An Approximation Algorithm for the Risk-Adjusted Two-Stage Stochastic Facility Location Problem with Penalties

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Abstract In this paper, we consider the *risk-adjusted two-stage stochastic facility location problem with penalties* (RSFLPP). Using the monotonicity and positive homogeneity of the risk measure function, we present an LP-rounding-based $\frac{1}{6}$ -approximation algorithm.

Keywords Facility location · Approximation algorithm · LP-rounding · Risk-adjusted

Mathematics Subject Classification (2010) 90C27 · 68W25

1 Introduction

In the classical *facility location problem* (FLP), there is a metric space which is made up of the nodes of facilities and clients. We want to open some facilities and assign each client to an open facility. The objective is to minimize the total cost which including the facility opening cost and assignment cost. In the Operations Research community, many researchers devote to design the approximation algorithms for the FLP [1, 4, 6, 8, 9, 11]. The currently best known approximation ratio is 1.488 which is due to Li [10]. The LP-rounding technique is one of the important techniques for designing the approximation algorithms [2, 13, 15].

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The *facility location problem with penalties* (FLPP) is introduced by Charikar et al. [3], who present a primal-dual 3-approximation algorithm. In this problem, each client can be either assigned to an open facility or reject for assignment with a penalty cost. Xu and Xu [16] further improve the approximation ratio to 1.8526. Hayrapetyan et al. [7] introduce the *facility location problem with submodular penalties* (FLPSP) and offer an LP-rounding 2.488-approximation algorithm. Du et al. [5] present a primal-dual combinatorial 3-approximation algorithm for the FLPSP.

In the *two-stage stochastic facility location problem* (SFLP), the set of clients is unknown in advance. But there are finite scenarios, each of which specified a subset of clients to be assigned. Moreover, each facility has two kinds of opening cost. One yields in the first stage and the other is scenario-dependent yielding in the second stage. The objective is to assign each client in each scenario to an open facility which is opened either in the first stage or in the second stage such that the total opening and assignment cost is minimized. Ravi and Sinha [12] give an LP-rounding 8-approximation algorithm for the SFLP.

The *risk-adjusted two-stage stochastic facility location problem* (RSFLP) is firstly presented by So et al. [14]. Considering that most of the recent works on the SFLP focus on minimization of the objectives of two extreme ways of handling risk, they introduce a family of risk measure functionals to interpolate between the two extreme ways and present an 8-approximation algorithm for the RSFLP.

Based on the above models, we introduce the *risk-adjusted two-stage stochastic facility location problem with penalty* (RSFLPP) which is an extension of the RSFLP. In the RSFLPP, a client can be assigned to an open facility (which is in the first stage or in the second stage) or rejected by paying a penalty. Integrating the LP-rounding techniques of [13, 17], we give a 6-approximation algorithm for this model.

The rest of this paper is organized as followed. In Sect. 2, we describe the formulation for the RSFLPP. In Sect. 3, we give an LP-rounding-based algorithm. In Sect. 4, we analyze the algorithm and prove its approximation ratio. Some discussions are given in Sect. 5.

2 Formulation

In the RSFLPP, we are given a set of facilities F and a set of potential clients D . We are also given a set of scenarios $\mathcal{S} = \{1, 2, \dots, N\}$ each of which specifies a subset of clients $D_s \subseteq D$ to be assigned. Each facility i has an opening cost f_i^0 if it is opened in the first stage and f_i^s if it is opened in the second stage in scenario s . Assigning a client j to an open facility i will yield an assignment cost c_{ij} which satisfies the triangle inequality. Denote p_j^s as the penalty incurred by the client $j \in D_s$ ($s \in \mathcal{S}$) for not being assigned to any open facility. Given a constant $\alpha \in [0, 1)$ which denotes the risk aversion level. We have a natural formulation for the RSFLPP as follows:

$$\begin{aligned} \min \quad & \sum_{i \in F} f_i^0 y_i^0 + \beta + \frac{1}{N(1-\alpha)} \sum_{s=1}^N \max\{q(y^0, s) - \beta, 0\} \\ \text{s.t.} \quad & y_i^0 \in \{0, 1\}, \quad \forall i \in F, \\ & \beta \geq 0, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
 q(y^0, s) &:= \min \sum_{i \in F} f_i^s y_i^s + \sum_{i \in F} \sum_{j \in D_s} c_{ij} (x_{ij}^{0s} + x_{ij}^{ss}) + \sum_{j \in D_s} p_j^s z_j^s \\
 \text{s.t. } &\sum_{i \in F} (x_{ij}^{0s} + x_{ij}^{ss}) + z_j^s \geq 1, \quad \forall j \in D_s, \\
 &x_{ij}^{0s} \leq y_i^0, \quad \forall i \in F, j \in D_s, \\
 &x_{ij}^{ss} \leq y_i^s, \quad \forall i \in F, j \in D_s, \\
 &x_{ij}^{0s}, x_{ij}^{ss}, y_i^s, z_j^s \in \{0, 1\}, \quad \forall i \in F, j \in D_s.
 \end{aligned} \tag{2.2}$$

Let us denote

$$\varphi_\alpha(q(y^0, s)) := \beta + \frac{1}{N(1-\alpha)} \sum_{s=1}^N \max\{q(y^0, s) - \beta, 0\}$$

as the risk measure function. Intuitively, $(1 - \alpha)^{-1}$ is the factor which boosts the costs of scenarios in the second stage. When $\alpha = 0$, the costs of scenarios in the second stage are not boosted; and as $\alpha \rightarrow 1$, the cost of scenario s which has the highest cost of $q(y^0, s)$ is to be boosted extraordinarily. Thus α captures the notion of risk. β can be viewed as the predicted cost of the second stage.

Lemma 2.1 ([14]) $\varphi_\alpha(q(y^0, s))$ satisfies monotonicity and positive homogeneity.

- (i) Monotonicity: $\varphi_\alpha(q_1) \leq \varphi_\alpha(q_2)$ for any random variables q_1 and q_2 with $q_1 \leq q_2$;
- (ii) Positive homogeneity: $\varphi_\alpha(q) = t\varphi_\alpha(q)$ for any $t > 0$ and any random variable q .

In formulation (2.1)–(2.2), we use y_i^0 (or y_i^s) to denote whether facility i is open or not in the first stage (or in the second stage scenario s), x_{ij}^{0s} (or x_{ij}^{ss}) to denote whether the client $j \in D_s$ is assigned to the facility i or not in the first stage (or in the second stage scenario s) and z_j^s to denote whether the client $j \in D_s$ is penalized or not. By relaxing the binary constraints, we obtain the following LP relaxation:

$$\begin{aligned}
 \min \sum_{i \in F} f_i^0 y_i^0 + \beta + \frac{1}{N(1-\alpha)} \sum_{s=1}^N \max \left\{ \sum_{i \in F} f_i^s y_i^s \right. \\
 \left. + \sum_{i \in F} \sum_{j \in D_s} c_{ij} (x_{ij}^{0s} + x_{ij}^{ss}) + \sum_{j \in D_s} p_j^s z_j^s - \beta, 0 \right\}
 \end{aligned}$$

$$\begin{aligned}
\text{s.t. } & \sum_{i \in F} (x_{ij}^{0s} + x_{ij}^{ss}) + z_j^s \geq 1, \quad \forall s \in \mathcal{S}, j \in D_s, \\
& x_{ij}^{0s} \leq y_i^0, \quad \forall i \in F, s \in \mathcal{S}, j \in D_s, \\
& x_{ij}^{ss} \leq y_i^s, \quad \forall i \in F, s \in \mathcal{S}, j \in D_s, \\
& x_{ij}^{0s}, x_{ij}^{ss}, y_i^0, y_i^s, z_j^s, \beta \geq 0, \quad \forall i \in F, s \in \mathcal{S}, j \in D_s.
\end{aligned} \tag{2.3}$$

In the remainder of this paper, we denote $\mathcal{D} := \{(j, s) : s \in \mathcal{S}, j \in D_s\}$.

3 Algorithm

In this section, we give our algorithm for the RSFLPP which proceeds as follows. Firstly, we classify the clients set into two subsets according to the optimal solution of (2.3). The first one includes the clients to be assigned to open facilities and the second one includes those to be penalized. Secondly, we consider two cases in the first clients subset to decide which ones are to be assigned to an open facility in the first stage and which ones are to be assigned in the second stage. Finally, we get an integral feasible solution of (2.1).

Algorithm 3.1

Step 1 Given $\gamma, \eta \in (0, 1)$ to be fixed later. Set $F_0 := \emptyset$, $F_s := \emptyset (s \in \mathcal{S})$, $\mathcal{D}_a := \mathcal{D}$, $\mathcal{D}_c^1 := \emptyset$, $\mathcal{D}_c^2 := \emptyset$, $(\bar{x}, \bar{y}, \bar{z}) := (0, 0, 0)$.

Step 2 Solve 2.3 to obtain the optimal fractional solution (x, y, z) . Set

$$\begin{aligned}
\mathcal{D}_r &:= \{(j, s) \in \mathcal{D} : z_j^s > 1 - \gamma\}, \quad \mathcal{D}_a := \mathcal{D}_a \setminus \mathcal{D}_r, \\
\bar{z}_j^s &:= 1, \quad \forall (j, s) \in \mathcal{D}_r.
\end{aligned}$$

Step 3 For each $(j, s) \in \mathcal{D}_a$, denote

$$\begin{aligned}
r_j^s &:= \min_{i \in F} \left\{ c_{ij} : \sum_{i' \in F : c_{i'j} \leq c_{ij}} (x_{i'j}^{0s} + x_{i'j}^{ss}) \geq \eta\gamma \right\}, \\
N_j^{s0} &:= \{i \in F : c_{ij} \leq r_j^s, x_{ij}^{0s} > 0\}, \\
N_j^{ss} &:= \{i \in F : c_{ij} \leq r_j^s, x_{ij}^{ss} > 0\}.
\end{aligned}$$

Step 4 Choose $(\bar{j}, \bar{s}) := \arg \min \{r_j^s : (j, s) \in \mathcal{D}_a\}$.

Case 1 $\sum_{i \in N_{\bar{j}}^{\bar{s}0}} x_{i\bar{j}}^{0\bar{s}} \geq \frac{1}{2}\eta\gamma$. Choose the facility $\bar{i} := \arg \min \{f_i^0 : i \in N_{\bar{j}}^{\bar{s}0}\}$ to open

in the first stage and assign the client $(\bar{j}, \bar{s})$ to it *directly*. Set $\bar{x}_{\bar{i}\bar{j}}^{0\bar{s}} := 1$, $\bar{y}_{\bar{i}}^0 := 1$,

$F_0 := F_0 \cup \{\bar{i}\}$, $\mathcal{D}_a := \mathcal{D}_a \setminus \{(\bar{j}, \bar{s})\}$, and $\mathcal{D}_c^1 := \mathcal{D}_c^1 \cup \{(\bar{j}, \bar{s})\}$.

If there exists $(j, s) \in \mathcal{D}_a$ such that $N_j^{s0} \cap N_{\bar{j}}^{\bar{s}0} \neq \emptyset$, we assign the client (j, s) to the facility $\bar{i}$ in the first stage *indirectly*. Set $\bar{x}_{\bar{i}j}^{0s} := 1$ and $\mathcal{D}_a := \mathcal{D}_a \setminus \{(j, s)\}$.

Case 2 Otherwise, we have $\sum_{i \in N_j^{\bar{s}\bar{s}}} x_{ij}^{\bar{s}\bar{s}} \geq \frac{1}{2}\eta\gamma$. Choose the facility $\bar{i} := \arg \min\{f_i^{\bar{s}} : i \in N_j^{\bar{s}\bar{s}}\}$ to open in the second stage scenario $\bar{s}$ and assign the client $(\bar{j}, \bar{s})$ to it *directly*. Set $\bar{x}_{\bar{i}\bar{j}}^{\bar{s}\bar{s}} := 1$, $\bar{y}_{\bar{i}}^{\bar{s}} := 1$, $F_{\bar{s}} := F_{\bar{s}} \cup \{\bar{i}\}$, $\mathcal{D}_a := \mathcal{D}_a \setminus \{(\bar{j}, \bar{s})\}$, and $\mathcal{D}_c^2 := \mathcal{D}_c^2 \cup \{(\bar{j}, \bar{s})\}$. If there exists $(j, \bar{s}) \in \mathcal{D}_a$ such that $N_j^{\bar{s}\bar{s}} \cap N_{\bar{j}}^{\bar{s}\bar{s}} \neq \emptyset$, we assign the client $(j, \bar{s})$ to the facility $\bar{i}$ in the second stage *indirectly*. Set $\bar{x}_{i\bar{j}}^{\bar{s}\bar{s}} := 1$ and $\mathcal{D}_a := \mathcal{D}_a \setminus \{(j, \bar{s})\}$.

Step 5 If $\mathcal{D}_a = \emptyset$, output $(\bar{x}, \bar{y}, \bar{z})$; otherwise, go to Step 4.

At the end of Algorithm 3.1, we obtain a set of open facilities F_0 which open in the first stage, facilities F_s which open in the second stage in scenario s , and a set of penalized clients $\mathcal{D}_r$.

4 Analysis

In this section, we give our main result addressing the approximation ratio of Algorithm 3.1. We bound the facility cost in Lemmas 4.1–4.2 and the connection cost in Lemmas 4.3–4.4. Then, by choosing suitable parameters, we draw our main theorem.

Lemma 4.1 *The cost of opening facilities in the first stage is*

$$\sum_{i \in F} f_i^0 \bar{y}_i^0 \leq \frac{2}{\eta\gamma} \sum_{i \in F} f_i^0 y_i^0.$$

Proof According to Step 4 of Algorithm 3.1, if the facility $\bar{i}$ is chosen to open in the first stage, there must exist a unique client $(\bar{j}, \bar{s}) \in \mathcal{D}_c^1$ such that

$$\sum_{i \in N_j^{\bar{s}0}} x_{ij}^{0\bar{s}} \geq \frac{1}{2}\eta\gamma.$$

It follows from Step 4 of Algorithm 3.1 and (2.3) that

$$f_{\bar{i}}^0 \leq \frac{2}{\eta\gamma} \sum_{i \in N_j^{\bar{s}0}} f_i^0 x_{ij}^{0\bar{s}} \leq \frac{2}{\eta\gamma} \sum_{i \in N_j^{\bar{s}0}} f_i^0 y_i^0.$$

Combining the disjointness of $\{N_j^{\bar{s}0}\}_{(\bar{j}, \bar{s}) \in \mathcal{D}_c^1}$, we conclude the lemma. $\square$

Similar to Lemma 4.1, we have

Lemma 4.2 *The cost of opening facilities in the second stage in scenario $s \in \mathcal{S}$ is*

$$\sum_{i \in F} f_i^s \bar{y}_i^s \leq \frac{2}{\eta\gamma} \sum_{i \in F} f_i^s y_i^s.$$

Lemma 4.3 For any $(j, s) \in \mathcal{D} \setminus \mathcal{D}_r$ and $\bar{x}_{ij}^{0s} = 1$, we have

$$c_{\bar{i}j} \leq \frac{3}{\gamma(1-\eta)} \sum_{i \in F} c_{ij}(x_{ij}^{0s} + x_{ij}^{ss}).$$

Proof We consider the following two possibilities.

Case 1. Client (j, s) is assigned to $(\bar{i}, 0)$ directly.

It follows from Step 4 of Algorithm 3.1 that $\bar{i} \in N_j^s$ which indicates that $c_{\bar{i}j} \leq r_j^s$.

Case 2. Client (j, s) is assigned to $(\bar{i}, 0)$ indirectly.

There must exist a client $(j', s') \in \mathcal{D}_c^1$ and a facility i' such that $r_{j'}^{s'} \leq r_j^s$, $\bar{i} \in N_{j'}^{s'0}$, and $i' \in N_j^{s0} \cap N_{j'}^{s'0}$. Thus we have

$$c_{\bar{i}j} \leq c_{i'j} + c_{i'j'} + c_{\bar{i}j'} \leq r_j^s + 2r_{j'}^{s'} \leq 3r_j^s.$$

Summing the above two cases, we have $c_{\bar{i}j} \leq 3r_j^s$.

From (2.3) and the definitions of r_j^s and $\mathcal{D}_r$, we have

$$\sum_{i \in F: c_{ij} < r_j^s} (x_{ij}^{0s} + x_{ij}^{ss}) < \eta\gamma$$

and

$$\sum_{i \in F} (x_{ij}^{0s} + x_{ij}^{ss}) = \sum_{i \in F: c_{ij} < r_j^s} (x_{ij}^{0s} + x_{ij}^{ss}) + \sum_{i \in F: c_{ij} \geq r_j^s} (x_{ij}^{0s} + x_{ij}^{ss}) \geq \gamma.$$

Thus, we have

$$r_j^s \leq \frac{1}{\gamma - \eta\gamma} \sum_{i \in F: c_{ij} \geq r_j^s} c_{ij}(x_{ij}^{0s} + x_{ij}^{ss}) \leq \frac{1}{\gamma(1-\eta)} \sum_{i \in F} c_{ij}(x_{ij}^{0s} + x_{ij}^{ss}),$$

which concludes the proof. $\square$

Similar to Lemma 4.3, we have

Lemma 4.4 For any $(j, s) \in \mathcal{D} \setminus \mathcal{D}_r$ and $\bar{x}_{ij}^{ss} = 1$, we have

$$c_{\bar{i}j} \leq \frac{3}{\gamma(1-\eta)} \sum_{i \in F} c_{ij}(x_{ij}^{0s} + x_{ij}^{ss}).$$

Now we are ready to state our main theorem.

Theorem 4.1 Setting $\eta := \frac{2}{5}$ and $\gamma := \frac{5}{6}$, Algorithm 3.1 is a 6-approximation algorithm for the RSFLPP.

Proof Noting that for any client $(j, s) \in \mathcal{D}_r$, we have

$$\bar{z}_j^s \leq \frac{1}{1-\gamma} z_j^s,$$

which implies that

$$\sum_{(j,s) \in \mathcal{D}} p_j^s \bar{z}_j^s \leq \frac{1}{1-\gamma} \sum_{(j,s) \in \mathcal{D}} p_j^s z_j^s. \quad (4.1)$$

For any scenario $s \in \mathcal{S}$, it follows from (2.2), (4.1), and Lemmas 4.2–4.4 that

$$\begin{aligned} q(\bar{y}^0, s) &= \sum_{i \in F} f_i^s \bar{y}_i^s + \sum_{i \in F} \sum_{(j,s) \in \mathcal{D}} c_{ij} (\bar{x}_{ij}^{0s} + \bar{x}_{ij}^{ss}) + \sum_{(j,s) \in \mathcal{D}} p_j^s \bar{z}_j^s \\ &\leq \frac{2}{\eta\gamma} \sum_{i \in F} f_i^s y_i^s + \frac{3}{\gamma(1-\eta)} \sum_{i \in F} \sum_{(j,s) \in \mathcal{D}} c_{ij} (x_{ij}^{0s} + x_{ij}^{ss}) + \frac{1}{1-\gamma} \sum_{(j,s) \in \mathcal{D}} p_j^s z_j^s \\ &\leq \max \left\{ \frac{2}{\eta\gamma}, \frac{3}{\gamma(1-\eta)}, \frac{1}{1-\gamma} \right\} \left(\sum_{i \in F} f_i^s y_i^s \right. \\ &\quad \left. + \sum_{i \in F} \sum_{(j,s) \in \mathcal{D}} c_{ij} (x_{ij}^{0s} + x_{ij}^{ss}) + \sum_{(j,s) \in \mathcal{D}} p_j^s z_j^s \right) \\ &= 6q(y^0, s). \end{aligned} \quad (4.2)$$

Let C_{alg} and C_{opt} be the objective values of the integral solution $(\bar{x}, \bar{y}, \bar{z})$ of (2.1) and the optimal fractional solution (x, y, z) of (2.3), respectively. From (4.2), Lemmas 2.1 and 4.1, we have

$$\begin{aligned} C_{\text{alg}} &= \sum_{i \in F} f_i^0 \bar{y}_i^0 + \varphi_\alpha(q(\bar{y}^0, s)) \\ &\leq 6 \sum_{i \in F} f_i^0 y_i^0 + 6\varphi_\alpha(q(y^0, s)) \\ &\leq 6C_{\text{opt}}. \end{aligned}$$

□

5 Discussion

In this paper, we present an LP-rounding 6-approximation algorithm for the RSFLPP. In the future research, it would be interesting to improve the approximation ratio for the RSFLPP. Another interesting question is to design a combinatorial approximation algorithm.

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