On Nondifferentiable Higher-Order Symmetric Duality in Multiobjective Programming Involving Cones

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Abstract In this paper, we point out some deficiencies in a recent paper (Lee and Kim in *J. Nonlinear Convex Anal.* 13:599–614, 2012), and we establish strong duality and converse duality theorems for two types of nondifferentiable higher-order symmetric duals multiobjective programming involving cones.

Keywords Multiobjective programming · Higher-order Mond–Weir symmetric dual model · Higher-order Wolfe symmetric dual model · Strong duality theorems · Converse duality theorems · Nondifferentiability

1 Introduction

Mangasarian [4] first formulated a class of higher-order dual problems for nonlinear programming problems. Later, in [5], Mond and Weir gave the conditions for duality and considered other higher-order duals. Mond and Zhang [6] obtained duality results for various higher-order dual problems under higher-order invexity assumptions. Higher-order duality in nonlinear programming has been studied in the

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last few years by many researchers. One practical advantage of higher-order duality is that it provides tighter bounds for the value of objective function of the primal problem when approximations are used because there are more parameters involved. Mishra and Rueda [8] considered higher-order duality for the nondifferentiable mathematical programming. They formulated a number of higher-order duals to a non-differentiable programming problems and established duality under the higher-order generalized invexity conditions introduced in [7]. In [9], Yang et al. extended the results in [8] to a class of nondifferentiable multiobjective programs. Chen [2] studied higher-order symmetric duality for multiobjective nondifferentiable programs by introducing higher-order F-convexity. Agarwal et al. [1] extended the results of [2] to arbitrary cones and proved appropriate duality relations under higher-order K-F-convexity assumptions.

Recently, Lee and Kim [3] have presented higher-order symmetric dual programs for multiobjective problems. In the literature strong and converse duality theorems have been established assuming conditions on known quantities. However, in strong and converse duality theorems in [3] an assumption involves the unknown Lagrange multiplier α for two types of symmetric duals. In this note we establish these results under appropriate assumptions, this fills some gaps in the work of Lee and Kim [3].

2 Notations and Preliminaries

First we consider the following multiobjective programming problem:

$$\begin{aligned} (\mathbf{P}) \quad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad -g(x) \in Q, \quad x \in C, \end{aligned}$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^l$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $C \subset \mathbb{R}^n$ and Q is a closed convex cone with nonempty interior in $\mathbb{R}^m$. We shall denote the feasible set of (P) by

$$X = \{x \mid -g(x) \in Q, x \in C\}.$$

Definition 2.1 A nonempty set K in $\mathbb{R}^n$ is said to be a cone with vertex zero, if $x \in K$ implies that $\lambda x \in K$ for all $\lambda \geq 0$. If, in addition, K is convex, then K is called a convex cone.

Definition 2.2 A feasible point $\bar{x}$ is a K -weakly efficient solution of (P), if there exists no other $x \in S$ such that $f(\bar{x}) - f(x) \in \text{int } K$.

Definition 2.3 The positive polar cone K^* of K is defined by

$$K^* = \{z \in \mathbb{R}^n \mid x^T z \geq 0 \text{ for all } x \in K\}.$$

Definition 2.4 Let $S \subseteq \mathbb{R}^n$ be open and $F: S \rightarrow \mathbb{R}$ be a differentiable function. The function $F: S \rightarrow \mathbb{R}$ is said to be higher-order invex at $u \in S$ with respect to $\eta: S \times S \rightarrow \mathbb{R}^n$ and $h: S \times \mathbb{R}^n \rightarrow \mathbb{R}$, if for all $(x, p) \in S \times \mathbb{R}^n$,

$$f(x) - f(u) \geq \eta(x, u)^T [\nabla_x f(u) + \nabla_p h(u, p)] + h(u, p) - p^T \nabla_p h(u, p).$$

Definition 2.5 Let C be a compact convex set in $\mathbb{R}^n$. The support function $s(x|C)$ of C is defined by

$$s(x|C) := \max\{x^T y : y \in C\}.$$

The support function $s(x|C)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|C) \geq s(x|C) + z^T(y - x), \quad \text{for all } x.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) := \{z \in C : z^T x = s(x|C)\}.$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z - x) \leq 0 \text{ for all } z \in S\}.$$

It is readily verified that for a compact convex set C , y is in $N_S(x)$ if and only if $s(y|C) = x^T y$, or equivalently, x is in the subdifferential of s at y .

Lemma 2.1 ([3]) *If $\bar{x}$ is a K -weakly efficient solution of (P), then there exist $\alpha \in K^*$, $\beta \in Q^*$ not both zero such that*

$$\begin{aligned} (\alpha^T \nabla f(\bar{x}) + \beta^T \nabla g(\bar{x}))(x - \bar{x}) &\geq 0, \quad \text{for all } x \in C, \\ \beta^T g(\bar{x}) &= 0. \end{aligned}$$

Equivalently, there exist $\alpha \in K^$, $\beta \in Q^*$, $\beta_1 \in C^*$, and $(\alpha, \beta, \beta_1) \neq 0$ such that*

$$\begin{aligned} \alpha^T \nabla f(\bar{x}) + \beta^T \nabla g(\bar{x}) - \beta_1 I &= 0, \\ \beta^T g(\bar{x}) &= 0, \\ \beta_1^T \bar{x} &= 0. \end{aligned}$$

3 Mond–Weir Type Higher-Order Symmetric Duality

We now establish duality theorems for the following pair of higher-order Mond–Weir type nondifferentiable multiobjective programming problems:

(MHNP)

$$\begin{aligned} \text{Minimize } P_M(x, y, \lambda, z, p) \\ = (f_1(x, y) + s(x|C_1) - y^T z_1 + h_1(x, y, p_1) \\ - p_1^T [\nabla_{p_1} h_1(x, y, p_1)], \dots, f_l(x, y) + s(x|C_l) \\ - y^T z_l + h_l(x, y, p_l) - p_l^T [\nabla_{p_l} h_l(x, y, p_l)]) \end{aligned}$$

$$\text{subject to} \quad - \left(\sum_{i=1}^l \lambda_i [\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i)] \right) \in Q_2^*, \quad (3.1)$$

$$y^T \sum_{i=1}^l \lambda_i [\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i)] \geq 0, \quad (3.2)$$

$$\begin{aligned} x &\in Q_1, & z_i &\in D_i, & i &= 1, 2, \dots, l, \\ \lambda &\in \text{int } K^*, & \lambda^T e &= 1 \end{aligned} \quad (3.3)$$

(MHND)

$$\begin{aligned} \text{Maximize} \quad & D_M(u, v, \lambda, \omega, r) \\ &= (f_1(u, v) - s(v|D_1) + u^T \omega_1 + g_1(u, v, r_1) \\ &\quad - r_1^T [\nabla_{r_1} g_1(u, v, r_1)], \dots, f_l(u, v) + s(v|D_l) - u^T \omega_l \\ &\quad + g_l(u, v, r_l) - r_l^T [\nabla_{r_l} g_l(u, v, r_l)]) \\ \text{subject to} \quad & \sum_{i=1}^l \lambda_i [\nabla_x f_i(u, v) + \omega_i + \nabla_{r_i} g_i(u, v, r_i)] \in Q_1^*, \end{aligned} \quad (3.4)$$

$$u^T \sum_{i=1}^l \lambda_i [\nabla_x f_i(u, v) + \omega_i + \nabla_{r_i} g_i(u, v, r_i)] \leq 0, \quad (3.5)$$

$$\begin{aligned} v &\in Q_2, & \omega_i &\in C_i, & i &= 1, 2, \dots, l, \\ \lambda &\in \text{int } K^*, & \lambda^T e &= 1 \end{aligned} \quad (3.6)$$

where C_i and D_i are compact convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively; Q_1 and Q_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively; $e = (1, 1, \dots, 1)^T$ is a vector in $\mathbb{R}^l$; and for each $i = 1, 2, \dots, l$, $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $h_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable functions.

Any problem, say (MHND), in which λ is fixed to be $\bar{\lambda}$, will be denoted by (MHND) $_{\bar{\lambda}}$.

Now we present a strong duality theorem for the dual pair (MHNP) and (MHND).

Theorem 3.1 (Strong Duality) *Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_l, \bar{p}_1, \bar{p}_2, \dots, \bar{p}_l)$ be a K -weakly efficient solution of (MHNP). Assume that for each $i = 1, \dots, l$, $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is thrice differentiable at $(\bar{x}, \bar{y})$, $h_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ is twice differentiable at $(\bar{x}, \bar{y}, \bar{p}_i)$, and $g_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable functions. Furthermore, suppose that the following conditions are satisfied.*

- (I) *For each $i = 1, 2, \dots, l$, $h_i(\bar{x}, \bar{y}, 0) = 0$, $g_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_{p_i} h_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_y h_i(\bar{x}, \bar{y}, 0) = 0$ and $\nabla_x h_i(\bar{x}, \bar{y}, 0) = \nabla_{r_i} g(\bar{x}, \bar{y}, 0)$;*
- (II) *For each $i = 1, 2, \dots, l$, the Hessian matrix $\nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)$ is nonsingular;*

- (III) The set of vectors $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\}_{i=1}^l$ is linearly independent;
- (IV) $\nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i) = \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i), i = 1, 2, \dots, l$;
- (V) K is a closed convex cone with $\mathbb{R}_+^l \subseteq K$.

Then, we have

- (i) $\bar{p}_i = 0, i = 1, 2, \dots, l$;
- (ii) there exists $\bar{\omega}_i \in C_i, i = 1, 2, \dots, l$, such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_l, \bar{r}_1 = \bar{r}_2 = \dots = \bar{r}_l = 0)$ is a feasible solution of (MHND).

Furthermore, if the hypotheses of Theorem 2.1 in [3] are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_l, \bar{r}_1 = \bar{r}_2 = \dots = \bar{r}_l = 0)$ is a K -weakly efficient solution to (MHND), and the two objective values are equal.

Proof Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ is a K -weakly efficient solution of (MHNP), by Lemma 2.1, then there exist $\alpha \in K^*, \beta \in Q_2, \mu \in \mathbb{R}_+, \delta \in Q_1^*$ and $\rho \in K$ such that

$$\begin{aligned} & \sum_{i=1}^l \alpha_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i)] + \sum_{i=1}^l \bar{\lambda}_i (\beta - \mu \bar{y})^T \nabla_{yx} f_i(\bar{x}, \bar{y}) \\ & + \sum_{i=1}^l (\bar{\lambda}_i \beta - \bar{\lambda}_i \mu \bar{y} - \alpha_i \bar{p}_i)^T \nabla_{p_i x} h_i(\bar{x}, \bar{y}, \bar{p}_i) - \delta = 0, \end{aligned} \quad (3.7)$$

$$\begin{aligned} & \sum_{i=1}^l \alpha_i [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i)] \\ & - \mu \sum_{i=1}^l \bar{\lambda}_i [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] \\ & + \sum_{i=1}^l (\bar{\lambda}_i \beta - \bar{\lambda}_i \mu \bar{y} - \alpha_i \bar{p}_i)^T \nabla_{p_i y} h_i(\bar{x}, \bar{y}, \bar{p}_i) \\ & + \sum_{i=1}^l \bar{\lambda}_i (\beta - \mu \bar{y})^T \nabla_{yy} f_i(\bar{x}, \bar{y}) = 0, \end{aligned} \quad (3.8)$$

$$(\beta - \mu \bar{y})^T [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] - \rho_i = 0, \quad i = 1, \dots, l, \quad (3.9)$$

$$(\bar{\lambda}_i \beta - \bar{\lambda}_i \mu \bar{y} - \alpha_i \bar{p}_i)^T \nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) = 0, \quad i = 1, \dots, l, \quad (3.10)$$

$$\beta^T \sum_{i=1}^l \bar{\lambda}_i [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0, \quad (3.11)$$

$$\mu \bar{y}^T \sum_{i=1}^l \bar{\lambda}_i [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0, \quad (3.12)$$

$$\bar{x}^T \delta = 0, \quad (3.13)$$

$$\bar{\lambda}^T \rho = 0, \quad (3.14)$$

$$\alpha_i \bar{y} + \bar{\lambda}_i \beta - \bar{\lambda}_i \mu \bar{y} \in N_{D_i}(\bar{z}_i), \quad i = 1, \dots, l, \quad (3.15)$$

$$\bar{x}^T w_i = s(\bar{x} | C_i), \quad w_i \in C_i, \quad i = 1, \dots, l, \quad (3.16)$$

$$(\alpha, \beta, \mu, \delta, \rho) \neq 0. \quad (3.17)$$

By the assumption (II), (3.10) gives

$$\bar{\lambda}_i \beta - \bar{\lambda}_i \mu \bar{y} - \alpha_i \bar{p}_i = 0, \quad i = 1, \dots, l. \quad (3.18)$$

Now Let $\alpha_i = 0$ for some $i \in \{1, 2, \dots, l\}$. Since $R_+^k \subseteq K$ implies $K^* \subseteq K_+^k$. Therefore $\bar{\lambda} > 0$. Then relation (3.18) yields $\beta = \mu \bar{y}$. From (3.8), the assumptions (IV) and (III), we obtain $\alpha_i = \mu \bar{\lambda}_i, i = 1, 2, \dots, l$. That is, $\mu = 0, \beta = 0$ and $\alpha_i = 0, i = 1, 2, \dots, l$. From (3.7) and (3.9), we have $\delta = 0$ and $\rho = 0$, respectively. These contradict (3.17). Thus, $\alpha_i \neq 0$, for all i . Since $\alpha \in K^*$ and $K^* \subseteq \mathbb{R}_+^k$, therefore,

$$\alpha_i > 0, \quad i = 1, 2, \dots, l.$$

From $\bar{\lambda} > 0$, (3.14) implies that $\rho = 0$. (3.9) and the assumption (III), we have $\beta = \mu \bar{y}$. From (3.18) and $\alpha > 0$, we know that $\bar{p}_i = 0, i = 1, 2, \dots, l$.

By (3.18) and the assumption (I), (3.7) and (3.8) give

$$\sum_{i=1}^l \alpha_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i)] - \delta = 0, \quad (3.19)$$

and

$$\sum_{i=1}^l (\alpha_i - \mu \bar{\lambda}_i) [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0. \quad (3.20)$$

From the assumption (III), (3.20) implies $\alpha_i = \mu \bar{\lambda}_i, i = 1, \dots, l$. Thus, $\mu > 0$, (3.19) holds, and we have

$$\sum_{i=1}^l \mu \bar{\lambda}_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i)] - \delta = 0$$

and from the assumption (I), it becomes

$$\sum_{i=1}^l \mu \bar{\lambda}_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{p}_i)] = \delta \in Q_1^*. \quad (3.21)$$

Also, since $\mu > 0$, it follows that

$$\sum_{i=1}^l \bar{\lambda}_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{p}_i)] \in Q_1^*.$$

And from (3.18), $\bar{y} = \frac{\bar{p}}{\mu} \in Q_2$. Multiplying (3.21) by $\bar{x}$ and using (3.13), it gives

$$\bar{x}^T \sum_{i=1}^l \bar{\lambda}_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0.$$

Taking $\bar{w}_i = w_i$, $i = 1, \dots, l$, we find that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ is a feasible solution of (MHND). Moreover, from (3.15), we get $\bar{y} \in N_{D_i}(\bar{z}_i)$, $i = 1, \dots, l$, so that $\bar{y}^T \bar{z}_i = s(\bar{y}|D_i)$, $i = 1, \dots, l$. Consequently, using (3.16),

$$\begin{aligned} f_i(\bar{x}, \bar{y}) + s(\bar{x}|C_i) - \bar{y}^T \bar{z}_i + h_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{p}_i^T \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) \\ = f_i(\bar{x}, \bar{y}) - s(\bar{y}|D_i) + \bar{x}^T \bar{w}_i \\ = f_i(\bar{x}, \bar{y}) - s(\bar{y}|D_i) + \bar{x}^T \bar{w}_i + g_i(\bar{x}, \bar{y}, \bar{r}_i) - \bar{r}_i^T \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i), \quad i = 1, \dots, l. \end{aligned}$$

Therefore, the objective values of (MHNP) and (MHND) are equal. Also, by Theorem 2.1 in [3], $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ are K -weakly efficient solutions of (MHNP) and (MHND), respectively. $\square$

We now state a converse duality theorem whose proof follows on the line of Theorem 3.1.

Theorem 3.2 (Converse Duality) *Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_l, \bar{r}_1, \bar{r}_2, \dots, \bar{r}_l)$ be a K -weakly efficient solution of (MHND). Assume that for each $i = 1, \dots, l$, $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is thrice differentiable at $(\bar{u}, \bar{v})$, $h_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ is twice differentiable at $(\bar{u}, \bar{v}, \bar{r}_i)$, and $g_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable functions. Furthermore, suppose that the following conditions are satisfied.*

- (I) *For each $i = 1, 2, \dots, l$, $h_i(\bar{u}, \bar{v}, 0) = 0$, $g_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_{r_i} h_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_v h_i(\bar{u}, \bar{v}, 0) = 0$ and $\nabla_u h_i(\bar{u}, \bar{v}, 0) = \nabla_{r_i} g_i(\bar{u}, \bar{v}, 0)$;*
- (II) *For each $i = 1, 2, \dots, l$, the Hessian matrix $\nabla_{r_i r_i} h_i(\bar{u}, \bar{v}, \bar{r}_i)$ is nonsingular;*
- (III) *The set of vectors $\{\nabla_v f_i(\bar{u}, \bar{v}) - \bar{\omega}_i + \nabla_{r_i} h_i(\bar{u}, \bar{v}, \bar{r}_i)\}_{i=1}^l$ is linearly independent;*
- (IV) $\nabla_v h_i(\bar{u}, \bar{v}, \bar{r}_i) = \nabla_{r_i} h_i(\bar{u}, \bar{v}, \bar{r}_i)$, $i = 1, 2, \dots, l$;
- (V) K is a closed convex cone with $\mathbb{R}_+^l \subseteq K$.

Then, we have

- (i) $\bar{r}_i = 0$, $i = 1, 2, \dots, l$; and
- (ii) *there exists $\bar{z}_i \in C_i$, $i = 1, 2, \dots, l$, such that $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_l, \bar{p}_1 = \bar{p}_2 = \dots = \bar{p}_l = 0)$ is a feasible solution of (MHNP).*

Furthermore, if the hypotheses of Theorem 2.1 in [3] are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_l, \bar{p}_1 = \bar{p}_2 = \dots = \bar{p}_l = 0)$ is a K -weakly efficient solution to (MHNP), and the two objective values are equal.

4 Wolfe Type Higher-Order Symmetric Duality

We now establish duality theorems for the following pair of higher-order Wolfe type nondifferentiable multiobjective programming problems:

(WHNP)

Minimize $P_W(x, y, \nabla, z, p)$

$$\begin{aligned}
&= \left(f_1(x, y) + s(x|C_1) - y^T z_1 + h_1(x, y, p_1) \right. \\
&\quad - p_1^T [\nabla_{p_1} h_1(x, y, p_1)] \\
&\quad - y^T \sum_{i=1}^l \lambda_i [\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i)], \dots, \\
&\quad f_l(x, y) + s(x|C_l) - y^T z_l + h_l(x, y, p_l) - p_l^T [\nabla_{p_l} h_l(x, y, p_l)] \\
&\quad \left. - y^T \sum_{i=1}^l \lambda_i [\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i)] \right)
\end{aligned}$$

$$\text{subject to } - \left(\sum_{i=1}^l \lambda_i [\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i)] \right) \in Q_2^*, \quad (4.1)$$

$$x \in Q_1, \quad z_i \in D_i, \quad i = 1, 2, \dots, k, \quad \lambda \in K^*, \quad \lambda^T e = 1 \quad (4.2)$$

(WHND)

Maximize $D_W(u, v, \nabla, w, r)$

$$\begin{aligned}
&= \left(f_1(u, v) - s(v|D_1) + u^T \omega_1 + g_1(u, v, r_1) \right. \\
&\quad - r_1^T [\nabla_{r_1} g_1(u, v, r_1)] \\
&\quad - u^T \sum_{i=1}^l \lambda_i [\nabla_x f_i(u, v) + \omega_i + \nabla_{r_i} f_i(u, v, r_i)], \dots, \\
&\quad f_l(u, v) - s(v|D_l) + u^T \omega_l + g_l(u, v, r_l) - r_l^T [\nabla_{r_l} g_l(u, v, r_l)] \left. \right) \\
&\quad - u^T \sum_{i=1}^l \lambda_i [\nabla_x f_i(u, v) + \omega_i + \nabla_{r_i} f_i(u, v, r_i)]
\end{aligned}$$

$$\text{subject to } \sum_{i=1}^k \lambda_i [\nabla_x f_i(u, v) + \omega_i + \nabla_{r_i} g_i(u, v, r_i)] \in Q_1^*, \quad (4.3)$$

$$v \in Q_2, \quad \omega_i \in C_i, \quad i = 1, 2, \dots, k, \quad \lambda \in K^*, \quad \lambda^T e = 1 \quad (4.4)$$

where C_i and D_i are compact convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively; Q_1 and Q_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively; $e = (1, 1, \dots, 1)^T$ is a vector in $\mathbb{R}^l$; and for each $i = 1, 2, \dots, l$, $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $h_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable functions.

Any problem, say (WHND), in which λ is fixed to be $\bar{\lambda}$, will be denoted by (WHND) $_{\bar{\lambda}}$.

Theorem 4.1 (Strong Duality) *Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_l, \bar{p}_1, \bar{p}_2, \dots, \bar{p}_l)$ be a K -weakly efficient solution of (WHNP). Fix $\lambda = \bar{\lambda}$ in (WHND). Assume that for each $i = 1, \dots, l$, $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is thrice differentiable at $(\bar{x}, \bar{y})$, $h_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ is twice differentiable at $(\bar{x}, \bar{y}, \bar{p}_i)$, and $g_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable at $(\bar{x}, \bar{y}, \bar{p}_i)$. Furthermore, suppose that the following conditions are satisfied.*

- (I) $h_i(\bar{x}, \bar{y}, 0) = 0$, $g_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_{p_i} h_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_y h_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_x h_i(\bar{x}, \bar{y}, 0) = \nabla_{r_i} g(\bar{x}, \bar{y}, 0)$, $i = 1, 2, \dots, l$;
- (II) for all $i \in \{1, 2, \dots, l\}$, the Hessian matrix $\nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)$ is nonsingular;
- (III) the set of vectors $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i), i = 1, 2, \dots, l\}$ is linearly independent;
- (IV) $\nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i) = \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)$, $i = 1, 2, \dots, l$;
- (V) K is a closed convex cone with $\mathbb{R}_+^k \subseteq K$.

Then, we have

- (i) $\bar{p}_i = 0, i = 1, 2, \dots, l$;
- (ii) there exists $\bar{\omega}_i \in C_i, i = 1, 2, \dots, k$, such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_l, \bar{r}_1 = \bar{r}_2 = \dots = \bar{r}_l = 0)$ is a feasible solution for (WHND) and the corresponding values of (WHNP) and (WHND) are equal.

Furthermore, if the hypotheses of Theorem 3.1 in [3] are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_l, \bar{r}_1 = \bar{r}_2 = \dots = \bar{r}_l = 0)$ is a K -weakly efficient solution of (WHND).

Proof Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ is a K -weakly efficient solution of (WHNP), by Lemma 2.1, then there exist $\alpha \in K^*$, $\beta \in Q_2$, $\delta \in Q_1^*$, and $\rho \in K$ such that

$$\begin{aligned} & \sum_{i=1}^l \alpha_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i)] + \sum_{i=1}^l \bar{\lambda}_i (\beta - (\alpha^T e) \bar{y})^T \nabla_{yx} f_i(\bar{x}, \bar{y}) \\ & + \sum_{i=1}^l (\bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} - \alpha_i \bar{p}_i)^T \nabla_{p_i x} h_i(\bar{x}, \bar{y}, \bar{p}_i) - \delta = 0, \\ & \sum_{i=1}^l \alpha_i [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i)] \\ & - \sum_{i=1}^l \bar{\lambda}_i (\alpha^T e) [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] \end{aligned} \quad (4.5)$$

$$\begin{aligned}
& + \sum_{i=1}^l (\bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} - \alpha_i \bar{p}_i)^T \nabla_{p_i y} h_i(\bar{x}, \bar{y}, \bar{p}_i) \\
& + \sum_{i=1}^l \bar{\lambda}_i (\beta - (\alpha^T e) \bar{y})^T \nabla_{yy} f_i(\bar{x}, \bar{y}) = 0,
\end{aligned} \tag{4.6}$$

$$(\beta - (\alpha^T e) \bar{y})^T [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] - \rho_i = 0, \quad i = 1, \dots, l, \tag{4.7}$$

$$(\bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} - \alpha_i \bar{p}_i)^T \nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) = 0, \quad i = 1, \dots, l, \tag{4.8}$$

$$\beta^T \sum_{i=1}^l \bar{\lambda}_i [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0, \tag{4.9}$$

$$\bar{x}^T \delta = 0, \tag{4.10}$$

$$\bar{\lambda}^T \rho = 0, \tag{4.11}$$

$$\alpha_i \bar{y} + \bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} \in N_{D_i}(\bar{z}_i), \quad i = 1, \dots, l, \tag{4.12}$$

$$\bar{x}^T w_i = s(\bar{x} | C_i), \quad w_i \in C_i, \quad i = 1, \dots, l, \tag{4.13}$$

$$(\alpha, \beta, \delta, \rho) \neq 0. \tag{4.14}$$

By the assumption (II), (4.8) gives

$$\bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} - \alpha_i \bar{p}_i = 0, \quad i = 1, \dots, l. \tag{4.15}$$

We claim that $\alpha = (\alpha_1, \dots, \alpha_l)^T \neq 0$. Suppose that $\alpha = 0$. Then, (4.15) yields $\beta = 0$. From (4.5) and (4.7), we have $\delta = 0$ and $\rho = 0$. These contradict (4.14). Hence $\alpha \neq 0$.

If some $\alpha_i = 0$, from $\bar{\lambda}_i > 0$, $i = 1, 2, \dots, l$, we know that (4.15) implies

$$\beta = (\alpha^T e) \bar{y}. \tag{4.16}$$

By the assumption (IV) and (4.16), (4.6) gives

$$\sum_{i=1}^l (\alpha_i - \bar{\lambda}_i (\alpha^T e)) [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0. \tag{4.17}$$

From the assumption (III), (4.17) implies that

$$\alpha_i = \bar{\lambda}_i (\alpha^T e), \quad i = 1, \dots, l.$$

So, $\alpha_i = 0$, $i = 1, 2, \dots, l$. This is a contradiction with $\alpha \neq 0$. Thus, $\alpha > 0$.

From $\bar{\lambda} > 0$, (4.11) implies $\rho = 0$. From (4.7) and the assumption (III), we have $\beta = (\alpha^T e) \bar{y}$, from (4.15), $\bar{\lambda} > 0$ and $\alpha > 0$ again, we obtain $\bar{p}_i = 0$, $i = 1, 2, \dots, l$.

By (4.15) and the assumption (I), (4.5) and (4.6) give

$$\sum_{i=1}^l \alpha_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{p}_i)] - \delta = 0, \quad (4.18)$$

and

$$\sum_{i=1}^l (\alpha_i - \bar{\lambda}_i(\alpha^T e)) [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0. \quad (4.19)$$

From the assumption (III), (4.19) implies that

$$\alpha_i = \bar{\lambda}_i(\alpha^T e), \quad i = 1, \dots, l.$$

Thus, (4.18) holds

$$\sum_{i=1}^l \bar{\lambda}_i(\alpha^T e) [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{p}_i)] = \delta \in Q_1^*. \quad (4.20)$$

And, since $\alpha^T e > 0$, it follows that

$$\sum_{i=1}^l \bar{\lambda}_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{p}_i)] \in Q_1^*.$$

Also, from (4.15), $\bar{y} = \frac{\beta}{\alpha^T e} \in Q_2$. Taking $\bar{w}_i = w_i$, $i = 1, \dots, l$, we find that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ is a feasible solution of (WHND). Moreover, from (4.12), we get $\bar{y} \in N_{D_i}(\bar{z}_i)$, $i = 1, \dots, l$, so that $\bar{y}^T \bar{z}_i = s(\bar{y} | D_i)$, $i = 1, \dots, l$. Multiplying (4.20) by $\bar{x}$ and using (4.10), we have

$$\bar{x}^T \sum_{i=1}^l \bar{\lambda}_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0.$$

And from (4.9) and (4.15), we obtain

$$\bar{y}^T \sum_{i=1}^l \bar{\lambda}_i [\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0.$$

Consequently, using (4.13),

$$\begin{aligned} & f_i(\bar{x}, \bar{y}) + s(\bar{x} | C_i) - \bar{y}^T \bar{z}_i + h_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{p}_i^T \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) \\ & - \bar{y}^T \sum_{i=1}^l \bar{\lambda}_i [\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] \\ & = f_i(\bar{x}, \bar{y}) - s(\bar{y} | D_i) + \bar{x}^T \bar{w}_i \end{aligned}$$

$$\begin{aligned}
&= f_i(\bar{x}, \bar{y}) - s(\bar{y}|D_i) + \bar{x}^T \bar{w}_i + g_i(\bar{x}, \bar{y}, \bar{r}_i) - \bar{r}_i^T \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i) \\
&\quad - \bar{x}^T \sum_{i=1}^l \bar{\lambda}_i [\nabla_x f_i(\bar{x}, \bar{y}) + w_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i)], \quad i = 1, \dots, l.
\end{aligned}$$

Therefore, the objective values of (WHNP) and (WHND) are equal. Also, by Theorem 3.1 in [3] $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ are K -weakly efficient solutions of (WHNP) and (WHND), respectively. $\square$

We now state a converse duality theorem whose proof follows on the line of Theorem 4.1.

Theorem 4.2 (Converse Duality) *Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_l, \bar{r}_1, \bar{r}_2, \dots, \bar{r}_l)$ be a K -weakly efficient solution of (WHND). Assume that for each $i = 1, \dots, l$, $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is thrice differentiable at $(\bar{u}, \bar{v})$, $h_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ is twice differentiable at $(\bar{u}, \bar{v}, \bar{r}_i)$, and $g_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable functions. Furthermore, suppose that the following conditions are satisfied.*

- (I) *For each $i = 1, 2, \dots, l$, $h_i(\bar{u}, \bar{v}, 0) = 0$, $g_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_{r_i} h_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_v h_i(\bar{u}, \bar{v}, 0) = 0$ and $\nabla_u h_i(\bar{u}, \bar{v}, 0) = \nabla_{r_i} g(\bar{u}, \bar{v}, 0)$;*
- (II) *For each $i = 1, 2, \dots, l$, the Hessian matrix $\nabla_{r_i r_i} h_i(\bar{u}, \bar{v}, \bar{r}_i)$ is nonsingular;*
- (III) *The set of vectors $\{\nabla_v f_i(\bar{u}, \bar{v}) - \bar{\omega}_i + \nabla_{r_i} h_i(\bar{u}, \bar{v}, \bar{r}_i)\}_{i=1}^l$ is linearly independent;*
- (IV) *$\nabla_v h_i(\bar{u}, \bar{v}, \bar{r}_i) = \nabla_{r_i} h_i(\bar{u}, \bar{v}, \bar{r}_i), i = 1, 2, \dots, l$;*
- (V) *K is a closed convex cone with $\mathbb{R}_+^l \subseteq K$.*

Then, we have

- (i) $\bar{r}_i = 0, i = 1, 2, \dots, l$;
- (ii) *there exists $\bar{z}_i \in C_i, i = 1, 2, \dots, l$, such that $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_l, \bar{p}_1 = \bar{p}_2 = \dots = \bar{p}_l = 0)$ is a feasible solution of (WHNP).*

Furthermore, if the hypotheses of Theorem 3.1 in [3] are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_l, \bar{p}_1 = \bar{p}_2 = \dots = \bar{p}_l = 0)$ is a K -weakly efficient solution to (WHNP), and the two objective values are equal.

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