A Large-Update Feasible Interior-Point Algorithm for Convex Quadratic Semi-definite Optimization Based on a New Kernel Function

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Abstract In this paper we present a large-update primal-dual interior-point algorithm for convex quadratic semi-definite optimization problems based on a new parametric kernel function. The goal of this paper is to investigate such a kernel function and show that the algorithm has the best complexity bound. The complexity bound is shown to be $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$.

Keywords Kernel function · Interior-point algorithm · Polynomial complexity · Large-update methods

Mathematics Subject Classification (2010) 90C22 · 90C25 · 90C33

1 Introduction

In this paper, we consider the standard form of the convex quadratic semi-definite optimization (CQSDO) problem,

$$\begin{aligned} \min \quad & C \bullet X + \frac{1}{2} X \bullet Q(X) \\ \text{s.t.} \quad & A_i \bullet X = b_i, \quad i = 1, 2, \dots, m \\ & X \succeq 0, \end{aligned} \tag{P}$$

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and its associated dual problem,

$$\begin{aligned} \max \quad & b^T y - \frac{1}{2} X \bullet Q(X) \\ \text{s.t.} \quad & \sum_{i=1}^m y_i A_i - Q(X) + S = C \\ & S \succeq 0, \end{aligned} \tag{D}$$

where each $A_i \in \mathbf{S}^n$ and they are linearly independent, $b \in \mathbb{R}^m$, $C \in \mathbf{S}^n$ and $Q(X) : \mathbf{S}^n \rightarrow \mathbf{S}^n$ is a given self-adjoint positive semi-definite linear operation on $\mathbf{S}^n$, i.e., for any $A, B \in \mathbf{S}^n$, then $Q(A) \bullet B = A \bullet Q(B)$ and $Q(A) \bullet A \geq 0$. Here, we use $\mathbf{S}^n$ to denote the set of all symmetric $n \times n$ matrices. By $G \bullet H$ we denote the trace of $G^T H$.

The CQSDO problem was first appeared in the researches of Kojima et al. [11]. This problem is a generalization of the SDO and convex quadratic optimization (CQO) problem. Some efficient interior-point algorithms have been proposed to CQSDO. For example, Nie et al. [14] presented an interior-point algorithm for CQSDO based on potential reduction approach and obtained an iteration bound $\mathcal{O}(n \log \frac{n}{\epsilon})$ in the worst case. They [15] also given a predictor-corrector interior-point algorithm for CQSDO using both Dikin-type and Newton centering steps, whose algorithm has an iteration bound $\mathcal{O}(\sqrt{n} \log \frac{n}{\epsilon})$. Toh proposed an inexact primal-dual path-following algorithm for CQSDO in [19]. Bai et al. [8] proposed a different technique for finding a class of search directions. Based on this technique, the authors designed a new primal-dual interior-point algorithm for CQSDO with small-update method, and obtained complexity bound.

Among the variants of interior-point methods (IPMs), the so-called primal-dual path-following methods are most efficient from the computational point of view. There are two types of path-following IPMs. The so-called large-update methods have superior practical performance but relatively weak theoretical results. While the so-called small-update methods enjoy the best known iteration bound but are inefficient at practice. Recently, Peng et al. [16, 17] replaced the primal-dual logarithmic barrier by a so-called self-regular barrier function, which is determined by a simple univariate self-regular function, called its kernel function. The search direction was modified accordingly, and a large-update method was obtained for which the theoretical iteration bound is $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$. Bai et al. [2–6, 20] proposed new primal-dual IPMs based on eligible kernel functions. They also proposed a scheme for analyzing the algorithms based on four conditions on the kernel function [2] and obtained an $\mathcal{O}(\sqrt{n} \log \frac{n}{\epsilon})$ iteration bound for small-update IPMs in all cases and an $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$ iteration bound for large-update methods with a specific kernel function [6]. Some new analytic tools were developed for the analysis of IPMs based on such kernel functions. As a result the analysis is much simpler than that in [16, 17], whereas the iteration bounds are at least as good. Recently, Kheirfam in [10] introduced a trigonometric kernel function and analyzed the feasible primal-dual IPMs for SDO based on this kernel function. He showed that the primal-dual IPMs for solving SDO enjoys $\mathcal{O}(n^{\frac{3}{4}} \log \frac{n}{\epsilon})$ as the worst case iteration complexity. Amini and

Haseli [1] presented an interior-point algorithm for LO based on a new kernel function and obtained the best known iteration bound for large-update methods. Recently, Wang and Zhu [21] proposed a unified kernel function approach to primal-dual IPMs for CQSDO based on kernel functions which were previously introduced for LO in [2]. Later on, Zhang [23] presented a large-update interior-point algorithm for CQSDO based on a new kernel function and obtained its iteration complexity as $\mathcal{O}(\sqrt{n}(\log n)^2 \log \frac{n}{\epsilon})$ for the CQSDO problems.

Motivated by the complexity results based on kernel functions, we present a large-update primal-dual interior-point algorithm for CQSDO based on a new parametric kernel function

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{t}{q} \left(e^{q(\frac{1}{t} - 1)} - 1 \right), \quad q \geq 2, t > 0, \quad (1.1)$$

where q is a parameter. Our goal is to deal with the primal-dual interior-point algorithm for CQSDO based on the kernel function $\psi(t)$ as defined by (1.1). Finally, we derive the currently best known iteration bound for the algorithm with large-update method, namely, $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$.

The paper is organized as follows: In Sect. 2 we recall some well-known results on matrices and matrix functions. In Sect. 3 we study the central path and the new search directions for CQSDO. The generic primal-dual interior-point algorithm for CQSDO is also presented. In Sect. 4 we develop some properties of the kernel function $\psi(t)$ and the corresponding barrier function $\Psi(V)$. In Sect. 5 we analyze the algorithm and derive the currently best known iteration bound with large-update method. We present a numerical example in Sect. 6. Finally, some conclusions and remarks follow in Sect. 7.

2 Preliminaries

Here, we outline some needed main results on matrices and matrix functions. For a comprehensive study, the reader is referred to [9].

Theorem 2.1 (Spectral Theorem for Symmetric Matrices) *The real $n \times n$ matrix A is symmetric if and only if there exists a matrix $Q \in \mathbb{R}^{n \times n}$ such that $Q^T Q = I$ and $Q^T A Q = \Lambda$ where Λ is a diagonal matrix.*

Due to Theorem 2.1, let $Q_V \in \mathbb{R}^{n \times n}$ be an orthogonal matrix that diagonalizes $V \in \mathbf{S}_{++}^n$, i.e., $V = Q_V^T \text{diag}(\lambda_1(V), \dots, \lambda_n(V)) Q_V$ and $Q_V^T Q_V = I$, where I denotes the identity matrix. Then, for the real function $\psi(t), t \in \mathbb{R}_+$, the matrix function $\psi : \mathbf{S}_{++}^n \rightarrow \mathbf{S}_{++}^n$ and the real valued matrix function $\Psi : \mathbf{S}_{++}^n \rightarrow \mathbb{R}_+$ are defined by

$$\psi(V) = Q_V^T \text{diag}(\psi(\lambda_1(V)), \dots, \psi(\lambda_n(V))) Q_V, \quad (2.1)$$

$$\Psi(V) := \text{Tr}(\psi(V)) = \sum_{i=1}^n \psi(\lambda_i(V)), \quad (2.2)$$

where $\mathbf{S}_{++}^n$ denotes the set of all symmetric positive definite $n \times n$ matrices and Tr stands for the trace operator.

Since the derivatives $\psi'(t)$ and $\psi''(t)$ are well defined, we can obtain the matrix functions $\psi'(V)$ and $\psi''(V)$ if $\psi(\lambda_i(V))$ in (2.2) is replaced by $\psi'(\lambda_i(V))$ and $\psi''(\lambda_i(V))$ for each i , respectively.

Definition 2.1 A matrix $M(t)$ is said to be a matrix of function (or a matrix-valued function) if each entry of $M(t)$ is a function of t , i.e., $M(t) = [M_{ij}(t)]$.

The usual concepts of continuity, differentiability, and integrability can be naturally extended to matrix-valued functions, by interpreting them component-wise. Let $M(t)$ and $N(t)$ be two matrices of functions. Then, we have

$$\frac{d}{dt}M(t) = M'(t), \quad (2.3)$$

$$\frac{d}{dt}\text{Tr}(M(t)) = \text{Tr}(M'(t)), \quad (2.4)$$

$$\frac{d}{dt}\text{Tr}(\psi(M(t))) = \text{Tr}(\psi'(M'(t))M'(t)), \quad (2.5)$$

$$\frac{d}{dt}(M(t)N(t)) = M'(t)N(t) + M(t)N'(t). \quad (2.6)$$

3 IPMs for CQSDO

We assume that both (P) and (D) satisfy the interior-point condition (IPC), i.e., there exist $X^0, S^0 \succ 0$ such that

$$A_i \bullet X^0 = b_i, \quad i = 1, 2, \dots, m, \quad \sum_{i=1}^m y_i^0 A_i - \mathcal{Q}(X^0) + S^0 = C.$$

If the IPC holds, then the optimality conditions for the problems (P) and (D) are (Theorem 4 in [15]):

$$\begin{aligned} A_i \bullet X &= b_i, \quad i = 1, 2, \dots, m, \quad X \succeq 0, \\ \sum_{i=1}^m y_i A_i - \mathcal{Q}(X) + S &= C, \quad S \succeq 0, \\ XS &= 0. \end{aligned} \quad (3.1)$$

The basic idea of primal-dual IPMs is to replace the third equation in (3.1), the so-called complementarity condition for (P) and (D), by the parameterized equation

$XS = \mu I$, with $\mu > 0$. Thus, one may consider

$$\begin{aligned} A_i \bullet X &= b_i, \quad i = 1, 2, \dots, m, \quad X \succeq 0, \\ \sum_{i=1}^m y_i A_i - Q(X) + S &= C, \quad S \succeq 0, \\ XS &= \mu I. \end{aligned} \quad (3.2)$$

Since the matrices A_i are linearly independent and the IPC holds, the parameterized system (3.2) has a unique solution for each $\mu > 0$, denoted by $(X(\mu), y(\mu), S(\mu))$. We call $X(\mu)$ the μ -center of (P) and $(y(\mu), S(\mu))$ the μ -center of (D). The set of μ -centers (with μ -running through positive real numbers) gives a homotopy path which is called the central path of (P) and (D). If $\mu \rightarrow 0$, then the limit of the central path exists and since the limit point satisfies the complementarity condition $XS = 0$, the limit yields an optimal solution for (P) and (D).

At a given feasible iterate (X, y, S) with $X \succ 0$ and $S \succ 0$, the search direction $(\Delta X, \Delta y, \Delta S)$ at the current iteration, to follow the Newton approach and to linearize the third equation in (3.2), is the unique solution of the following system:

$$\begin{aligned} A_i \bullet \Delta X &= 0, \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m \Delta y_i A_i - Q(\Delta X) + \Delta S &= 0, \\ X\Delta S + \Delta XS &= \mu I - XS. \end{aligned} \quad (3.3)$$

The system (3.3) can be rewritten as

$$\begin{aligned} A_i \bullet \Delta X &= 0, \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m \Delta y_i A_i - Q(\Delta X) + \Delta S &= 0, \\ \Delta X + X\Delta SS^{-1} &= \mu S^{-1} - X. \end{aligned} \quad (3.4)$$

It is clear that ΔS is symmetric due to the second equation in (3.4). However, a crucial observation is that ΔX is not necessarily symmetric because $X\Delta SS^{-1}$ may be not symmetric. Many researchers have proposed methods for symmetrizing the third equation in the Newton system (3.4) such that the resulting new system has a unique symmetric solution. In this paper, we use the Nesterov–Todd symmetrization scheme [12, 13] which leads us to the so-called NT-direction. Let

$$P := X^{\frac{1}{2}} (X^{\frac{1}{2}} S X^{\frac{1}{2}})^{-\frac{1}{2}} X^{\frac{1}{2}} \left[S^{-\frac{1}{2}} (S^{\frac{1}{2}} X S^{\frac{1}{2}})^{\frac{1}{2}} S^{-\frac{1}{2}} \right], \quad (3.5)$$

and $D = P^{\frac{1}{2}}$, where for any symmetric positive definite matrix G , the exponent $G^{\frac{1}{2}}$ denotes its symmetric square root. The matrix D can be used to scale X and S to the

same matrix V as follows:

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} \left[= \frac{1}{\sqrt{\mu}} D S D \right]. \quad (3.6)$$

It follows that

$$V^2 = \frac{1}{\mu} D^{-1} X S D. \quad (3.7)$$

Note that the matrices D and V are symmetric and positive definite. In NT-symmetrization scheme, the term $X \Delta S S^{-1}$ in the third equation of (3.4) is replaced by $P \Delta S P^T$. Thus the system (3.4) becomes

$$\begin{aligned} A_i \bullet \Delta X &= 0, \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m \Delta y_i A_i - \mathcal{Q}(\Delta X) + \Delta S &= 0, \end{aligned} \quad (3.8)$$

$$\Delta X + P \Delta S P^T = \mu S^{-1} - X.$$

Obviously, now ΔX is a symmetric matrix and system (3.8) still has a unique solution (see [22]).

Let us further define

$$D_X := \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}; \quad D_S := \frac{1}{\sqrt{\mu}} D \Delta S D. \quad (3.9)$$

Then it follows from (3.8) that the scaled NT-search directions $(D_X, \Delta y, D_S)$ are obtained from the system

$$\begin{aligned} \bar{A}_i \bullet D_X &= 0, \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m \Delta y_i \bar{A}_i - \bar{\mathcal{Q}}(D_X) + D_S &= 0, \end{aligned} \quad (3.10)$$

$$D_X + D_S = V^{-1} - V,$$

where

$$\bar{A}_i := \frac{1}{\sqrt{\mu}} D A_i D, \quad \bar{\mathcal{Q}}(D_X) := D \mathcal{Q}(D D_X D) D.$$

So far, we have described the scheme that defines the classical NT-direction. Note that the right-hand side of the third equation in (3.10) is the negative gradient of the matrix barrier function $\Psi_c(V)$ with the classical kernel function $\psi_c(t) = \frac{t^2-1}{2} - \log(t)$, while $\psi_c(t)$ satisfies

$$\begin{aligned} \psi'_c(1) &= \psi_c(1) = 0, \\ \psi''_c(t) &> 0, \quad t > 0, \\ \lim_{t \rightarrow 0^+} \psi_c(t) &= \lim_{t \rightarrow +\infty} \psi_c(t) = +\infty. \end{aligned} \quad (3.11)$$

We replace the right-hand-side of the third equation in (3.10) by $-\nabla\Psi(V)$, where $-\nabla\Psi(V)$ is the negative gradient of the matrix barrier function $\Psi(V)$ with the kernel function (1.1). Thus this system can be rewritten as

$$\begin{aligned} \bar{A}_i \bullet D_X &= 0, \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m \Delta y_i \bar{A}_i - \bar{Q}(D_X) + D_S &= 0, \end{aligned} \quad (3.12)$$

$$D_X + D_S = -\nabla\Psi(V).$$

The new search direction $(D_X, \Delta y, D_S)$ is obtained by solving (3.12) so that $(\Delta X, \Delta y, \Delta S)$ is computed via (3.9). By taking a step along the search directions with the step size α defined by some line searches, we can construct a new iterate (X_+, y_+, S_+) according to

$$X_+ = X + \alpha \Delta X, \quad y_+ = y + \alpha \Delta y, \quad S_+ = S + \alpha \Delta S. \quad (3.13)$$

Since $\mathcal{Q}(\cdot)$ is self-adjoint positive semi-definite linear operation, we have

$$D_X \bullet D_S = \left(\frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1} \right) \bullet \left(\frac{1}{\sqrt{\mu}} D \Delta S D \right) = \frac{1}{\mu} \Delta X \bullet \mathcal{Q}(\Delta X) \geq 0.$$

Note that it is one of the difference between the CQSDO problem and the SDO problem in the analysis of our algorithm.

Algorithm 3.1 Primal-Dual Algorithm for CQSDO

Input : Accuracy parameter $\epsilon > 0$;
 barrier update parameter θ , $0 < \theta < 1$;
 threshold parameter $\tau \geq 1$;
 $X^0 > 0$, $S^0 > 0$, and $\mu^0 = 1$ such that $\Psi(X^0, S^0, \mu^0) \leq \tau$

begin :
 $X := X^0, S := S^0, \mu := \mu^0$;
while $n\mu \geq \epsilon$ **do**
begin
 μ – update :
 $\mu := (1 - \theta)\mu$;
while $\Psi(X, S, \mu) > \tau$ **do**
begin
 Solve the system (3.12) and use (3.9) for $\Delta X, \Delta y, \Delta S$;
 Determine a step size α ;
 $X := X + \alpha \Delta X$;
 $y := y + \alpha \Delta y$;
 $S := S + \alpha \Delta S$;
end
end
end

Suppose that the current iterate (X, y, S) is in τ -neighborhood of the corresponding μ -center; i.e., $\Psi(V) \leq \tau$. Then, μ is reduced by the factor $1 - \theta$ with $0 < \theta < 1$, which changes the value of V according to (3.6) and defines a new μ -center $(X(\mu), y(\mu), S(\mu))$. This is the cause that $\Psi(V) \geq \tau$. Now we start the inner iteration by calculating new iterate (3.13) where ΔX and ΔS are calculated from (3.12) and (3.9) and the step size α is chosen appropriately with the goal of reducing the value of barrier function $\Psi(V)$. If necessary, the procedure is repeated until we find the iterate that again belongs to the τ -neighborhood of the current μ -center, that is, until $\Psi(V) \leq \tau$. At this point we start a new outer iteration by reducing the value of μ again. This process is repeated until μ is small enough, say until $n\mu \leq \epsilon$ for a certain accuracy parameter ϵ , at this stage we have found an ϵ -approximate solution of (P) and (D). The generic IPM outlined above is summarized in Algorithm 3.1.

The parameters τ , θ and the step size α play an important role in the algorithm. These parameters should be chosen in such a way that the algorithm is optimized in the sense that the number of iterations required by the algorithm is as small as possible. Usually, if θ is a constant independent of the dimension of the problem, for instance $\theta = \frac{1}{2}$, then we call the algorithm a large-update method.

4 Properties of the New Kernel (Barrier) Function

In this section, we present some basic properties of the kernel function (1.1) and the corresponding barrier function $\Psi(V)$. For ψ we have the first three derivatives as follows:

$$\psi'(t) = t + \left(\frac{1}{q} - \frac{1}{t} \right) e^{q(\frac{1}{t}-1)} - \frac{1}{q}, \quad (4.1)$$

$$\psi''(t) = 1 + \frac{q}{t^3} e^{q(\frac{1}{t}-1)}, \quad (4.2)$$

$$\psi'''(t) = -\frac{q(3t+q)}{t^5} e^{q(\frac{1}{t}-1)} < 0. \quad (4.3)$$

The next lemma is essential in the analysis of the algorithm.

Lemma 4.1 For $\psi(t)$ defined in (1.1), we have

$$\psi''(t) > 1, \quad (4.4)$$

$$t\psi''(t) + \psi'(t) > 0, \quad (4.5)$$

$$t\psi''(t) - \psi'(t) > 0, \quad (4.6)$$

$$\psi'''(t) < 0. \quad (4.7)$$

Proof Clearly, the relation (4.2) follows (4.4). For the proof of (4.5), consider two cases:

Case 1. Assume that $0 < t < 1$.

By using (4.1), (4.2), and $e^z \geq 1 + z$, $\forall z$, we have

$$\begin{aligned} t\psi''(t) + \psi'(t) &= 2t + \left(\frac{1}{q} - \frac{1}{t} + \frac{q}{t^2}\right)e^{q(\frac{1}{t}-1)} - \frac{1}{q} \\ &\geq 2t + \left(\frac{1}{q} - \frac{1}{t} + \frac{q}{t^2}\right)\left(1 + q\left(\frac{1}{t} - 1\right)\right) - \frac{1}{q} \\ &= 2t + \frac{q}{t} - 1 + \frac{q^2}{t^2}\left(\frac{1}{t} - 1\right) > 2t > 0. \end{aligned}$$

Case 2. Assume that $t \geq 1$.

Then $\psi'(1) = 0$ and using (4.4), we see that $\psi'(t)$ is strictly increasing. Hence

$$t\psi''(t) + \psi'(t) > t + \psi'(t) > t + \psi'(1) = t > 0.$$

The two cases together prove (4.5).

To prove (4.6), by using (4.1) and (4.2) we have

$$t\psi''(t) - \psi'(t) = \left(\frac{q}{t^2} - \frac{1}{q} + \frac{1}{t}\right)e^{q(\frac{1}{t}-1)} + \frac{1}{q}.$$

Define

$$f(t) = \left(\frac{q}{t^2} - \frac{1}{q} + \frac{1}{t}\right)e^{q(\frac{1}{t}-1)} + \frac{1}{q}.$$

We have

$$f'(t) = -\left(\frac{3qt + t^2 + q^2}{t^4}\right)e^{q(\frac{1}{t}-1)} < 0.$$

This implies that $f(t)$ is strictly decreasing and hence $f(t) > \lim_{t \rightarrow \infty} f(t) = \frac{1}{q}(1 - e^{-q}) > 0$. This completes the proof. $\square$

It follows that $\psi(1) = \psi'(1) = 0$. It is also quite straightforward that

$$\lim_{t \rightarrow 0^+} \psi(t) = \lim_{t \rightarrow +\infty} \psi(t) = +\infty.$$

Due to conditions $\psi'(1) = \psi(1) = 0$, one can describe the function $\psi(t)$ by its second derivative according to

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi. \quad (4.8)$$

In the analysis of the algorithm, we also use the norm-based proximity measure defined by

$$\delta := \delta(V) = \frac{1}{2} \|\nabla \Psi(V)\| = \frac{1}{2} \sqrt{\sum_{i=1}^n \psi'(\lambda_i(V))^2}. \quad (4.9)$$

The next lemma is very useful in the analysis of interior-point algorithms based on the kernel functions (see for example [2, 16]).

Lemma 4.2 (Lemma 2.1.2 in [17]) *Let $\psi(t)$ be a twice differentiable function for $t > 0$. Then the following three properties are equivalent:*

- (i) $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2))$ for $t_1, t_2 > 0$.
- (ii) $\psi'(t) + t\psi''(t) \geq 0$, $t > 0$.
- (iii) $\psi(e^{\frac{t}{s}})$ is convex.

Following [17], the property described in Lemma 4.2 is called exponential convexity, or shortly e -convexity. Therefore, Lemma 4.2 and (4.5) show that our new kernel function (1.1) is e -convex for $t > 0$.

Lemma 4.3 (Lemma 9 in [10]) *Let the kernel function $\psi(t)$ be defined as in (1.1). Then, we have*

$$\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2.$$

Lemma 4.4 *Let $\varrho: [0, \infty) \rightarrow [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then*

$$\sqrt{1+2s} \leq \varrho(s) \leq 1 + \sqrt{2s}.$$

Proof The inverse function of $\psi(t)$ for $t \geq 1$ is obtained by solving t from

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{t}{q} \left(e^{q(\frac{1}{t} - 1)} - 1 \right) = s, \quad t \geq 1.$$

For $t \geq 1$, $e^{q(\frac{1}{t} - 1)} \leq 1$. One has $\frac{t}{q}(e^{q(\frac{1}{t} - 1)} - 1) \leq 0$. Therefore,

$$\frac{t^2 - 1}{2} \geq s,$$

this implies that $t = \varrho(s) \geq \sqrt{1+2s}$. This proves the first inequality. For the proof of second inequality, by Lemma 4.3, we have

$$s = \psi(t) \geq \frac{1}{2}(t-1)^2,$$

whence

$$t = \varrho(s) \leq 1 + \sqrt{2s}.$$

This completes the proof. □

The next result, establishes a lower bound of $\delta(V)$ in terms of $\Psi(V)$.

Lemma 4.5 Let $V \in \mathbf{S}_{++}^n$. Then

$$\delta(V) \geq \sqrt{\frac{\Psi(V)}{2}}.$$

Proof Using (4.9), the second inequality of Lemma 4.3 and (2.2), we have

$$\begin{aligned} \delta(V) &= \frac{1}{2} \sqrt{\sum_{i=1}^n \psi'(\lambda_i(V))^2} \geq \frac{1}{2} \sqrt{\sum_{i=1}^n 2\psi(\lambda_i(V))} = \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^n \psi(\lambda_i(V))} \\ &= \sqrt{\frac{\Psi(V)}{2}}, \end{aligned}$$

which is the desired inequality. $\square$

Corollary 4.1 If $\Psi(V) \geq 1$, then $\delta(V) \geq \frac{1}{\sqrt{2}}$.

At the start of each outer iteration, just before the update of μ with the factor $1 - \theta$, we have $\Psi(V) \leq \tau$. Due to the update of μ the matrix V , defined by (3.6), is divided by the factor $\sqrt{1 - \theta}$, with $0 < \theta < 1$, which leads to an increasing in the value of $\Psi(V)$. Then, during the inner iterations, $\Psi(V)$ decreases until it passes the threshold τ again. Hence, during the course of the algorithm the largest values of $\Psi(V)$ occur just after the updates of μ . In the rest this section, we derive an estimate for the effect of a μ -update on the value of $\Psi(V)$. We start with an important theorem. This is due to the fact that $\psi(t)$ satisfies (4.6) and (4.7).

Theorem 4.1 (Theorem 3.1 in [7]) Let ϱ be the inverse function of $\psi(t)$ for $t \geq 1$. Then for any positive definite matrix V , and any $\beta \geq 1$,

$$\Psi(\beta V) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(V)}{n}\right)\right).$$

An immediate consequence of Theorem 4.1 for the real valued matrix function $\Psi(V)$ is given by the following lemma.

Corollary 4.2 Let $0 \leq \theta < 1$ and $V_+ = \frac{V}{\sqrt{1-\theta}}$. If $\Psi(V) \leq \tau$, then

$$\Psi(V_+) \leq \frac{n\theta + 2\tau + \sqrt{8n\tau}}{2(1-\theta)}.$$

Proof Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho\left(\frac{\Psi(V)}{n}\right) \geq 1$, we have $\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(V)}{n}\right) \geq 1$. Using Theorem 4.1 with $\beta = \frac{1}{\sqrt{1-\theta}}$ and the function ϱ is monotonically increasing since $\psi(t)$ for $t \geq 1$ is monotonically increasing because of its definition, we have

$$\Psi(V_+) \leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(V)}{n}\right)\right) \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{1 + \sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right).$$

According to the proof of Lemma 4.4, it is clear that $\psi(t) \leq \frac{t^2-1}{2}$ for $t \geq 1$. Therefore, we get

$$\psi(V_+) \leq \frac{n}{2} \left(\left(\frac{1 + \sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}} \right)^2 - 1 \right) = \frac{n\theta + 2\tau + \sqrt{8n\tau}}{2(1-\theta)}. \quad \square$$

5 Analysis of the Algorithm

In this section, we determine a default step size and obtain an upper bound to the decrease of the barrier function $\Psi(V)$ during an inner iteration.

5.1 Decrease the Value of $\psi(V)$ and Choose a Default Step Size α

In each iteration the search directions ΔX , Δy and ΔS are obtained by solving the system (3.12) and via (3.9). After a step with size α and due to (3.9), the new iterate is obtained by

$$X_+ = X + \alpha \Delta X = X + \alpha \sqrt{\mu} D D_X D = \sqrt{\mu} D (V + \alpha D_X) D,$$

and

$$S_+ = S + \alpha \Delta S = S + \alpha \sqrt{\mu} D^{-1} D_S D^{-1} = \sqrt{\mu} D^{-1} (V + \alpha D_S) D^{-1}.$$

According to (3.6), we have

$$V_+ = \frac{1}{\sqrt{\mu}} [D^{-1} X_+ S_+ D]^{\frac{1}{2}}.$$

Therefore, V_+^2 is similar to the matrix $\frac{1}{\mu} X_+^{\frac{1}{2}} S_+ X_+^{\frac{1}{2}}$ and thus to $(V + \alpha D_X)^{\frac{1}{2}} (V + \alpha D_S) (V + \alpha D_X)^{\frac{1}{2}}$. Consequently, the eigenvalues of the matrix V_+ are the same as those of $[(V + \alpha D_X)^{\frac{1}{2}} (V + \alpha D_S) (V + \alpha D_X)^{\frac{1}{2}}]^{\frac{1}{2}}$. Since the proximity after one step is defined by $\Psi(V_+)$, it follows from (2.2) that

$$\Psi(V_+) = \Psi \left([(V + \alpha D_X)^{\frac{1}{2}} (V + \alpha D_S) (V + \alpha D_X)^{\frac{1}{2}}]^{\frac{1}{2}} \right).$$

Hence, by Lemma 4.2,

$$\Psi(V_+) \leq \frac{1}{2} (\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)).$$

Let us denote the difference between the proximity before and after one step by a function of the step size, that is,

$$f(\alpha) := \Psi(V_+) - \Psi(V).$$

Then $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2} (\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)) - \Psi(V).$$

Obviously

$$f(0) = f_1(0) = 0.$$

By using (2.4)–(2.6), the derivative of $f_1(\alpha)$ with respect to α is given by

$$f'_1(\alpha) = \frac{1}{2} \text{Tr}(\psi'(V + \alpha D_X) D_X + \psi'(V + \alpha D_S) D_S). \quad (5.1)$$

This, using (4.9) and the third equation of (3.12), gives

$$f'_1(0) = \frac{1}{2} \text{Tr}(\psi'(V)(D_X + D_S)) = \frac{1}{2} \text{Tr}(-\psi'(V)^2) = -2\delta(V)^2. \quad (5.2)$$

If we differentiate f'_1 with respect to α , then we obtain

$$f''_1(\alpha) = \frac{1}{2} \text{Tr}(\psi''(V + \alpha D_X) D_X^2 + \psi''(V + \alpha D_S) D_S^2). \quad (5.3)$$

In what follows, we use the short notation $\delta := \delta(V)$ and state four important lemmas without proofs. These are due to the fact that $\psi''(t)$ is monotonically decreasing.

Lemma 5.1 (Lemma 4.2 in [20]) *One has*

$$f''_1(\alpha) \leq 2\delta^2 \psi''(\lambda_{\min}(V) - 2\alpha\delta).$$

Lemma 5.2 (Lemma 4.2 in [2]) *If the step size α satisfies*

$$-\psi'(\lambda_{\min}(V) - 2\alpha\delta) + \psi'(\lambda_{\min}(V)) \leq 2\delta, \quad (5.4)$$

then $f'_1(\alpha) \leq 0$.

Lemma 5.3 (Lemma 4.3 in [2]) *Let $\rho : [0, \infty) \rightarrow (0, 1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ on the interval $(0, 1]$, then the largest possible value of the step size of α satisfying (5.4) is given by*

$$\bar{\alpha} := \frac{1}{2\delta} (\rho(\delta) - \rho(2\delta)).$$

Lemma 5.4 (Lemma 4.4 in [2]) *Let ρ and $\bar{\alpha}$ be the same as defined in Lemma 5.3. Then*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

For the purpose of finding an upper bound for $f(\alpha)$, we need a default step size $\tilde{\alpha}$ that is the lower bound of the $\bar{\alpha}$ and consists of δ .

Lemma 5.5 *One has*

$$\frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{1 + q(1 + \frac{4q\delta}{q-1})(1 + q^{-1} \log(1 + \frac{4q\delta}{q-1}))^3}.$$

Proof To obtain the inverse function $t = \rho(s)$ of $-\frac{1}{2}\psi'(t)$ for $t \in (0, 1]$, we need to solve the equation

$$-\psi'(t) = -t + \left(\frac{1}{t} - \frac{1}{q}\right)e^{q(\frac{1}{t}-1)} + \frac{1}{q} = 2s.$$

By setting $t = \rho(2\delta)$, we have

$$-\psi'(t) = 4\delta.$$

Hence, we have

$$\left(\frac{1}{t} - \frac{1}{q}\right)e^{q(\frac{1}{t}-1)} = 4\delta + t - \frac{1}{q} \leq 4\delta + 1 - \frac{1}{q} = 4\delta + \frac{q-1}{q}.$$

For $t \leq 1$ we have $1 - \frac{1}{q} \leq \frac{1}{t} - \frac{1}{q}$. Therefore

$$\left(1 - \frac{1}{q}\right)e^{q(\frac{1}{t}-1)} \leq 4\delta + \frac{q-1}{q}.$$

This implies that

$$e^{q(\frac{1}{t}-1)} \leq \frac{4q\delta}{q-1} + 1. \quad (5.5)$$

By taking the logarithm of both sides of (5.5), we obtain

$$q\left(\frac{1}{t} - 1\right) \leq \log\left(1 + \frac{4q\delta}{q-1}\right), \quad (5.6)$$

and hence

$$t = \rho(2\delta) \geq \frac{1}{1 + q^{-1} \log\left(1 + \frac{4q\delta}{q-1}\right)}. \quad (5.7)$$

It follows from (4.2), (5.5) and (5.7) that

$$\frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{1 + q\left(1 + \frac{4q\delta}{q-1}\right)(1 + q^{-1} \log\left(1 + \frac{4q\delta}{q-1}\right))^3}.$$

This completes the proof. $\square$

In the sequel, we use the notation

$$\tilde{\alpha} = \frac{1}{1 + q\left(1 + \frac{4q\delta}{q-1}\right)(1 + q^{-1} \log\left(1 + \frac{4q\delta}{q-1}\right))^3}, \quad (5.8)$$

and we will use $\tilde{\alpha}$ as the default step size. By Lemma 5.4, $\bar{\alpha} \geq \tilde{\alpha}$.

Lemma 5.6 (Lemma 4.5 in [2]) *If the step size α is such that $\alpha \leq \tilde{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

The following lemma customizes the result of Lemma 5.6 for the default step size.

Lemma 5.7 *If $\tilde{\alpha}$ is the default step size as given by (5.4) and $\Psi \geq \tau \geq 1$, then*

$$f(\tilde{\alpha}) \leq \frac{-\sqrt{\Psi}}{2 + q\sqrt{2}(\sqrt{2} + \frac{4q}{q-1})(1 + q^{-1} \log(1 + \frac{4q}{q-1} \sqrt{\frac{\Psi_0}{2}}))^3}.$$

Proof Using Lemma 5.6 with $\alpha = \tilde{\alpha}$ and (5.4), we have

$$f(\tilde{\alpha}) \leq \frac{-\delta^2}{1 + q(1 + \frac{4q\delta}{q-1})(1 + q^{-1} \log(1 + \frac{4q\delta}{q-1}))^3}.$$

This expresses the decrease in one inner iteration in terms of δ . Since the right-hand side of the above expression is monotonically decreasing in δ , we can express the decrease in terms of $\Psi := \Psi(V)$ by Lemma 4.5 and after some elementary reductions as follows:

$$\begin{aligned} f(\tilde{\alpha}) &\leq \frac{-\delta}{\sqrt{2} + q(\sqrt{2} + \frac{4q}{q-1})(1 + q^{-1} \log(1 + \frac{4q\delta}{q-1}))^3} \\ &\leq \frac{-\sqrt{\Psi}}{2 + q\sqrt{2}(\sqrt{2} + \frac{4q}{q-1})(1 + q^{-1} \log(1 + \frac{4q}{q-1} \sqrt{\frac{\Psi_0}{2}}))^3}, \end{aligned}$$

where the last inequality follows from $\Psi_0 \geq \Psi$, and therefore the proof is complete. $\square$

5.2 Iteration Bound

We need to count how many inner iterations are required to return to the situation where $\Psi(V) \leq \tau$ after a μ -update. We define the value of $\Psi(V)$ after μ -update as Ψ_0 and the subsequent values in the same outer iteration are denoted as $\Psi_k, k = 1, 2, \dots, K$, where K denotes the total number of inner iterations in the outer iteration. Then, we have

$$\Psi_0 \leq \frac{n\theta + 2\tau + \sqrt{8n\tau}}{2(1 - \theta)}, \quad \Psi_{K-1} > \tau, \quad 0 \leq \Psi_K \leq \tau. \quad (5.9)$$

Since the large-update methods use $\theta = \Theta(1)$ and $\tau = \mathcal{O}(n)$, we have $\Psi_0 = \mathcal{O}(n)$.

According to decrease of $f(\tilde{\alpha})$, for $k = 1, 2, \dots, K - 1$, we obtain

$$\begin{aligned} \Psi_{k+1} &\leq \Psi_k - \frac{\Psi_k^{\frac{1}{2}}}{2 + q\sqrt{2}(\sqrt{2} + \frac{4q}{q-1})(1 + q^{-1} \log(1 + \frac{4q}{q-1} \sqrt{\frac{\Psi_0}{2}}))^3} \\ &\leq \Psi_k - \frac{\Psi_k^{\frac{1}{2}}}{2 + q\sqrt{2}(\sqrt{2} + 8)(1 + q^{-1} \log(1 + 8\sqrt{\frac{\Psi_0}{2}}))^3}, \end{aligned} \quad (5.10)$$

where the last inequality follows by $\frac{q}{q-1} \leq 2$.

The following lemma is crucial in deriving the number of inner iterations in an outer iteration.

Lemma 5.8 (Lemma 14 in [16]) *Suppose $t_0, t_1, \dots, t_k$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \quad k = 0, 1, \dots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lceil \frac{t_0^\gamma}{\beta\gamma} \rceil$.

Now, using Lemma 5.8, letting $t_k = \psi_k$, $\gamma = \frac{1}{2}$ and

$$\beta = \frac{1}{2 + q\sqrt{2}(\sqrt{2} + 8)(1 + q^{-1} \log(1 + 8\sqrt{\frac{\psi_0}{2}}))^3},$$

we can provide the worst case upper bound for the total number of inner iterations in an outer iteration as follows:

Lemma 5.9 *Let K be the total number of inner iterations in the outer iteration. Then*

$$K \leq \left(4 + 2q\sqrt{2}(\sqrt{2} + 8) \left(1 + q^{-1} \log \left(1 + 8\sqrt{\frac{\psi_0}{2}} \right) \right)^3 \right) \psi_0^{\frac{1}{2}}.$$

Now, we can derive the iteration complexity of our algorithm.

Theorem 5.1 *Given that $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, which are characteristics of the large-update methods, Algorithm 3.1 will obtain an ϵ -approximate solution of problems (P) and (D) in at most $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$ iterations.*

Proof It is well known that the number of outer iterations is bounded above by $\frac{1}{\theta} \log(\frac{n}{\epsilon})$ [Lemma II.17 in [18]]. By multiplying the number of outer iterations and the number of inner iterations we get an upper bound for the total number of iterations, namely,

$$\frac{4 + q(16\sqrt{2} + 4)(1 + q^{-1} \log(1 + 8\sqrt{\frac{n\theta + 2\tau + \sqrt{8n\tau}}{4(1-\theta)}}))^3}{\theta} \times \left(\frac{n\theta + 2\tau + \sqrt{8n\tau}}{2(1-\theta)} \right)^{\frac{1}{2}} \log\left(\frac{n}{\epsilon}\right).$$

Choosing $q = \log(1 + 8\sqrt{\frac{n\theta + 2\tau + \sqrt{8n\tau}}{4(1-\theta)}}) = \mathcal{O}(\log n)$, the iteration bound becomes

$$\mathcal{O}\left(\sqrt{n} \log n \log \frac{n}{\epsilon}\right).$$

□

6 Numerical Example

In this section, we illustrate that the proposed algorithm is promising by a numerical experiment. The numerical experiment is implemented on MATLAB 7.8 under Windows XP. We consider the CQSDO problems whose primal-dual pair of (P) and (D) have the following date:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & -1 & -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 2 & 1 & 2 \\ -2 & 1 & -2 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 2 & -1 & -1 \\ 2 & 0 & 2 & 1 \\ -1 & 2 & 0 & 1 \\ -1 & 1 & 1 & -2 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 3 & -3 & -1 \\ 3 & 4 & 3 & 2 \\ -3 & 3 & -4 & 1 \\ -1 & 2 & 1 & -2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}.$$

One may easily verify that $X^0 = I$ is feasible for (P) and $y = (1, 1, 1)^T$ and $S^0 = I$ is feasible for (D). We obtain a primal-dual optimal solution

$$X^* = \begin{bmatrix} 0.0574 & -0.0368 & -0.0554 & -0.0304 \\ -0.0368 & 0.0648 & 0.0536 & 0.1540 \\ -0.0554 & 0.0536 & 0.2056 & 0.1688 \\ -0.0304 & 0.1540 & 0.1688 & 0.4996 \end{bmatrix}, \quad y^* = \begin{bmatrix} 0.8458 \\ 1.0559 \\ 0.9747 \end{bmatrix},$$

$$S^* = \begin{bmatrix} 0.1081 & 0.1681 & 0.0311 & -0.0557 \\ 0.1681 & 0.2615 & 0.0483 & -0.0867 \\ 0.0311 & 0.0483 & 0.0089 & -0.0160 \\ -0.0557 & -0.0867 & -0.0160 & 0.0287 \end{bmatrix}.$$

We have a tolerance $\epsilon = 10^{-5}$ and $\theta = \frac{1}{\sqrt{4}}$ that the algorithm reaches this solution in 23 iterations.

7 Conclusion

We have analyzed a primal-dual path-following interior-point algorithm for CQSDO based on a new kernel function and derived the currently best known iteration bound for the algorithm with large-update method, $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$. The numerical test is an interesting topic for investigating the behavior of the algorithm.

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