On the ℓ_1 -Norm Invariant Convex k -Sparse Decomposition of Signals

Guangwu Xu · Zhiqiang Xu

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Abstract Inspired by an interesting idea of Cai and Zhang, we formulate and prove the convex k -sparse decomposition of vectors that is invariant with respect to the ℓ_1 norm. This result fits well in discussing compressed sensing problems under the Restricted Isometry property, but we believe it also has independent interest. As an application, a simple derivation of the RIP recovery condition $\delta_k + \theta_{k,k} < 1$ is presented.

Keywords Convex k -sparse decomposition · ℓ_1 minimization · Restricted isometry property · Sparse recovery

1 Introduction

The *Restricted Isometry Property (RIP)* of Candès and Tao [7] is one of the most commonly used frameworks for sparse recovery via ℓ_1 minimization. For an $n \times p$ matrix $\Phi \in \mathbb{R}^{n \times p}$ and an integer k , $1 \leq k \leq p$, the k -restricted isometry constant δ_k is the smallest constant such that

$$\sqrt{1 - \delta_k} \|c\|_2 \leq \|\Phi c\|_2 \leq \sqrt{1 + \delta_k} \|c\|_2$$

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G. Xu
Department of EE & CS, University of Wisconsin-Milwaukee, Milwaukee, WI 53211, USA
e-mail: gxu4uwm@uwm.edu

Z. Xu (✉)
Inst. Comp. Math., Academy of Mathematics and Systems Science, Chinese Academy of Sciences,
Beijing, China
e-mail: xuzq@lsec.cc.ac.cn

for every k -sparse vector c (namely, c has at most k nonzero components). If $k + k' \leq p$, the k, k' -restricted orthogonality constant $\theta_{k,k'}$, is the smallest number that satisfies

$$|\langle \Phi c, \Phi c' \rangle| \leq \theta_{k,k'} \|c\|_2 \|c'\|_2,$$

for all k -sparse vector c and k' -sparse vector c' with disjoint supports.

It has been shown that ℓ_1 minimization can recover a sparse signal with a small or zero error under various conditions on δ_k and $\theta_{k,k'}$, such as the condition $\delta_k + \theta_{k,k} + \theta_{k,2k} < 1$ in Candès and Tao [7], and the condition $\delta_k < 0.307$ of Cai, Wang and Xu [6]. Recently, Cai and Zhang [2] established a sharp condition on δ_k for k -sparse recovery:

$$\delta_k < \frac{1}{3}.$$

In the same paper, they also proved that $\delta_{2k} < \frac{1}{2}$ is sufficient for k -sparse signal reconstruction. Cai and Zhang developed a marvelous technique in the proof of their results.

Inspired by the *division lemma* of Cai and Zhang [2], we formulate and prove the ℓ_1 -norm invariant convex k -sparse decomposition of vectors in this note. This result (Theorem 2.1) asserts that every vector is a convex combination of k -sparse vectors with invariant ℓ_1 norm. Such decomposition fits well in treating compressed sensing problems under RIP, as a tighter conversion between ℓ_1 -norm and ℓ_2 -norm is desired. We shall demonstrate this by showing how to use the decomposition to derive the sparse recovery condition $\delta_k + \theta_{k,k} < 1$ of Cai and Zhang [3] in a simple manner. However, we believe that this result is of independent interest for other applications.

After the early appearance of this note (arXiv:1305.6021, May 2013), we learned that Cai and Zhang [4] also established a similar decomposition and using it derived some good RIP conditions (e.g., $\delta_{2k} < \frac{1}{\sqrt{2}}$). Using the ℓ_1 -norm invariant convex k -sparse decomposition, under the tight frame sparsification, Baker [1] obtained the condition $\delta_{2k} < \frac{1}{\sqrt{2}}$ for the Dictionary-Restricted Isometry Property.

The paper is organized as follows. Section 2 presents the ℓ_1 -norm invariant k -sparse convex decomposition. As a consequence of this decomposition, we prove a useful result for comparing ℓ_p -norms. In Sect. 3, the k -sparse convex decomposition is used to give a simple derivation of the sparse recovery condition $\delta_k + \theta_{k,k} < 1$ of Cai and Zhang.

2 Convex k -Sparse Decomposition

In this section, we prove that every vector is a convex combination of k -sparse vectors with invariant ℓ_1 -norm. The formulation is inspired by the celebrated ideas from Cai and Zhang [2]. We also show that the ℓ_∞ -norm of the summand vectors is well behaved. More specifically, we have

Theorem 2.1 *For positive integers $k \leq n$, and positive constant C , let $\mathbf{v} \in \mathbb{R}^n$ be a vector with*

$$\|\mathbf{v}\|_1 \leq C \quad \text{and} \quad \|\mathbf{v}\|_\infty \leq \frac{C}{k}.$$

Then there are k -sparse vectors $\mathbf{w}_1, \dots, \mathbf{w}_M$ with

$$\|\mathbf{w}_t\|_1 = \|\mathbf{v}\|_1 \quad \text{and} \quad \|\mathbf{w}_t\|_\infty \leq \frac{C}{k} \quad \text{for } t = 1, \dots, M, \quad (2.1)$$

such that

$$\mathbf{v} = \sum_{t=1}^M x_t \mathbf{w}_t \quad (2.2)$$

for some nonnegative real numbers $x_1, \dots, x_M$ with $\sum_{t=1}^M x_t = 1$.

Proof If $k = n$, or $\mathbf{v}$ is already k -sparse, then there is nothing to do. Assume now $n > k$. Without loss of generality, we may consider the case that all components of $\mathbf{v}$ are positive¹ (the general case can be argued easily, as (2.2) still holds by multiplying -1 to the i th components of both sides):

$$\mathbf{v}(1) \geq \mathbf{v}(2) \geq \dots \geq \mathbf{v}(n) > 0.$$

For each $j = 1, \dots, k$, let $\eta_j := \frac{C}{k} - \mathbf{v}(j)$. Since $\sum_{j=1}^k \eta_j = C - \mathbf{v}(1) - \dots - \mathbf{v}(k) \geq \mathbf{v}(k+1) + \dots + \mathbf{v}(n)$, so we have $\eta_j \geq 0$ and $\sum_{j=1}^k \eta_j > 0$. Let

$$\lambda_i := \frac{\eta_i}{\sum_{j=1}^k \eta_j}, \quad i = 1, 2, \dots, k.$$

Then $\sum_{i=1}^k \lambda_i = 1$.

We shall construct $k+1$ vectors $\mathbf{g}_0, \dots, \mathbf{g}_k$: each has $n-1$ nonzero components and satisfies

$$\|\mathbf{g}_t\|_1 = \|\mathbf{v}\|_1 \quad \text{and} \quad \|\mathbf{g}_t\|_\infty \leq \frac{C}{k} \quad \text{for } t = 0, 1, \dots, k.$$

Furthermore, $\mathbf{v}$ is a convex combination of $\mathbf{g}_0, \dots, \mathbf{g}_k$. In the following construction, we will use $\mathbf{v}_{\{j\}}$ to denote the vector whose j th component is $\mathbf{v}(j)$ and other components are zero, and use $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to denote the canonical basis. The $k+1$ vectors $\mathbf{g}_0, \dots, \mathbf{g}_k$ are

$$\begin{aligned} \mathbf{g}_0 &= \sum_{j=1}^k (\mathbf{v}(j) + \lambda_j \mathbf{v}(n)) \mathbf{e}_j + \mathbf{v}_{\{k+1\}} + \dots + \mathbf{v}_{\{n-1\}}, \\ \mathbf{g}_t &= \sum_{\substack{j=1 \\ j \neq t}}^k (\mathbf{v}(j) + \lambda_j \mathbf{v}(n)) \mathbf{e}_j + (\mathbf{v}(t) + \lambda_t \mathbf{v}(n)) \mathbf{e}_n + \mathbf{v}_{\{k+1\}} + \dots + \mathbf{v}_{\{n-1\}}, \quad 1 \leq t \leq k. \end{aligned}$$

Let

$$y_1 := \frac{\lambda_1 \mathbf{v}(n)}{\mathbf{v}(1) + \lambda_1 \mathbf{v}(n)}, \quad y_2 := \frac{\lambda_2 \mathbf{v}(n)}{\mathbf{v}(2) + \lambda_2 \mathbf{v}(n)}, \quad \dots, \quad y_k := \frac{\lambda_k \mathbf{v}(n)}{\mathbf{v}(k) + \lambda_k \mathbf{v}(n)}.$$

¹In this note, we will treat a vector of $\mathbb{R}^n$ as a function from $\{1, \dots, n\}$ to $\mathbb{R}$.

Then we see that $y_i < \lambda_i$. So, by setting $y_0 = 1 - y_1 - \cdots - y_k$, we have $y_0 > 0$ and

$$y_0 + y_1 + \cdots + y_k = 1.$$

It is also straightforward to verify

$$\mathbf{v} = y_0 \mathbf{g}_0 + y_1 \mathbf{g}_1 + \cdots + y_k \mathbf{g}_k.$$

For example, the first component of $y_0 \mathbf{g}_0 + y_1 \mathbf{g}_1 + \cdots + y_k \mathbf{g}_k$ is

$$\begin{aligned} & y_0(\mathbf{v}(1) + \lambda_1 \mathbf{v}(n)) + y_2(\mathbf{v}(1) + \lambda_1 \mathbf{v}(n)) + \cdots + y_k(\mathbf{v}(1) + \lambda_1 \mathbf{v}(n)) \\ &= (\mathbf{v}(1) + \lambda_1 \mathbf{v}(n))(1 - y_1) = \mathbf{v}(1). \end{aligned}$$

The other requirements for $\mathbf{g}_t$ ($t = 0, 1, \dots, k$) are:

1. $\|\mathbf{g}_t\|_1 = \|\mathbf{v}\|_1$. This is certainly true.
2. $|\mathbf{g}_t(i)| \leq \frac{C}{k}$. To see this, we note that

$$\mathbf{g}_t(i) = \begin{cases} 0 & \text{if } i = t \text{ or if } t = 0, i = n; \\ \mathbf{v}(i) + \lambda_i \mathbf{v}(n) & \text{if } 1 \leq i \leq k \text{ and } i \neq t; \\ \mathbf{v}(i) & \text{if } k < i < n; \\ \mathbf{v}(t) + \lambda_t \mathbf{v}(n) & \text{if } t > 0, i = n. \end{cases}$$

For $1 \leq i \leq n$,

$$\begin{aligned} \frac{C}{k} - |\mathbf{g}_t(i)| &\geq \frac{C}{k} - (\mathbf{v}(i) + \lambda_i \mathbf{v}(n)) = \eta_i - \lambda_i \mathbf{v}(n) \\ &= \eta_i - \frac{\eta_i \mathbf{v}(n)}{\sum_{j=1}^k \eta_j} \geq \eta_i - \frac{\eta_i \mathbf{v}(n)}{\mathbf{v}(k+1) + \cdots + \mathbf{v}(n)} \geq 0. \end{aligned}$$

If $n-1 > k$, we repeat this process for each $\mathbf{g}_t$, and so on, until a k -sparse convex combination is reached. $\square$

Remark 2.2 The proof of Theorem 2.1 in fact presents a method to construct the vectors $\mathbf{w}_t$, $t = 1, \dots, M$, with $M = \binom{n}{k}$. Using this method, the time complexity to construct the M vectors is $O(k^{n-k})$. It will be interesting to design efficient algorithms to construct the vectors.

3 RIP Conditions in Compressed Sensing

In this section, we shall use the k -sparse convex decomposition to describe a short proof of the following results of Cai and Zhang [3]. Our proof follows an approach similar to that in [5, 6]. We first consider the recovery of k -sparse signal:

Theorem 3.1 Let β be a k -sparse signal and $y = \Phi\beta$ where Φ satisfies

$$\delta_k + \theta_{k,k} < 1. \quad (3.1)$$

Let

$$\hat{\beta} = \arg \min_{\gamma \in \mathbb{R}^p} \{\|\gamma\|_1 \text{ subject to } y = \Phi\gamma\}.$$

Then $\beta = \hat{\beta}$.

Proof Let $h := \hat{\beta} - \beta$. We need to show that $h = 0$. Otherwise, we assume $|h(1)| \geq |h(2)| \geq \dots \geq |h(p)| > 0$. Denote

$$T := \{1, 2, \dots, k\}, \quad S := \{k+1, k+2, \dots, p\},$$

then as in [6], the minimality of $\hat{\beta}$ yields

$$\|h_S\|_1 \leq \|h_T\|_1,$$

where $h_Q = h \mathbb{I}_Q$ and $\mathbb{I}_Q$ denotes the indicator function of the set Q (namely, $\mathbb{I}_Q(j) = 1$ if $j \in Q$ and 0 if $j \notin Q$).

From the assumption, we also have $|h_S(j)| \leq \frac{\|h_T\|_1}{k}$ for all $j \in S$, i.e.

$$\|h_S\|_\infty \leq \frac{\|h_T\|_1}{k}.$$

Therefore, by Theorem 2.1, h_S can be written as

$$h_S = \sum_{j=1}^M x_j \mathbf{w}_j \quad \text{where } x_j \geq 0 \text{ and } \sum_{j=1}^q x_j = 1,$$

where each $\mathbf{w}_j$ is k -sparse and supported on S , and $\|\mathbf{w}_j\|_1 = \|h_S\|_1$, $\|\mathbf{w}_j\|_\infty \leq \frac{\|h_T\|_1}{k}$.

As h_T and $\mathbf{w}_j$ have disjoint supports and $\|\mathbf{w}_j\|_2 \leq \sqrt{k(\frac{\|h_T\|_1}{k})^2} \leq \|h_T\|_2$, we get

$$\begin{aligned} (1 - \delta_k) \|h_T\|_2^2 &\leq \|\Phi h_T\|_2^2 = |\langle \Phi h_T, \Phi h_S \rangle| \leq \sum_{j=1}^M x_j |\langle \Phi h_T, \Phi \mathbf{w}_j \rangle| \\ &\leq \sum_{j=1}^M x_j \theta_{k,k} \|h_T\|_2 \|\mathbf{w}_j\|_2 \leq \sum_{j=1}^M x_j \theta_{k,k} \|h_T\|_2 \|h_T\|_2 \\ &= \theta_{k,k} \|h_T\|_2^2. \end{aligned}$$

We have reached a contradiction. Hence $h = 0$. $\square$

Remark 3.2 We state the proof for the k -sparse signal. In fact, one also can extend the proof to the noise case easily. In [1], Baker stated such a proof for the case where the signals are sparse in a redundant dictionary.

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