1.2 ([12]) *If $p_i \geq 2$ for each i , then K_n^* has the 2-pebbling property.*

Theorem 1.3 ([12]) *If G has the 2-pebbling property and $p_i \geq 2$ for each i , then*

$$f(K_n^* \times G) \leq f(K_n^*)f(G).$$

The purpose of this paper is to extend the above theorems as follows:

Theorem 1.4 *If $n \geq 2$ and $p_i \geq 1$ for each i , then $f_t(K_n^*) = \sum_{i=1}^n p_i + 2n + 8t - 6$.*

Theorem 1.5 *If $p_i \geq 1$ for each i , then K_n^* has the $2t$ -pebbling property.*

Theorem 1.6 *If G has the $2t$ -pebbling property and $p_i \geq 1$ for each i , then*

$$f_{st}(K_n^* \times G) \leq f_s(K_n^*)f_t(G).$$

Corollary 1.1 $f_{st}(K_n^* \times K_m^*) \leq f_s(K_n^*)f_t(K_m^*) = \left(\sum_{i=1}^n p_i + 2n + 8s - 6\right) \left(\sum_{i=1}^m p_i + 2m + 8t - 6\right).$

2 Lemmas

In this section, we present some lemmas that will be used in later proof. We always assume that $u_{i,1}, u_{i,2}, \dots, u_{i,p_i}$ are the p_i vertices of degree 1 attached to v_i for each i .

Given a configuration of pebbles placed on G , a transmitting subgraph of G is defined to be a path $x_1, \dots, x_n$ such that there are at least two pebbles on x_1 and at least one pebble on each of the other vertices in the path except x_n , so that one pebble can be moved to x_n . A tree T can be viewed as a directed tree T_v with edges directed toward a root vertex v . A path-partition $P = \{P_1, \dots, P_n\}$ of T_v is a set of edge-disjoint directed paths, the union of which is T_v . We will assume that $|E(P_i)| \geq |E(P_j)|$ whenever $i \leq j$. A path-partition $\{P_1, \dots, P_n\}$ of T_v is said to *majorize* another $\{P'_1, \dots, P'_n\}$ if the non-increasing sequence of its path size majorizes that of the other. That is, if $a_i = |E(P_i)|$ and $b_j = |E(P'_j)|$, then $(a_1, \dots, a_n) > (b_1, \dots, b_n)$ if and only if $a_i > b_i$ where $i = \min \{j : a_j \neq b_j\}$. A path-partition of T_v is said to be maximum if it majorizes all other path-partitions of T_v .

Lemma 2.1 ([1, 10]) *If $a_1, \dots, a_m$ is the sequence of the path sizes in a maximum path-partition of T_v , then $f(T, v) = \sum_{i=1}^m 2^{a_i} - m + 1$, and $f_i(T, v) = t2^{a_1} + \sum_{i=2}^m 2^{a_i} - m + 1$.*

Lemma 2.2 ([9]) $f_t(K_n) = 2t + n - 2$, where $n \geq 2$.

Lemma 2.3 ([5]) *Let $x_i x_j \in E(G)$. Suppose that in $G \times H$, we have p_i pebbles occupying vertex $\{x_i\} \times H$, and r_i of these vertices have an odd number of pebbles. If $r_i \leq k \leq p_i$ and if k and p_i have the same parity, then k pebbles can be retained on $\{x_i\} \times H$, while moving $\frac{p_i - k}{2}$ pebbles onto $\{x_j\} \times H$. If k and p_i have the opposite parity, we must leave $k + 1$ pebbles on $\{x_i\} \times H$, while moving $\frac{p_i - k - 1}{2}$ pebbles onto $\{x_j\} \times H$. In particular, we can always move at least $\frac{p_i - r_i}{2}$ pebbles onto $\{x_j\} \times H$, since p_i and r_i have the same parity.*

Lemma 2.4 ([4]) *Let T be a tree and H have the $2t$ -pebbling property. For all positive integers s and t , all vertices $x \in V(G)$ and $y \in V(H)$, we have*

$$f_{st}(T \times H, (x, y)) \leq f_s(T, x)f_t(H).$$

Lemma 2.5 ([4]) *Let K_n be a complete graph and G have the $2t$ -pebbling property. For all positive integers s and t , we have*

$$f_{st}(K_n \times G) \leq f_s(K_n)f_t(G).$$

3 Proofs of Theorem 1.4–1.6

Proof of Theorem 1.4 Considering the following distribution of $\sum_{i=1}^n p_i + 2n + 8t - 7$ pebbles on K_n^* as follows: $p(u_{1,1}) = 8t - 1$, $p(u_{1,j}) = 1$ for all $j \in \{2, \dots, p_1\}$, $p(u_{i,1}) = 3$ for all $i \in \{2, \dots, n - 1\}$, $p(u_{n,1}) = 0$, $p(u_{i,j}) = 1$ for all $i \in \{2, \dots, n - 1\}$ and all $j \in \{2, \dots, p_i\}$, $p(u_{n,j}) = 1$ for all $j \in \{2, \dots, p_n\}$, and $p(v_i) = 0$ for all $i \in \{1, \dots, n\}$. Then, t pebbles can not be moved to $u_{n,1}$. Thus, $f_t(K_n^*) \geq \sum_{i=1}^n p_i + 2n + 8t - 6$. Now considering any distribution of $\sum_{i=1}^n p_i + 2n + 8t - 6$ pebbles on K_n^* . Let s be the number of occupied vertices of $\{u_{1,1}, \dots, u_{1,p_1}, u_{2,1}, \dots, u_{2,p_2}, u_{n,1}, \dots, u_{n,p_n}\}$. Clearly, $s \leq \sum_{i=1}^n p_i$. We will proceed by induction on t .

First, we assume $t = 1$. Without loss of generality, we consider the following cases:

Case 1.1 The target vertex is v_1 and $p(v_1) = 0$.

Let $p(K_n) = m$, we can assume that $0 \leq m \leq n - 1$ by Lemma 2.2. By pebbling moves, we can move $\frac{1}{2}(\sum_{i=1}^n p_i + 2n + 2 - m - s)$ pebbles to v_i from $u_{i,j}$ for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, p_i\}$, and we have

$$m + \frac{1}{2} \left(\sum_{i=1}^n p_i + 2n + 2 - m - s \right) \geq \frac{1}{2} \left(\sum_{i=1}^n p_i + 2n + 2 + m - s \right) > n$$

pebbles on K_n . Hence, we can move one pebble to v_1 by Lemma 2.2.

Case 1.2 The target vertex is $u_{1,1}$ and $p(u_{1,1}) = 0$.

Clearly, $s \leq \sum_{i=1}^n p_i - 1$. By Lemma 2.2, we can assume that $p(K_n) = m$, $0 \leq m \leq n + 2 - 1$. By pebbling moves, we can move $\frac{1}{2}(\sum_{i=1}^n p_i + 2n + 2 - m - s)$ pebbles to K_n from $u_{i,j}$ for each i and each j except $i = j = 1$. Thus, at least

$$m + \frac{1}{2} \left(\sum_{i=1}^n p_i + 2n + 2 - m - s \right) \geq \frac{1}{2} \left(\sum_{i=1}^n p_i + 2n + 2 + m - s \right) \geq n + \frac{3}{2}$$

pebbles can be moved to K_n , this implies that $n + 2$ pebbles can be moved to K_n , hence, we can move two pebbles to v_1 by Lemma 2.2. Furthermore, we can move one pebble to $u_{1,1}$.

Now, assume that Theorem 1.4 is true for $t \leq k$ ($k \geq 1$). We will prove that Theorem 1.4 is true for $t = k + 1$. Without loss of generality, we consider the following cases:

Case 2.1 The target vertex is v_1 and $p(v_1) = \ell$ for $0 \leq \ell \leq k$.

If $1 \leq \ell \leq k$, then we can move $k + 1 - \ell$ additional pebbles to v_1 by the induction hypothesis, since $p(K_n^* - v_1) = \sum_{i=1}^n p_i + 2n + 8(k + 1) - 6 - \ell \geq \sum_{i=1}^n p_i + 2n + 8(k + 1 - \ell) - 6$. Hence, we can move $k + 1$ pebbles to v_1 . Assume that $\ell = 0$. By

Lemma 2.2, we can assume that $p(K_n) = m$, $0 \leq m \leq 2(k+1) + n - 3$. By pebbling moves, we can move $\frac{1}{2}(\sum_{i=1}^n p_i + 2n + 8(k+1) - 6 - m - s)$ pebbles to K_n from $u_{i,j}$ for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, p_i\}$. Thus, at least

$$\begin{aligned} m + \frac{1}{2} \left(\sum_{i=1}^n p_i + 2n + 8(k+1) - 6 - m - s \right) &\geq \frac{1}{2} \left(\sum_{i=1}^n p_i + 2n + 8(k+1) - 6 + m - s \right) \\ &\geq n + 4(k+1) - 3 \\ &> n + 2(k+1) - 2 \end{aligned}$$

pebbles can be moved to K_n . Hence, we can move $k+1$ pebbles to v_1 by Lemma 2.2.

Case 2.2 The target vertex is $u_{1,1}$ and $p(u_{1,1}) = \ell$.

If $1 \leq \ell \leq k$, then we can move additional $k+1-\ell$ pebbles to $u_{1,1}$ by the induction hypothesis, since $p(K_n^* - u_{1,1}) = \sum_{i=1}^n p_i + 2n + 8(k+1) - 6 - \ell \geq \sum_{i=1}^n p_i + 2n + 8(k+1-\ell) - 6$. Hence, we can move $k+1$ pebbles to v_1 . Now suppose that $p(u_{1,1}) = 0$. Clearly, $s \leq \sum_{i=1}^n p_i - 1$. Let $p(K_n) = m$, we can assume that $0 \leq m \leq 4(k+1) + n - 3$ by Lemma 2.2. By pebbling moves, we can move $\frac{1}{2}(\sum_{i=1}^n p_i + 2n + 8(k+1) - 6 - m - s)$ pebbles to K_n from $u_{i,j}$ for each i and j except $i=j=1$. Thus,

$$\begin{aligned} m + \frac{1}{2} \left(\sum_{i=1}^n p_i + 2n + 8(k+1) - 6 - m - s \right) &\geq \frac{1}{2} \left(\sum_{i=1}^n p_i + 2n + 8(k+1) - 6 + m - s \right) \\ &\geq 4(k+1) + n - \frac{5}{2}. \end{aligned}$$

This implies that $4(k+1) + n - 2$ pebbles can be moved to K_n , and hence we can move $2(k+1)$ pebbles to v_1 by Lemma 2.2. Further, we can move $k+1$ pebbles to $u_{1,1}$. $\square$

Proof of Theorem 1.5 Clearly, K_n^* is a star graph when $n = 1$, and the result is true. Now, suppose that $n \geq 2$. Given a distribution of pebbles on K_n^* , let p be the number of pebbles on K_n^* , q be the number of vertices with at least one pebble, and $p = 2(\sum_{i=1}^n p_i + 2n + 8t - 6) - q + 1$. Let s be the number of occupied vertices of $\{u_{1,1}, \dots, u_{1,p_1}, u_{2,1}, \dots, u_{2,p_2}, u_{n,1}, \dots, u_{n,p_n}\}$, and s' be the number of occupied vertices of $\{v_1, v_2, \dots, v_n\}$. Clearly, $s \leq \sum_{i=1}^n p_i$, $s' \leq p(K_n)$, $s + s' = q$, and $q \leq \sum_{i=1}^n p_i + n$. Without loss of generality, we can assume that the target vertices are v_1 and $u_{1,1}$.

Case 1 The target vertex is v_1 and $p(v_1) = \ell$ for $0 \leq \ell \leq 2t - 1$.

If $\ell \geq 1$, then $p(K_n^* - v_1) = 2(\sum_{i=1}^n p_i + 2n + 8t - 6) - q + 1 - \ell \geq \sum_{i=1}^n p_i + 2n + 8(2t - \ell) - 6$. Thus, we can move $2t - \ell$ additional pebbles to v_1 by Theorem 1.4. Hence, we can move $2t$ pebbles to v_1 . Assume that $p(v_1) = 0$. Note that $q \leq \sum_{i=1}^n p_i + n - 1$. Let $p(K_n) = m$, we can assume that $0 \leq m \leq 4t + n - 3$ by Lemma 2.2. By pebbling moves, we can move $\frac{1}{2}(2(\sum_{i=1}^n p_i + 2n + 8t - 6) - q + 1 - m - s)$ pebbles to K_n from $u_{i,j}$ for each i and j . Thus, at least

$$m + \frac{1}{2} \left(2 \left(\sum_{i=1}^n p_i + 2n + 8t - 6 \right) - q + 1 - m - s \right) \geq \frac{1}{2} \left(2 \left(\sum_{i=1}^n p_i + 2n + 8t - 6 \right) - q + 1 + m - s \right) \geq 4t + n - 2$$

pebbles on K_n . Hence, we can move $2t$ pebbles to v_1 by Lemma 2.2.

Case 2 The target vertex is $u_{1,1}$ and $p(u_{1,1}) = \ell$ for $0 \leq \ell \leq 2t - 1$.

If $\ell \geq 1$, then $p(K_n^* - u_{1,1}) = 2(\sum_{i=1}^n p_i + 2n + 8t - 6) - q + 1 - \ell \geq \sum_{i=1}^n p_i + 2n + 8(2t - \ell) - 6$. Thus, we can move $2t - \ell$ additional pebbles to $u_{1,1}$ by Theorem 1.4. Assume that $p(u_{1,1}) = 0$. Then, $s \leq \sum_{i=1}^n p_i - 1$. Let $p(K_n) = m$, we can assume that $0 \leq m \leq 8t + n - 3$ by Lemma 2.2. By pebbling moves, then at least

$$\begin{aligned} m + \sum_{i=1}^n p_i + 2n + 8t - 6 + \frac{1 - q - m - s}{2} &\geq \sum_{i=1}^n p_i + 2n + 8t - 6 \\ &\quad + \frac{1 + m - (s + s') - s}{2} \\ &\geq 8t + n - \frac{5}{2} \end{aligned}$$

pebbles can be moved to K_n , and this implies that $n + 8t - 2$ pebbles can be moved to K_n . Hence, we can move $4t$ pebbles to v_1 by Lemma 2.2. Therefore, we can move $2t$ pebbles to $u_{1,1}$. $\square$

Proof of of Theorem 1.6 Clearly, K_n^* is a tree when $n = 1$, and so the result is true by Lemma 2.4. Now, suppose that $n \geq 2$. We consider any distribution of $(\sum_{i=1}^n p_i + 2n + 8s - 6)f_t(G)$ pebbles on the vertices of $K_n^* \times G$. Let $G_{ij} = \{u_{ij}\} \times G$, m_{ij} be the number of pebbles on G_{ij} , and r_{ij} be the number of vertices with an odd number of pebbles in G_{ij} for each i and each j .

Case 1 The target vertex is (v_i, y) for some $y \in V(G)$.

Without loss of generality, assume that the target vertex is (v_1, y) . Clearly, $K_n^* - \{u_{1,1}, \dots, u_{1,p_1}, u_{2,1}, \dots, u_{2,p_2}, u_{n,1}, \dots, u_{n,p_n}\} = K_n$. By $r_{ij} \leq |V(G)| \leq f(G) \leq f_t(G)$, $\sum_{i=1}^n \sum_{j=1}^{p_i} m_{ij} \leq (\sum_{i=1}^n p_i + 2n + 8s - 6)f_t(G)$, we get

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^{p_i} (m_{ij} + r_{ij}) &= \sum_{i=1}^n \sum_{j=1}^{p_i} m_{ij} + \sum_{i=1}^n \sum_{j=1}^{p_i} r_{ij} \\ &\leq \left(\sum_{i=1}^n p_i + 2n + 8s - 6 \right) f_t(G) + \left(\sum_{i=1}^n p_i \right) f_t(G) \\ &= 2 \left(\sum_{i=1}^n p_i + n + 4s - 3 \right) f_t(G). \end{aligned}$$

By Lemma 2.3, we can move at least $\sum_{i=1}^n \sum_{j=1}^{p_i} \left(\frac{m_{ij} - r_{ij}}{2} \right)$ pebbles to $K_n \times G$ from all the vertices in $G_{11}, \dots, G_{1p_1}, \dots, G_{n1}, \dots, G_{np_n}$. So, we have at least

$$\begin{aligned}
& \left(\sum_{i=1}^n p_i + 2n + 8s - 6 \right) f_t(G) - \sum_{i=1}^n \sum_{j=1}^{p_i} m_{ij} + \sum_{i=1}^n \sum_{j=1}^{p_i} \left(\frac{m_{ij} - r_{ij}}{2} \right) \\
&= \left(\sum_{i=1}^n p_i + 2n + 8s - 6 \right) f_t(G) - \sum_{i=1}^n \sum_{j=1}^{p_i} \left(\frac{m_{ij} + r_{ij}}{2} \right) \\
&\geq \left(\sum_{i=1}^n p_i + 2n + 8s - 6 \right) f_t(G) - \left(\sum_{i=1}^n p_i + n + 4s - 3 \right) f_t(G) \\
&= (4s + n - 3) f_t(G) \\
&\geq (2s + n - 2) f_t(G)
\end{aligned}$$

pebbles on $K_n \times G$. Therefore, we can move st pebbles to (v_1, y) by Lemma 2.5.

Case 2 The target vertex is (u_{ij}, y) for some $y \in V(G)$.

Without loss of generality, assume that the target vertex is $(u_{1,1}, y)$. Let T be a tree obtained from K_n^* by deleting all edges between v_i and v_j for $i, j \in \{2, \dots, n\}$. By Lemma 2.1, we have $f_s(T, u_{1,1}) = \sum_{i=1}^n p_i + 2n + 8s - 6$. Note that $T \times G$ is a spanning subgraph of $K_n^* \times G$ and $f_{st}(K_n^* \times G, (u_{1,1}, y)) \leq f_{st}(T \times G, (u_{1,1}, y))$. Hence, by Lemma 2.4, we have

$$\begin{aligned}
f_{st}(K_n^* \times G, (u_{1,1}, y)) &\leq f_{st}(T \times G, (u_{1,1}, y)) \leq f_s(T, u_{1,1}) f_t(G) \\
&= \left(\sum_{i=1}^n p_i + 2n + 8s - 6 \right) f_t(G).
\end{aligned}$$

Therefore, we can move st pebbles to $(u_{1,1}, y)$. $\square$

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