Analysis of Stationary Queue Length Distribution for Geo/T-IPH/1 Queue

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Received: 11 June 2013 / Revised: 1 August 2013 / Accepted: 27 August 2013 /
Published online: 17 September 2013
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Abstract In this paper we study a Geo/T-IPH/1 queue model, where T-IPH denotes the discrete time phase type distribution defined on a birth-and-death process with countably many states. The queue model can be described by a quasi-birth-and-death (QBD) process with countably phases. Using the operator-geometric solution method, we first give the expression of the operator and the joint stationary distribution. Then we obtain the probability generating function (PGF) for stationary queue length distribution and sojourn time distribution, respectively.

Keywords Geo/T-IPH/1 queue · QBD process · Probability generating function · Stationary queue length · Stationary sojourn time

Mathematics Subject Classification (2000) 60K25 · 90B22

1 Introduction

During the last two decades, discrete-time queue models have been well investigated and have been applied extensively to various areas such as control system, communication system and so forth; since such systems are more digital than analogue, our work in discrete modeling has become more appropriate.

This work was supported by the National Natural Science Foundation of China (No. 61174160) and Post-Doctoral Science Foundation of Central South University (No. 125011).

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It is well known that the discrete PH (phase type) distribution [12] is an absorbing time distribution of discrete-time Markov chain defined on a finite state space. The PH distributions play an important role in discrete-time stochastic models. A comprehensive review of the models, methods and results can be found in either the expository paper of Alfa [1], or the monographs of Neuts [12] or Alfa [2].

Shi et al. [14] extend the discrete PH distributions to a countable state space and call the distributions so constructed IPH distributions (discrete-time PH distribution with infinite phase), which also play an important role in modern stochastic models, for example, see Alfa [1, 2].

In this paper, we consider a special class of the IPH distributions with representation $(\alpha, \mathbf{T})$ where

$$\alpha = (1, 0, 0, \dots)$$

and

$$\mathbf{T} = \begin{bmatrix} b & a & & & \\ c & b & a & & \\ & c & b & a & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

with $a, b, c \in (0, 1)$, $a + b + c = 1$ and $c > a$ (this condition ensures that the distribution has a finite positive mean value, see proof of Lemma 2.1 for details). We call this special class the class of T-IPH distributions. So T-IPH distribution is discrete-time phase type distribution defined on a birth-and-death process with countably many states.

Now we consider the Geo/T-IPH/1 queue model, which is special case of classical Geo/G/1 queue. There were many studies on Geo/G/1 queueing model. Pioneering work on such model was given by Meisling [9], and a comprehensive review of such model, related methods and results can be found in the monographs of Takagi [16]. One of the important special case of Geo/G/1 queue is Geo/PH/1 queue, which can be analyzed by quasi-birth-and-death (QBD) process with finite phases, see Alfa [1, 2] or Breuer [4] for details. Since T-IPH distribution is more general than discrete-time PH distribution, the model we considered in this paper is different from Geo/PH/1 queue and is another important special case of Geo/G/1 queue. The new model can be analyzed by QBD process with countably many phases [10]. A QBD process is a special two-dimensional Markov process for which the transition probability matrix has a block tri-diagonal structure. The first component of the process is called the level and the second the phase.

QBD processes with finitely many phases have been studied extensively. A comprehensive discussion can be found in the monographs of Neuts [12] or Breuer [4]. But when the phase number is infinite, the computation of the exact stationary probability distribution is usually very difficult. Although the stationary distribution has an operator-geometric solution [10, 18], it is a challenging problem to determine the closed-form of the rate operator, because in this case, the rate operator is an infinite matrix, which is determined by a quadratic matrix equation. In order to avoid those difficulties, many researchers have focused their research interest for such process mainly on tail analysis or numerical solution for the corresponding stationary indices.

For example, Takahashi [17], Haque [5], Motyer [11] and Li [7] considered the sufficient conditions when the stationary distribution of queue length has the characteristic of geometric decay along certain direction. As for numerical methods, one can see in Blanc [3] for power-series algorithm, in Rao [13] or Lian [8] for dimension-reduction method, or in Shi [15] for BRI algorithm.

Our main contributions in this paper include two parts, on the one hand, for the QBD process defined by Geo/T-IPH/1 queue, although it has infinite phases number, we show that both the rate operator and the joint stationary distribution can be obtained analytically. On the other hand, based on the formula of joint stationary distribution, we further give the probability generating function (PGF) of stationary distributions for queue length and sojourn time.

The rest of the paper is organized as follows. In Sect. 2, we present the QBD model for the queue we considered. In Sect. 3, we give the concrete form of the rate operator and joint stationary distribution. Moreover, the PGF of stationary queue length and waiting time distributions are given in Sect. 4. In the end, Sect. 5 is the conclusion part, we consider some open problems which are worthy of further study.

2 Model Description

In this section we give the Geo/T-IPH/1 queue model. Now the potential customer arrivals at discrete time instants $t = n^+$, $n = 1, 2, \dots$. The inter-arrival time A is an i.i.d. sequence and follows a geometric distribution with parameter p .

The beginning and ending of potential service occur at discrete time instants $t = n^-$, $n = 1, 2, \dots$. The distribution of service time B follows a T-IPH distribution with the representation $(\alpha, \mathbf{T})$.

We assume that inter-arrival times A and service times B are mutually independent. The service order is assumed to be first in first out (FIFO). Moreover, we further suppose that a customer who arrives at time $t = n^+$ can enter the system which is empty at $t = n$, so this model is referred as the early arrival system, see Hunter [6] for details.

Let L_n denote the total number of customers in the queue at time $t = n$, according to above assumption, a customer who finishes service at $t = n^-$ does not reckon in L_n . Let J_n denote the service phase at time $t = n$. Then the two dimensional stochastic process $\{(J_n, L_n), n \geq 0\}$ is a Markov chain with the following state space:

$$E = \{0\} \cup \{(i, j) : i, j = 1, 2, \dots\}.$$

The random variable J_n is called the level of the process at time n and the random variable L_n is called the phase of the process at time n . Thus the number of possible value of the phase variable is infinite. The state transition diagram of this process is shown in Fig. 1.

Fig. 1 State transition diagram of the QBD process

If we order the states in lexicographic order, then we can get the block tri-diagonal transition probability matrix of the process as follows:

$$\mathbb{P} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & & \\ \mathbf{C}_0 & \mathbf{B}_1 & \mathbf{A} & & \\ & \mathbf{C} & \mathbf{B} & \mathbf{A} & \\ & & \mathbf{C} & \mathbf{B} & \mathbf{A} \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \quad (2.1)$$

where the sub-matrices are given as follows:

$$\mathbf{B}_0 = (\bar{p}), \quad \mathbf{A}_0 = (p, 0, 0, \dots), \quad \mathbf{C}_0 = (\bar{p}c, 0, 0, \dots)^\top,$$

$$\mathbf{B}_1 = \begin{bmatrix} pc + \bar{p}b & pb & & & \\ \bar{p}c & pc + \bar{p}b & pb & & \\ & \bar{p}c & pc + \bar{p}b & pb & \\ & & & \ddots & \ddots \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \bar{p}b & pb & & & \\ \bar{p}b & \bar{p}b & pb & & \\ & \bar{p}b & \bar{p}b & pb & \\ & & & \ddots & \ddots \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} \bar{p}a & pa & & & \\ \bar{p}a & \bar{p}a & pa & & \\ & \bar{p}a & \bar{p}a & pa & \\ & & & \ddots & \ddots \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \bar{p}c & pc & & & \\ \bar{p}c & \bar{p}c & pc & & \\ & \bar{p}c & \bar{p}c & pc & \\ & & & \ddots & \ddots \end{bmatrix}$$

with $\bar{p} \triangleq 1 - p$. Moreover, superscript $\top$ denotes the transpose of a matrix.

The Markov chain with transition probability matrix (2.1) is called the quasi-birth-and-death (QBD) process, because in block it behaves like a birth-and-death process, i.e. up and down only by a maximum of one level at a time.

We begin our analysis of the QBD process $\{(J_n, L_n), n \geq 0\}$ by finding a simple stability condition for the queueing system, which is given as follows.

Lemma 2.1 *The stability condition for Geo/T-IPH/1 queue is*

$$\rho \triangleq \frac{p}{c-a} < 1. \quad (2.2)$$

Proof Firstly, we know that the mean value of a IPH distribution with representation $(\alpha, \mathbf{T})$ is $\alpha(\mathbf{I} - \mathbf{T})^{-1}\mathbf{e}$ (see Shi [14] for details). For the T-IPH distribution we considered, it is easily to verify that the first row of matrix $(\mathbf{I} - \mathbf{T})^{-1}$ is $(\frac{1}{c}, \frac{a}{c^2}, \frac{a^2}{c^3}, \dots)$, so the mean value m of the T-IPH distribution is

$$\begin{aligned} m &= \alpha(\mathbf{I} - \mathbf{T})^{-1}\mathbf{e} \\ &= \left(\frac{1}{c}, \frac{a}{c^2}, \frac{a^2}{c^3}, \dots\right)\mathbf{e} \\ &= \frac{1}{c-a}. \end{aligned}$$

Now, by the stability condition for the classical Geo/GI/1 queue, together with the mean value for geometric distribution with parameter p is $\frac{1}{p}$, we can get the conclusion. $\square$

Throughout the paper, we assume that stability condition (2.2) holds, so the stationary distribution of the QBD process with transition probability matrix (2.1) exists. And by Miller [10], the stationary distribution $(\pi_0, \pi_1, \pi_2, \dots)$ has the operator-geometric form

$$\pi_i = \pi_1 \mathbf{R}^{i-1}, \quad i = 1, 2, \dots, \quad (2.3)$$

where

$$\pi_i = (\pi_{i1}, \pi_{i2}, \pi_{i3}, \dots), \quad i = 1, 2, \dots,$$

are infinite vectors and $\mathbf{R}$ is an infinite matrix, which is the minimal non-negative solution to the quadratic matrix equation

$$\mathbf{R} = \mathbf{A} + \mathbf{R}\mathbf{B} + \mathbf{R}^2\mathbf{C}. \quad (2.4)$$

Moreover, vector (π_0, π_1) is determined by equation

$$(\pi_0, \pi_1) \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_0 \\ \mathbf{C}_0 & \mathbf{B}_1 + \mathbf{R}\mathbf{C} \end{bmatrix} = (\pi_0, \pi_1) \quad (2.5)$$

and the normalization condition

$$\pi_0 + \sum_{k=1}^{\infty} \pi_k \mathbf{e} = 1. \quad (2.6)$$

Remark 2.1 The matrix $\mathbf{R} = (r_{ij})$ is called the rate operator, it has the following probabilistic interpretation [10, 12]. For any level $k = 1, 2, \dots$, the entry r_{ij} equals the expected number of visits to state $(k+1, j)$ before the first return to level k for a process starting in (k, i) . This interpretation will be used in sequel to identify zero entries of $\mathbf{R}$.

3 Operator-Geometric Solution

In this section, we will derive the operator-geometric solution of the QBD process $\{(J_n, L_n), n \geq 0\}$. To this aim, we first give the expression for rate operator. By the probabilistic interpretation for the rate operator in Remark 2.1, together with transition diagram given in Fig. 1, we know $\mathbf{R}$ has the following special form:

$$\mathbf{R} = \begin{bmatrix} r_0 & r_1 & r_2 & r_3 & \cdots \\ & r_0 & r_1 & r_2 & \cdots \\ & & r_0 & r_1 & \cdots \\ & & & r_0 & \cdots \\ & & & & \cdots \end{bmatrix},$$

where for any $j \geq 0$, the non-zero entry r_j is the expected number of visits to state $(k+1, i+j)$ before the first return to any level $k \geq 1$, given that the process starting in the state (k, i) , and the value is independent of k and i . Moreover, since the process is positive recurrent, each r_j has finite value, which can be determined by the following lemma.

Lemma 3.1 *Define*

$$R(z) = \sum_{k=0}^{\infty} r_k z^k, \quad |z| \leq 1$$

as the generating function of sequence $\{r_0, r_1, \dots\}$, then we have

$$R(z) = \frac{1 - b(pz + \bar{p}) - \sqrt{[1 - b(pz + \bar{p})]^2 - 4ac(pz + \bar{p})^2}}{2c(pz + \bar{p})}. \quad (3.1)$$

Proof Since $\mathbf{R}$ satisfies (2.4), by the expression of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, we can write up the component form of (2.4) as follows:

$$\begin{cases} \bar{p}a - (1 - \bar{p}b)r_0 + \bar{p}cr_0^2 = 0, \\ pa + pbr_0 - (1 - \bar{p}b)r_1 + \bar{p}c(r_0r_1 + r_1r_0) + pcr_0^2 = 0, \\ pbr_k - (1 - \bar{p}b)r_{k+1} + \bar{p}c \sum_{i=0}^{k+1} r_i r_{k-i+1} + pc \sum_{i=0}^k r_i r_{k-i} = 0, \quad k \geq 1. \end{cases} \quad (3.2)$$

Now, by routine algebraic operations, we can get

$$c(\bar{p} + pz)R^2(z) - (1 - b(\bar{p} + pz))R(z) + a(\bar{p} + pz) = 0. \quad (3.3)$$

Since $r_0, r_1, \dots$ are the minimal non-negative solutions to systems of (3.2), by the first equation in (3.2), we have

$$r_0 = r \triangleq \frac{1 - \bar{p}b - \sqrt{(1 - \bar{p}b)^2 - 4ac\bar{p}^2}}{2\bar{p}c} \in (0, 1),$$

so we can get (3.1) by solving quadratic equation (3.3) evaluated in $R(z)$. $\square$

Remark 3.1 By (3.1) and a simple calculation, we can easily get

$$\lim_{z \rightarrow 1} R(z) = \frac{a}{c}, \quad (3.4)$$

so we have

$$r_0 + r_1 + \dots = \frac{a}{c} < 1.$$

Moreover, (3.2) enables us to calculate r_i recursively. In fact, by (3.2), we can easily get

$$\begin{cases} r_0 = r, \\ r_1 = \frac{p}{d}(cr^2 + br + a), \\ r_{k+1} = \frac{1}{d}(pbr_k + \bar{p}c \sum_{i=1}^k r_i r_{k+1-i} + pc \sum_{i=0}^k r_i r_{k-i}), \quad k \geq 1, \end{cases}$$

where $d \triangleq \sqrt{(1 - \bar{p}b)^2 - 4ac\bar{p}^2}$. However, we will not use this relationship in the sequel.

Since we have obtained the expression of $\mathbf{R}$, to use the operator-geometric solution given in (2.3), we need the expression of π_1 , which is given as follows.

Lemma 3.2 *Define*

$$G_1(z) = \sum_{k=1}^{\infty} \pi_{1k} z^{k-1}, \quad |z| \leq 1$$

as the generating function for π_1 . Then we have

$$G_1(z) = \frac{\pi_{11}\bar{p}c(1-z)}{pbz^2 - [1 - pc - \bar{p}b - c(pz + \bar{p})R(z)]z + \bar{p}c}, \quad (3.5)$$

where $R(z)$ is given by (3.1).

Proof By the concrete expression of blocks in matrix $\mathbb{P}$ and $\mathbf{R}$, linear equation (2.5) can be rewritten in the following components form:

$$\begin{cases} 0 = -p\pi_0 + \bar{p}c\pi_{11}, \\ 0 = p\pi_0 - [1 - (pc + \bar{p}b + \gamma_0)]\pi_{11} + \bar{p}c\pi_{12}, \\ 0 = pb\pi_{1,k-2} - (1 - pc - \bar{p}b)\pi_{1,k-1} + \bar{p}c\pi_{1k} + \sum_{i=0}^{k-2} \gamma_i \pi_{1,k-i-1}, \quad k \geq 3 \end{cases}$$

where

$$\gamma_0 = \bar{p}cr_0, \quad \gamma_k = \bar{p}cr_k + pcr_{k-1}, \quad k \geq 1.$$

Based on the above equations and routine calculation, we can easily get

$$G_1(z) = \frac{\pi_{11}\bar{p}c(1-z)}{pbz^2 - [1 - pc - \bar{p}b - \Gamma(z)]z + \bar{p}c},$$

where

$$\Gamma(z) = \sum_{k=0}^{\infty} \gamma_k z^k = c(pz + \bar{p})R(z).$$

So we can get (3.5). □

Furthermore, define

$$G_n(z) = \sum_{k=1}^{\infty} \pi_{nk} z^{k-1}, \quad |z| \leq 1, \quad n = 2, 3, \dots$$

as the generating function of vector π_n , then we have the following lemma.

Lemma 3.3 *We have*

$$G_n(z) = G_1(z)R^{n-1}(z), \quad n = 1, 2, \dots, \quad (3.6)$$

which is the functional analogue of operator-geometric solution (2.3).

Proof By (2.3) and the special form of $\mathbf{R}$, we have $\pi_{2i} = \sum_{k=0}^i \pi_{1i} r_{k-i}$, so we can further get $G_2(z) = G_1(z)R(z)$ by routine method. In the end, using induction method, we can get (3.6). □

4 Stationary Queue Length

Now, let L be the numbers of customers in steady state, then by Lemmas 3.2 and 3.3 given above, we can obtain the following important theorem.

Theorem 4.1 *The probability generating function of L is given by*

$$L(z) = (1 - \rho) \left[1 + \frac{pz}{\bar{p}c - p(a+b)z - c(pz + \bar{p})R(z)} \right]. \quad (4.1)$$

Proof By definition of $L(z)$ and (3.6), we have

$$\begin{aligned} L(z) &= \sum_{k=0}^{\infty} P(L=k)z^k = \pi_0 + \sum_{k=1}^{\infty} z^k \sum_{n=1}^{\infty} \pi_{nk} \\ &= \pi_0 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \pi_{nk} z^k = \pi_0 + z \sum_{n=1}^{\infty} G_n(z) \\ &= \pi_0 + zG_1(z) \sum_{n=1}^{\infty} R^{n-1}(z) \\ &= \pi_0 + \frac{zG_1(z)}{1-R(z)}. \end{aligned}$$

So, substituting (3.1) and (3.5) into the last equation, by the fact that $R(z)$ satisfies (3.3) and simple calculation, we further have

$$L(z) = \pi_0 \left[1 + \frac{pz}{\bar{p}c - p(a+b)z - c(pz + \bar{p})R(z)} \right].$$

Finally, by normalization condition (2.6) and (3.4), we have

$$1 = \lim_{z \uparrow 1} L(z) = \pi_0 \left[1 + \frac{p}{\bar{p}c - p(a+b) - a} \right] = \pi_0 \frac{c-a}{c-a-p},$$

so $\pi_0 = 1 - \rho$, this yields the conclusion of the theorem. $\square$

Denote the sojourn time of a customer in steady state and its PGF by S and $S(z)$, respectively. Between S and L has the following classical relationship (see Alfa [1] or Hunter [6] for details):

$$L(z) = S(pz + \bar{p}).$$

So by above relationship, (4.1) and corresponding properties of PGF, we can get the following two corollaries.

Corollary 4.1 *The probability generating function for stationary sojourn time S is given by*

$$S(z) = \frac{(1-\rho)[cz - T(z)]}{\bar{p} - (a+b)z - T(z)}, \quad (4.2)$$

where

$$T(z) = \frac{1}{2} \left(1 - bz - \sqrt{(1-bz)^2 - 4acz^2} \right).$$

Corollary 4.2 *The mean value for stationary queue length and sojourn time are given by*

$$E(L) = \frac{p\bar{p} + a\rho^2}{(1-\rho)(c-a)}, \quad E(S) = \frac{E(L)}{p}, \quad (4.3)$$

respectively.

5 Conclusion

In this paper, we analyze the discrete-time Geo/T-IPH/1 queue by QBD process and operator-geometric method. For such a queue model, we get the closed-form of the rate operator. Based on the rate operator, we further get PGF for stationary queue length and waiting time distribution.

However, the busy period distribution is another important performance index, so how to analyze the busy period based on the results we get is a problem worthy of further study.

Acknowledgements The authors are grateful to the three anonymous referees and the editor for their careful reading and invaluable comments and suggestions, which are helpful to improve the paper. This research completed when the first author was a post-doctoral fellow in School of Mathematics, Central South University.

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