An Exact l_1 Exponential Penalty Function Method for Multiobjective Optimization Problems with Exponential-Type Invexity

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Abstract The purpose of this paper is to devise exact l_1 exponential penalty function method to solve multiobjective optimization problems with exponential-type invexity. The conditions governing the equivalence of the (weak) efficient solutions to the vector optimization problem and the (weak) efficient solutions to associated unconstrained exponential penalized multiobjective optimization problem are studied. Examples are given to illustrate the obtained results.

Keywords Exact l_1 exponential penalty method · Exponential penalized vector optimization problems · (p, r) -invexity

Mathematics subject classification (2010) 90C26 · 90C30

1 Introduction

Penalty method is one of the most powerful techniques for solving constrained nonlinear optimization problems. This method involves transforming the constrained optimization problem into either a single unconstrained problem or into a sequence of unconstrained problems by adding a term to the objective function that reflects the measure of violation of constraints.

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Instead of considering a sequence of unconstrained problems, exact penalty methods are used to solve nonlinear programming problems by minimization of a single unconstrained problem. The term “exact” indicates that the variables used in both the constrained and unconstrained optimization problems are in the same space and for certain choices of the penalty parameter, the single unconstrained problem can yield exact solution of the nonlinear optimization problem.

Penalty function methods have been studied by many researchers, see [1, 2, 4–13]. One of the most popular exact penalty functions is absolute value penalty function, also called as l_1 exact penalty function due to the fact that the penalty term involves the l_1 norm of the violated constraint. The l_1 exact penalty function is nondifferentiable and has been extensively studied in the last few decades.

Fletcher [6, 7] showed that an exact penalty function with a local unconstrained minimum can be constructed at any local minimum of a nonlinear programming problem with inequality constraints. Meng et al. [12] presented a penalty function with objective parameters for inequality constrained optimization problems. Thereafter, Correia et al. [5] classified some of the existing penalty methods and described their assumptions and limitations.

Antczak [3] introduced the notion of (p, r) -invexity for multiobjective programming to further generalize the concept of invexity. Later, Antczak [2] applied the l_1 exact exponential penalty approach for solving mathematical programming problems. Recently, Mandal and Nahak [11] used the l_1 exact exponential penalty function method to solve nonlinear mathematical programming problems with $(p, r) - \rho - (\eta, \theta)$ -invex functions.

Most of the literature on penalty function methods focuses on single objective nonlinear programming problems. However, many realistic problems in decision theory, engineering, and planning, demand optimization of multiple objectives with conflicting nature to be considered simultaneously. Hence, this work is devoted to extend the results in [2] to multiobjective optimization problems involving exponential-type invex functions which subsumes the class of functions considered in [2].

This paper is organized as follows: In Sect. 2, we recall some definitions which will be useful, while proving the results. In Sect. 3, we extend the use of exact l_1 exponential penalty method to vector optimization problem involving exponential-type invex functions and study the conditions under which the (weak) efficient solutions of the constrained vector optimization problem are also the (weak) efficient solutions to the exponential penalized multiobjective optimization problem and vice versa. Finally, in Sect. 4, we conclude our paper.

2 Notations and Preliminaries

The following convention for equalities and inequalities will be used throughout this paper. Let $x, y \in \mathbb{R}^n$, where $\mathbb{R}^n$ denotes the n -dimensional Euclidean space. We define:

$$(i) \quad x = y \Leftrightarrow x_i = y_i, \forall i = 1, \dots, n;$$

- (ii) $x < y \Leftrightarrow x_i < y_i, \forall i = 1, \dots, n;$
- (iii) $x \leq y \Leftrightarrow x_i \leq y_i, \forall i = 1, \dots, n;$
- (iv) $x \leq y \Leftrightarrow x \leq y$ and $x \neq y$.

Throughout this paper, let X be a nonempty subset of $\mathbb{R}^n$, unless specified otherwise.

Definition 2.1 [3] Let $f : X \mapsto \mathbb{R}^k$ be a differentiable function on X and p, r be arbitrary real numbers. The function f is said to be (p, r) -invex (strictly (p, r) -invex) function with respect to $\eta : X \times X \mapsto \mathbb{R}^n$ at u on X , if for all $x \in X$ and $i = 1, \dots, k$, one of the relations

$$\frac{1}{r} e^{rf_i(x)} \geq \frac{1}{r} e^{rf_i(u)} \left[1 + \frac{r}{p} (e^{p\eta(x,u)} - 1)^T \nabla f_i(u) \right] \quad (> \text{ if } x \neq u) \quad \text{for } p \neq 0, r \neq 0,$$

$$\frac{1}{r} e^{rf_i(x)} \geq \frac{1}{r} e^{rf_i(u)} \left[1 + r[\eta(x, u)]^T \nabla f_i(u) \right] \quad (> \text{ if } x \neq u) \quad \text{for } p = 0, r \neq 0,$$

$$f_i(x) - f_i(u) \geq \frac{1}{p} (e^{p\eta(x,u)} - 1)^T \nabla f_i(u) \quad (> \text{ if } x \neq u) \quad \text{for } p \neq 0, r = 0,$$

$$f_i(x) - f_i(u) \geq [\eta(x, u)]^T \nabla f_i(u) \quad (> \text{ if } x \neq u) \quad \text{for } p = 0, r = 0,$$

holds. The function f is said to be (p, r) -invex with respect to η on X , if f is (p, r) -invex with respect to the same η at each $u \in X$.

Also, the exponentials appearing on the right hand sides of the above inequalities are understood to be taken componentwise and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. For more details on (p, r) -invex function, we refer to [3].

Remark 2.1 All the results in this paper will be proved only for the case when $p \neq 0, r \neq 0$. The theorems in other cases can be proved likewise and are easier than the considered case. Moreover, without loss of generality, we assume that $r > 0$. For $r < 0$, some of the inequalities in the proofs of the given theorems should be reversed.

We consider the following vector optimization problem:

$$\begin{aligned} (\text{VP}) \quad & \text{V-minimize } f(x) = (f_1(x), \dots, f_k(x)) \\ & \text{s.t.} \\ & g_j(x) \leq 0, j \in J = \{1, \dots, m\}, \\ & x \in X, \end{aligned}$$

where $f_i : X \mapsto \mathbb{R}, i \in I = \{1, \dots, k\}$ and $g_j : X \mapsto \mathbb{R}$ are differentiable functions on X . Let D denotes the set of all feasible solutions to the vector optimization problem (VP), i.e.,

$$D = \{x \in X : g_j(x) \leq 0, j \in J\}.$$

In multiobjective programming problems, there may not exist a point which simultaneously optimizes each objective. Therefore, the concept of efficient (Pareto optimal) solution is used.

Definition 2.2 A point $\bar{x} \in D$ is said to be an efficient solution to (VP) if there exists no $y \in D$ such that

$$f(y) \leq f(\bar{x}).$$

Definition 2.3 A point $\bar{x} \in D$ is said to be a weakly efficient solution to (VP) if there exists no $y \in D$ such that

$$f(y) < f(\bar{x}).$$

Clearly, every efficient solution is a weakly efficient solution to the vector optimization problem.

The following Karush–Kuhn–Tucker (KKT) conditions are necessary for a feasible solution to be an (weakly) efficient solution for the considered vector optimization problem.

Theorem 2.1 (KKT Necessary Conditions) [4] *Let $\bar{x}$ be an (weakly) efficient solution to vector optimization problem (VP) and the suitable constraint qualification is satisfied at $\bar{x}$. Then there exist Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^k$ and $\bar{\mu} \in \mathbb{R}^m$ such that*

$$\sum_{i=1}^k \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j(\bar{x}) = 0, \quad (2.1)$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, \quad j \in J, \quad (2.2)$$

$$\bar{\lambda} \geq 0, \quad \sum_{i=1}^k \bar{\lambda}_i = 1, \quad \bar{\mu} \geq 0. \quad (2.3)$$

Let us denote by $J(\bar{x})$ the set

$$J(\bar{x}) = \{j \in J : \bar{\mu}_j \neq 0\}.$$

Definition 2.4 The point $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in D \times \mathbb{R}^k \times \mathbb{R}^m$ satisfying the KKT necessary conditions (2.1)–(2.3) is said to be a KKT point in problem (VP).

Let Y denote the set of all feasible solutions to the vector optimization problem (VP) at which the KKT necessary conditions are satisfied, i.e.,

$$Y = \{\bar{x} \in D : \exists \bar{\lambda} \in \mathbb{R}^k, \bar{\mu} \in \mathbb{R}^m, (\bar{x}, \bar{\lambda}, \bar{\mu}) \text{ is a KKT point in problem (VP)}\}.$$

Penalty function methods transform the constrained problem into unconstrained ones by placing the constraint into the objective function via a penalty parameter in a way that penalizes any violation of the constraints.

3 Exact l_1 Exponential Penalty Method for Vector Optimization Problem

We construct the following exponential penalized multiobjective optimization problem $(VP_r(c))$ involving exact l_1 penalty function associated with vector optimization problem (VP).

$$(\mathbf{VP}_r(\mathbf{c})) \text{ Min } VP_r(x, c) = \left(\frac{1}{r} e^{rf_1(x)} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(x)} - 1), \dots, \frac{1}{r} e^{rf_k(x)} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(x)} - 1) \right),$$

where r is a finite real number not equal to zero and $c > 0$ is a penalty parameter. For a given constraint $g_j(x) \leq 0$, the function $\frac{1}{r}(e^{rg_j^+(x)} - 1)$ is defined as follows:

$$\frac{1}{r}(e^{rg_j^+(x)} - 1) = \begin{cases} 0, & \text{if } g_j(x) \leq 0, \\ \frac{1}{r}(e^{rg_j(x)} - 1), & \text{if } g_j(x) > 0. \end{cases} \quad (3.1)$$

Definition 3.1 A point $x \in X$ is said to be an efficient solution to the exponential penalized multiobjective optimization problem $(\mathbf{VP}_r(\mathbf{c}))$ if there exists no $y \in X$ such that

$$VP_r(y, c) \leq VP_r(x, c).$$

Definition 3.2 A point $x \in X$ is said to be a weakly efficient solution to the exponential penalized multiobjective optimization problem $(\mathbf{VP}_r(\mathbf{c}))$ if there exists no $y \in X$ such that

$$VP_r(y, c) < VP_r(x, c).$$

Now, we shall show that a KKT point to vector optimization problem (VP) gives an efficient solution to the exponential penalized multiobjective optimization problem $(\mathbf{VP}_r(\mathbf{c}))$ when the penalty parameter exceeds some suitable threshold value.

Theorem 3.1 Let $\bar{x}$ be a feasible solution to the vector optimization problem (VP) such that the KKT conditions (2.1)–(2.3) are satisfied at $\bar{x}$ with Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^k$, $\bar{\mu} \in \mathbb{R}^m$. Assume that the objective function f and the constraint functions g_j , $j \in J(\bar{x})$ are (p, r) -invex at $\bar{x}$ with respect to the same function η .

Furthermore, if the penalty parameter c is assumed to be sufficiently large (it is sufficient to set $c \geq \max\{e^{rf_i(\bar{x})} : i \in I\} \max\{\bar{\mu}_j : j \in J\}$), then $\bar{x}$ is a weakly efficient solution to the exponential penalized multiobjective optimization problem $(\mathbf{VP}_r(\mathbf{c}))$ involving exact l_1 exponential penalty function.

Proof Suppose contrary to the result that $\bar{x}$ is not a weakly efficient solution to the exponential penalized multiobjective optimization problem $[\mathbf{VP}_r(\mathbf{c})]$ involving exact l_1 exponential penalty function. Then there exists $x \in X$ such that

$$VP_r(x, c) < VP_r(\bar{x}, c). \\ \Rightarrow \frac{1}{r} e^{rf_i(x)} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(x)} - 1) < \frac{1}{r} e^{rf_i(\bar{x})} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1), \quad \forall i \in I.$$

Since $\bar{x}$ is a feasible solution to (VP), the above inequality together with (3.1) gives

$$\frac{1}{r}e^{rf_i(x)} + c \sum_{j=1}^m \frac{1}{r}(e^{rg_j^+(x)} - 1) < \frac{1}{r}e^{rf_i(\bar{x})}, \quad \forall i \in I.$$

By assumption $c \geq \max\{e^{rf_i(\bar{x})} : i \in I\} \max\{\bar{\mu}_j : j \in J\}$, it follows that

$$\frac{1}{r}e^{rf_i(x)} + \frac{1}{r}e^{rf_i(\bar{x})} \sum_{j=1}^m \bar{\mu}_j(e^{rg_j^+(x)} - 1) < \frac{1}{r}e^{rf_i(\bar{x})}, \quad \forall i \in I.$$

Again using (3.1), the above inequality yields

$$\begin{aligned} & \frac{1}{r}e^{rf_i(x)} + \frac{1}{r}e^{rf_i(\bar{x})} \sum_{j=1}^m \bar{\mu}_j(e^{rg_j(x)} - 1) < \frac{1}{r}e^{rf_i(\bar{x})}, \quad \forall i \in I. \\ \Rightarrow & \frac{1}{r}(e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j(e^{rg_j(x)} - 1) < 0, \quad \forall i \in I. \end{aligned}$$

From the feasibility of $\bar{x}$ to (VP) and KKT condition (2.2), the above inequality gives

$$\frac{1}{r}(e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j \in J(\bar{x})} \bar{\mu}_j(e^{\frac{r}{\bar{\mu}_j}(\bar{\mu}_j g_j(x) - \bar{\mu}_j g_j(\bar{x}))} - 1) < 0, \quad \forall i \in I.$$

Thus, it follows that

$$\frac{1}{r}(e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j(e^{r(g_j(x)-g_j(\bar{x}))} - 1) < 0, \quad \forall i \in I.$$

Since $\bar{\lambda} \geq 0$ and $\sum_{i=1}^k \bar{\lambda}_i = 1$, we have

$$\frac{1}{r} \bar{\lambda}_i(e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \bar{\lambda}_i \sum_{j=1}^m \bar{\mu}_j(e^{r(g_j(x)-g_j(\bar{x}))} - 1) \leq 0, \quad \forall i \in I$$

and for at least one $i \in I$,

$$\frac{1}{r} \bar{\lambda}_i(e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \bar{\lambda}_i \sum_{j=1}^m \bar{\mu}_j(e^{r(g_j(x)-g_j(\bar{x}))} - 1) < 0.$$

Taking summation over $i \in I$, it follows from the above inequalities that

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i(e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j(e^{r(g_j(x)-g_j(\bar{x}))} - 1) \sum_{i=1}^k \bar{\lambda}_i < 0.$$

Again since $\sum_{i=1}^k \bar{\lambda}_i = 1$, we get

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i(e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j(e^{r(g_j(x)-g_j(\bar{x}))} - 1) < 0. \quad (3.2)$$

On the other hand, from the hypothesis, f is (p, r) -invex at $\bar{x}$ w.r.t. η , the relation

$$\frac{1}{r} e^{rf_i(x)} \geq \frac{1}{r} e^{rf_i(\bar{x})} \left[1 + \frac{r}{p} (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}) \right], \quad \forall i \in I$$

holds for all $x \in X$.

As $\bar{\lambda} \geq 0$, we have

$$\begin{aligned} \frac{1}{r} \bar{\lambda}_i e^{rf_i(x)} &\geq \frac{1}{r} \bar{\lambda}_i e^{rf_i(\bar{x})} \left[1 + \frac{r}{p} (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}) \right], \quad \forall i \in I. \\ \Rightarrow \frac{1}{r} \bar{\lambda}_i (e^{r(f_i(x) - f_i(\bar{x}))} - 1) &\geq \frac{\bar{\lambda}_i}{p} (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}), \quad \forall i \in I. \end{aligned}$$

Taking summation over $i \in I$, it follows from the above inequality that

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (e^{r(f_i(x) - f_i(\bar{x}))} - 1) \geq \frac{1}{p} \sum_{i=1}^k \bar{\lambda}_i (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}). \quad (3.3)$$

Again, by the hypothesis, $g_j, j \in J(\bar{x})$ are (p, r) -invex at $\bar{x}$ w.r.t. the same function η , the relation

$$\frac{1}{r} e^{rg_j(x)} \geq \frac{1}{r} e^{rg_j(\bar{x})} \left[1 + \frac{r}{p} (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \nabla g_j(\bar{x}) \right], \quad \forall j \in J(\bar{x})$$

holds for all $x \in X$.

As $\bar{\mu} \geq 0$, we have

$$\frac{1}{r} \bar{\mu}_j e^{rg_j(x)} \geq \frac{1}{r} \bar{\mu}_j e^{rg_j(\bar{x})} \left[1 + \frac{r}{p} (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \nabla g_j(\bar{x}) \right], \quad \forall j \in J(\bar{x}).$$

Taking into account the constraints for which the associated Lagrange multipliers are zero, the above inequality yields

$$\frac{1}{r} \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) \geq \frac{\bar{\mu}_j}{p} (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \nabla g_j(\bar{x}), \quad \forall j \in J.$$

Taking summation over $j \in J$, it follows that

$$\frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) \geq \frac{1}{p} \sum_{j=1}^m \bar{\mu}_j (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \nabla g_j(\bar{x}). \quad (3.4)$$

On adding inequalities (3.3) and (3.4), we get

$$\begin{aligned} &\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (e^{r(f_i(x) - f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) \\ &\geq \frac{1}{p} (e^{p\eta(x, \bar{x})} - \mathbf{1})^T \left[\sum_{i=1}^k \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j(\bar{x}) \right]. \end{aligned}$$

Finally, by using KKT condition (2.1), the above inequality reduces to

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (e^{r(f_i(x) - f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) \geq 0,$$

which contradicts (3.2). This completes the proof. $\square$

Corollary 3.1 *Let $\bar{x}$ be a weakly efficient solution to the vector optimization problem (VP) such that the KKT conditions (2.1)–(2.3) are satisfied at $\bar{x}$ with Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^k$, $\bar{\mu} \in \mathbb{R}^m$. Assume that the objective function f and the constraint functions $g_j, j \in J(\bar{x})$ are (p, r) -invex at $\bar{x}$ with respect to the same function η .*

Furthermore, if the penalty parameter c is assumed to be sufficiently large (it is sufficient to set $c \geq \max\{e^{rf_i(\bar{x})} : i \in I\} \max\{\bar{\mu}_j : j \in J\}$), then $\bar{x}$ is a weakly efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(c))$ involving exact l_1 exponential penalty function.

Proof The result follows directly from Theorem 3.1, since every weakly efficient solution is a feasible solution. $\square$

Theorem 3.2 *Let $\bar{x}$ be a feasible solution to the vector optimization problem (VP) such that the KKT conditions (2.1)–(2.3) are satisfied at $\bar{x}$ with Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^k$, $\bar{\mu} \in \mathbb{R}^m$. Assume that the objective function f and the constraint functions $g_j, j \in J(\bar{x})$ are (p, r) -invex at $\bar{x}$ with respect to the same function η and the Lagrange multipliers associated with the objective functions f_i are positive real numbers, i.e., $\bar{\lambda} > 0$.*

Moreover, if the penalty parameter c is assumed to be sufficiently large (it is sufficient to set $c \geq \max\{e^{rf_i(\bar{x})} : i \in I\} \max\{\bar{\mu}_j : j \in J\}$), then $\bar{x}$ is an efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(c))$ involving exact l_1 exponential penalty function.

Proof Suppose contrary to the result that $\bar{x}$ is not an efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(c))$ involving exact l_1 exponential penalty function. Then there exists $x \in X$ such that

$$VP_r(x, c) \leq VP_r(\bar{x}, c).$$

$$\Rightarrow \frac{1}{r} e^{rf_i(x)} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(x)} - 1) \leq \frac{1}{r} e^{rf_i(\bar{x})} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1), \quad \forall i \in I.$$

Since $\bar{x}$ is a feasible solution to (VP), the above inequality together with (3.1) gives

$$\frac{1}{r} e^{rf_i(x)} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(x)} - 1) \leq \frac{1}{r} e^{rf_i(\bar{x})}, \quad \forall i \in I.$$

By assumption $c \geq \max\{e^{rf_i(\bar{x})} : i \in I\} \max\{\bar{\mu}_j : j \in J\}$, it follows that

$$\frac{1}{r} e^{rf_i(x)} + \frac{1}{r} e^{rf_i(\bar{x})} \sum_{j=1}^m \bar{\mu}_j (e^{rg_j^+(x)} - 1) \leq \frac{1}{r} e^{rf_i(\bar{x})}, \quad \forall i \in I.$$

Again using (3.1), the above inequality yields

$$\begin{aligned} & \frac{1}{r} e^{rf_i(x)} + \frac{1}{r} e^{rf_i(\bar{x})} \sum_{j=1}^m \bar{\mu}_j (e^{rg_j(x)} - 1) \leq \frac{1}{r} e^{rf_i(\bar{x})}, \quad \forall i \in I. \\ \Rightarrow & \frac{1}{r} (e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{rg_j(x)} - 1) \leq 0, \quad \forall i \in I. \end{aligned}$$

From the feasibility of $\bar{x}$ to problem (VP) and KKT condition (2.2), the above inequality gives

$$\frac{1}{r} (e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j \in J(\bar{x})} \bar{\mu}_j (e^{\frac{r}{\bar{\mu}_j}(\bar{\mu}_j g_j(x) - \bar{\mu}_j g_j(\bar{x}))} - 1) \leq 0, \quad \forall i \in I.$$

Thus, it follows that

$$\frac{1}{r} (e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x)-g_j(\bar{x}))} - 1) \leq 0, \quad \forall i \in I.$$

Using the hypothesis $\bar{\lambda} > 0$ and $\sum_{i=1}^k \bar{\lambda}_i = 1$, we have

$$\frac{1}{r} \bar{\lambda}_i (e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \bar{\lambda}_i \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x)-g_j(\bar{x}))} - 1) \leq 0, \quad \forall i \in I$$

and for at least one $i \in I$,

$$\frac{1}{r} \bar{\lambda}_i (e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \bar{\lambda}_i \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x)-g_j(\bar{x}))} - 1) < 0.$$

Taking summation over $i \in I$, it follows from the above inequalities that

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x)-g_j(\bar{x}))} - 1) \sum_{i=1}^k \bar{\lambda}_i < 0.$$

Again since $\sum_{i=1}^k \bar{\lambda}_i = 1$, we get

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (e^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x)-g_j(\bar{x}))} - 1) < 0.$$

The rest of the proof is similar to that of Theorem 3.1 and hence being omitted. $\square$

Now we present an example to illustrate the above result.

Example 3.1 Let $X = \mathbb{R}$ and consider the following vector optimization problem

$$\begin{aligned} \text{(VP)} \quad & \text{Minimize} \quad f(x) = (\ln(4x^2 + x + 2), \ln(3x^2 - 2x + 3)) \\ \text{s.t.} \quad & g(x) = \ln(x^2 - x + 1) \leq 0. \end{aligned}$$

We observe that the feasible region of (VP) is given by

$$D = \{x \in \mathbb{R} : 0 \leq x \leq 1\}.$$

Clearly, $\bar{x} = 0$ is a feasible solution to (VP) and the KKT conditions are satisfied at the point $\bar{x} = 0$ with the Lagrange multipliers $\bar{\lambda} = (5/7, 2/7)$ and $\bar{\mu} = 1/6$.

Let $\eta : X \times X \mapsto \mathbb{R}$ be defined by

$$\eta(x, u) = \ln(x - u + 1).$$

It is easy to verify that the functions $f(x)$ and $g(x)$ are (1,1)-invex with respect to η .

Further, the exponential penalized optimization problem with exact l_1 exponential penalty function $(VP_1(c))$ associated with (VP) is given by

$$(VP_1(c)) \text{ Min } VP_1(x, c) = (4x^2 + x + 2 + c \max\{0, x^2 - x + 1\}, \\ 3x^2 - 2x + 3 + c \max\{0, x^2 - x + 1\}).$$

For any parameter $c \geq 1/2$, it is easy to see that $\bar{x} = 0$ is an efficient solution to the exponential penalized optimization problem $(VP_1(c))$ with exact l_1 exponential penalty function.

Corollary 3.2 *Let $\bar{x}$ be an efficient solution to the vector optimization problem (VP) such that the KKT conditions (2.1)–(2.3) are satisfied at $\bar{x}$ with Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^k$, $\bar{\mu} \in \mathbb{R}^m$. Assume that the objective function f and the constraint functions $g_j, j \in J(\bar{x})$ are (p, r) -invex at $\bar{x}$ with respect to the same function η and the Lagrange multipliers associated with the objective functions f_i are positive real numbers, i.e., $\bar{\lambda} > 0$.*

Moreover, if the penalty parameter c is assumed to be sufficiently large (it is sufficient to set $c \geq \max\{e^{rf_i(\bar{x})} : i \in I\} \max\{\bar{\mu}_j : j \in J\}$), then $\bar{x}$ is an efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(c))$ involving exact l_1 exponential penalty function.

Proof The result follows directly from Theorem 3.2. $\square$

Theorem 3.3 *Let $\bar{x}$ be a feasible solution to the vector optimization problem (VP) such that the KKT conditions (2.1)–(2.3) are satisfied at $\bar{x}$ with Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^k$, $\bar{\mu} \in \mathbb{R}^m$. Assume that the objective function f is strictly (p, r) -invex at $\bar{x}$ with respect to η and the constraint functions $g_j, j \in J(\bar{x})$ are (p, r) -invex at $\bar{x}$ with respect to the same function η .*

Moreover, if the penalty parameter c is assumed to be sufficiently large (it is sufficient to set $c \geq \max\{e^{rf_i(\bar{x})} : i \in I\} \max\{\bar{\mu}_j : j \in J\}$), then $\bar{x}$ is an efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(c))$ involving exact l_1 exponential penalty function.

Proof Proceeding similarly as in the proof of Theorem 3.2, we have

$$\frac{1}{r}(e^{r(f_i(x) - f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) \leq 0, \quad \forall i \in I.$$

Since $\bar{\lambda} \geq 0$, we have

$$\frac{1}{r} \bar{\lambda}_i (\mathbf{e}^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \bar{\lambda}_i \sum_{j=1}^m \bar{\mu}_j (\mathbf{e}^{r(g_j(x)-g_j(\bar{x}))} - 1) \leq 0, \quad \forall i \in I.$$

Taking summation over $i \in I$, it follows from the above inequality that

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (\mathbf{e}^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (\mathbf{e}^{r(g_j(x)-g_j(\bar{x}))} - 1) \sum_{i=1}^k \bar{\lambda}_i \leq 0.$$

Again since $\sum_{i=1}^k \bar{\lambda}_i = 1$, we get

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (\mathbf{e}^{r(f_i(x)-f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (\mathbf{e}^{r(g_j(x)-g_j(\bar{x}))} - 1) \leq 0. \quad (3.5)$$

On the other hand, from the hypothesis, f is strictly (p, r) -invex at $\bar{x}$ w.r.t. η , the relation

$$\frac{1}{r} \mathbf{e}^{rf_i(x)} > \frac{1}{r} \mathbf{e}^{rf_i(\bar{x})} \left[1 + \frac{r}{p} (\mathbf{e}^{p\eta(x,\bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}) \right], \quad \forall i \in I$$

holds for all $x \in X$.

$$\Rightarrow \frac{1}{r} (\mathbf{e}^{r(f_i(x)-f_i(\bar{x}))} - 1) > \frac{1}{p} (\mathbf{e}^{p\eta(x,\bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}), \quad \forall i \in I.$$

As $\bar{\lambda} \geq 0$, we have

$$\frac{1}{r} \bar{\lambda}_i (\mathbf{e}^{r(f_i(x)-f_i(\bar{x}))} - 1) \geq \frac{\bar{\lambda}_i}{p} (\mathbf{e}^{p\eta(x,\bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}), \quad \forall i \in I$$

and for at least one $i \in I$, we have

$$\frac{1}{r} \bar{\lambda}_i (\mathbf{e}^{r(f_i(x)-f_i(\bar{x}))} - 1) > \frac{\bar{\lambda}_i}{p} (\mathbf{e}^{p\eta(x,\bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}).$$

Taking summation over $i \in I$, it follows from the above inequalities that

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (\mathbf{e}^{r(f_i(x)-f_i(\bar{x}))} - 1) > \frac{1}{p} \sum_{i=1}^k \bar{\lambda}_i (\mathbf{e}^{p\eta(x,\bar{x})} - \mathbf{1})^T \nabla f_i(\bar{x}). \quad (3.6)$$

Again, by the hypothesis, $g_j, j \in J(\bar{x})$ are (p, r) -invex at $\bar{x}$ w.r.t. the same function η , the relation

$$\frac{1}{r} \mathbf{e}^{rg_j(x)} \geq \frac{1}{r} \mathbf{e}^{rg_j(\bar{x})} \left[1 + \frac{r}{p} (\mathbf{e}^{p\eta(x,\bar{x})} - \mathbf{1})^T \nabla g_j(\bar{x}) \right], \quad \forall j \in J(\bar{x})$$

holds for all $x \in X$. As $\bar{\mu} \geq 0$, we have

$$\frac{1}{r} \bar{\mu}_j e^{r g_j(x)} \geq \frac{1}{r} \bar{\mu}_j e^{r g_j(\bar{x})} \left[1 + \frac{r}{p} (e^{p \eta(x, \bar{x})} - 1)^T \nabla g_j(\bar{x}) \right], \quad \forall j \in J(\bar{x}).$$

Taking into account the constraints for which the associated Lagrange multipliers are zero, the above inequality yields

$$\frac{1}{r} \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) \geq \frac{\bar{\mu}_j}{p} (e^{p \eta(x, \bar{x})} - 1)^T \nabla g_j(\bar{x}), \quad \forall j \in J.$$

Taking summation over $j \in J$, it follows that

$$\frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) \geq \frac{1}{p} \sum_{j=1}^m \bar{\mu}_j (e^{p \eta(x, \bar{x})} - 1)^T \nabla g_j(\bar{x}). \quad (3.7)$$

On adding inequalities (3.6) and (3.7), we get

$$\begin{aligned} \frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (e^{r(f_i(x) - f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) \\ > \frac{1}{p} (e^{p \eta(x, \bar{x})} - 1)^T \left[\sum_{i=1}^k \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j(\bar{x}) \right]. \end{aligned}$$

Finally, by using KKT condition (2.1), the above inequality reduces to

$$\frac{1}{r} \sum_{i=1}^k \bar{\lambda}_i (e^{r(f_i(x) - f_i(\bar{x}))} - 1) + \frac{1}{r} \sum_{j=1}^m \bar{\mu}_j (e^{r(g_j(x) - g_j(\bar{x}))} - 1) > 0,$$

which contradicts (3.5). This completes the proof. $\square$

Corollary 3.3 *Let $\bar{x}$ be an efficient solution to the vector optimization problem (VP) such that the KKT conditions (2.1)–(2.3) are satisfied at $\bar{x}$ with Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^k$, $\bar{\mu} \in \mathbb{R}^m$. Assume that the objective function f is strictly (p, r) -invex at $\bar{x}$ with respect to η and the constraint functions g_j , $j \in J(\bar{x})$ are (p, r) -invex at $\bar{x}$ with respect to the same function η .*

Moreover, if the penalty parameter c is assumed to be sufficiently large (it is sufficient to set $c \geq \max\{e^{r f_i(\bar{x})} : i \in I\} \max\{\bar{\mu}_j : j \in J\}$), then $\bar{x}$ is an efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(c))$ involving exact l_1 exponential penalty function.

Proof The result follows directly from Theorem 3.3. $\square$

Proposition 3.1 *Let $\bar{x}$ be a weakly efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(\bar{c}))$ involving exact l_1 exponential penalty function. Then,*

$$\sim \exists_{x \in D} \text{ s.t. } f(x) < f(\bar{x}),$$

i.e., there exists no $x \in D$ such that $f(x) < f(\bar{x})$.

Proof Since $\bar{x}$ is a weakly efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(\bar{c}))$ involving exact l_1 exponential penalty function, then

$$\sim \exists_{x \in X} \text{ s.t. } VP_r(x, \bar{c}) < VP_r(\bar{x}, \bar{c}),$$

i.e.,

$$\begin{aligned} \sim \exists_{x \in X} \text{ s.t. } \frac{1}{r} e^{rf_i(x)} + \bar{c} \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(x)} - 1) &< \frac{1}{r} e^{rf_i(\bar{x})} \\ &+ \bar{c} \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1), \forall i \in I. \end{aligned}$$

Since $D \subseteq X$, the above inequality implies,

$$\begin{aligned} \sim \exists_{x \in D} \text{ s.t. } \frac{1}{r} e^{rf_i(x)} + \bar{c} \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(x)} - 1) &< \frac{1}{r} e^{rf_i(\bar{x})} \\ &+ \bar{c} \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1), \forall i \in I. \end{aligned}$$

Equivalently,

$$\forall_{x \in D} \exists_{i \in I} \text{ s.t. } \frac{1}{r} e^{rf_i(x)} + \bar{c} \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(x)} - 1) \geq \frac{1}{r} e^{rf_i(\bar{x})} + \bar{c} \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1).$$

Since $x \in D$, using (3.1), we have

$$\forall_{x \in D} \exists_{i \in I} \text{ s.t. } \frac{1}{r} e^{rf_i(x)} \geq \frac{1}{r} e^{rf_i(\bar{x})} + \bar{c} \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1).$$

Again by (3.1), it follows that

$$\begin{aligned} \forall_{x \in D} \exists_{i \in I} \text{ s.t. } \frac{1}{r} e^{rf_i(x)} &\geq \frac{1}{r} e^{rf_i(\bar{x})}. \\ \Rightarrow \forall_{x \in D} \exists_{i \in I} \text{ s.t. } f_i(x) &\geq f_i(\bar{x}). \end{aligned}$$

Equivalently,

$$\sim \exists_{x \in D} \text{ s.t. } f(x) < f(\bar{x}).$$

This completes the proof. $\square$

Proposition 3.2 Let $\bar{x}$ be an efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(\bar{c}))$ involving exact l_1 exponential penalty function. Then,

$$\sim \exists_{x \in D} \text{ s.t. } f(x) \leq f(\bar{x}).$$

Proof The proof of this proposition is similar to that of Proposition 3.1 and hence being omitted. $\square$

Theorem 3.4 *Let $\bar{x}$ be a weakly efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(\bar{c}))$ involving exact l_1 exponential penalty function. Assume that*

- (i) D is a compact subset of $\mathbb{R}^n$,
- (ii) $VP_r(x, c) \geq VP_r(\bar{x}, c)$ is satisfied for all $x \in D$ and any $c > \bar{c}$.

Then $\bar{x}$ is also a weakly efficient solution to the vector optimization problem (VP).

Proof Since $\bar{x}$ is a weakly efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(\bar{c}))$ involving exact l_1 exponential penalty function, therefore, by Proposition 3.1, we have,

$$\sim \exists_{x \in D} f(x) < f(\bar{x}),$$

equivalently,

$$\forall_{x \in D} \exists_{i \in I} \text{ s.t. } f_i(x) \geq f_i(\bar{x}). \quad (3.8)$$

Remaining, we have to show that $\bar{x}$ is a feasible solution to the vector optimization problem (VP). Suppose contrary to the result $\bar{x}$ is not a feasible solution to (VP), then by (3.1), it follows that

$$\frac{1}{r}(e^{rg_j^+(\bar{x})} - 1) > 0. \quad (3.9)$$

Let $\hat{x}$ be any feasible solution to the problem (VP). By hypothesis (ii), for any $c > \bar{c}$, we have

$$\begin{aligned} & VP_r(\hat{x}, c) \geq VP_r(\bar{x}, c), \\ \Rightarrow & \frac{1}{r}e^{rf_i(\hat{x})} + c \sum_{j=1}^m \frac{1}{r}(e^{rg_j^+(\hat{x})} - 1) \geq \frac{1}{r}e^{rf_i(\bar{x})} + c \sum_{j=1}^m \frac{1}{r}(e^{rg_j^+(\bar{x})} - 1), \quad \forall i \in I. \end{aligned} \quad (3.10)$$

On the other hand, we set

$$c > \max \left\{ \frac{\frac{1}{r}(e^{rf_i(\hat{x})} - e^{rf_i(\bar{x})})}{\sum_{j=1}^m \frac{1}{r}(e^{rg_j^+(\bar{x})} - 1)}, \bar{c}; i \in I \right\}. \quad (3.11)$$

As $\hat{x} \in D$ and D is a compact subset of $\mathbb{R}^n$, it follows by inequalities (3.8), (3.9), and (3.11) that c is a finite positive real number. Hence

$$\begin{aligned} & c \sum_{j=1}^m \frac{1}{r}(e^{rg_j^+(\bar{x})} - 1) > \frac{1}{r}(e^{rf_i(\hat{x})} - e^{rf_i(\bar{x})}), \quad \forall i \in I. \\ \Rightarrow & \frac{1}{r}e^{rf_i(\bar{x})} + c \sum_{j=1}^m \frac{1}{r}(e^{rg_j^+(\bar{x})} - 1) > \frac{1}{r}e^{rf_i(\hat{x})}, \quad \forall i \in I. \end{aligned} \quad (3.12)$$

Again using the feasibility of $\hat{x}$ to vector optimization problem (VP), it follows from (3.1) that

$$c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1) = 0. \quad (3.13)$$

From (3.12) and (3.13), we conclude that

$$\frac{1}{r} e^{rf_i(\bar{x})} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1) > \frac{1}{r} e^{rf_i(\bar{x})} + c \sum_{j=1}^m \frac{1}{r} (e^{rg_j^+(\bar{x})} - 1), \forall i \in I,$$

which contradicts (3.10). Hence $\bar{x}$ is a feasible solution to vector optimization problem (VP). $\square$

Theorem 3.5 *Let $\bar{x}$ be an efficient solution to the exponential penalized multiobjective optimization problem $(VP_r(\bar{c}))$ involving exact l_1 exponential penalty function. Assume that*

- (i) D is a compact subset of $\mathbb{R}^n$,
- (ii) $VP_r(x, c) \geq VP_r(\bar{x}, c)$ is satisfied for all $x \in D$ and any $c > \bar{c}$.

Then, $\bar{x}$ is also an efficient solution to the vector optimization problem (VP).

Proof The proof of this theorem is similar to that of Theorem 3.4 and hence being omitted. $\square$

Now, we present an example to elucidate the above result.

Example 3.2 Let $X = \mathbb{R}_+$ and consider the following vector optimization problem

$$\begin{aligned} \text{(VP)} \quad & \text{Minimize} \quad f(x) = (\ln(x^3 + x + 3), \ln(5x^2 + x + 6)) \\ & \text{subject to} \quad g(x) = \ln(x^2 - x + 1/2) \leq 0. \end{aligned}$$

The feasible region of (VP) is given by

$$D = \{x \in \mathbb{R}_+ : 0 \leq x \leq 1.36\}.$$

Clearly, D is a compact subset of $\mathbb{R}$.

The associated exponential penalized optimization problem with exact l_1 exponential penalty function $(VP_1(\bar{c}))$ is given by

$$\begin{aligned} \text{(VP}_1(\bar{c})) \text{ Min } VP_1(x, \bar{c}) = & (x^3 + x + 3 + \bar{c} \max\{0, x^2 - x + 1/2\}, 5x^2 \\ & + x + 6 + \bar{c} \max\{0, x^2 - x + 1/2\}). \end{aligned}$$

In the feasible region D , we have

$$\max\{0, x^2 - x + 1/2\} = x^2 - x + 1/2.$$

Let $\bar{c} = 1/2$, then $(VP_1(\bar{c}))$ becomes

$$\begin{aligned}
 (\text{VP}_1(\bar{c})) \text{ Min } VP_1\left(x, \frac{1}{2}\right) &= \left(x^3 + x + 3 + \frac{1}{2}\left(x^2 - x + \frac{1}{2}\right), \right. \\
 &\quad \left. 5x^2 + x + 6 + \frac{1}{2}\left(x^2 - x + \frac{1}{2}\right)\right) \\
 &= \left(x^3 + \frac{x^2}{2} + \frac{x}{2} + \frac{13}{4}, \frac{11x^2}{2} + \frac{x}{2} + \frac{25}{4}\right).
 \end{aligned}$$

It is easy to see that $\bar{x} = 0$ is an efficient solution to $(\text{VP}_1(\bar{c}))$.

Further, for $c = 1$ ($> \bar{c}$), we have

$$\begin{aligned}
 VP_1(x, 1) &= \left(x^3 + x + 3 + x^2 - x + \frac{1}{2}, 5x^2 + x + 6 + x^2 - x + \frac{1}{2}\right) \\
 &= \left(x^3 + x^2 + \frac{7}{2}, 6x^2 + \frac{13}{2}\right).
 \end{aligned}$$

At $\bar{x} = 0$, we have

$$VP_1(\bar{x}, 1) = \left(\frac{7}{2}, \frac{13}{2}\right).$$

Therefore, it follows that $VP_1(x, 1) \geq VP_1(\bar{x}, 1)$ for all $x \in D$. Hence, by Theorem 3.5, $\bar{x} = 0$ is an efficient solution to (VP) .

4 Conclusion

In this paper, we have constructed a penalized exponential multiobjective optimization problem associated to vector optimization problem so that it can be solved easily by known methods of solving unconstrained optimization problems. We have derived the conditions for which the (weak) efficient solutions to the vector optimization problem are equivalent to the (weak) efficient solutions to the constructed penalized exponential multiobjective optimization problem and vice versa.

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