Uniform Parallel-Machine Scheduling with Time Dependent Processing Times

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Abstract We consider several uniform parallel-machine scheduling problems in which the processing time of a job is a linear increasing function of its starting time. The objectives are to minimize the total completion time of all jobs and the total load on all machines. We show that the problems are polynomially solvable when the increasing rates are identical for all jobs; we propose a fully polynomial-time approximation scheme for the standard linear deteriorating function, where the objective function is to minimize the total load on all machines. We also consider the problem in which the processing time of a job is a simple linear increasing function of its starting time and each job has a delivery time. The objective is to find a schedule which minimizes the time by which all jobs are delivered, and we propose a fully polynomial-time approximation scheme to solve this problem.

Keywords Scheduling · Uniform machine · Linear deterioration · Fully polynomial time approximation scheme

1 Introduction

For most scheduling problems, the processing times of jobs are considered to be constant and independent of their starting time. However, this assumption is not appropriate for the modeling of many modern industrial processes, we often encounter

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situations in which processing time increases over time, when the machines gradually lose efficiency. Such problems are generally known as scheduling with deterioration effects. Scheduling with linear deterioration was first considered by Browne and Yechiali [1] who assumed that the processing times of jobs are nondecreasing, start time dependent linear functions. They provided the optimal solution for the single machine when the objective is to minimize the makespan. In addition, they solved a special case when the objective function is to minimize the total weighted completion time. Mosheiov [2] considered simple linear deterioration where jobs have a fixed job-dependent growth rate but no basic processing time. He showed that most commonly applied performance criteria, such as the makespan, the total flow time, the total lateness, the sum of weighted completion time, the maximum lateness, the maximum tardiness, and the number of tardy jobs, remain polynomially solvable.

In the standard form of linear models, the actual processing time of job J_j is given by $p_j = a_j + b_j t_j$, where a_j , b_j , and t_j denote the basic processing time, the deteriorating rate, and the starting time of job J_j , respectively. Kuo and Yang [3] studied several parallel-machine scheduling problems in which the actual processing time of job J_j is given by $p_j = a_j + b_j t_j$ or $p_j = a_j - b_j t_j$. The objectives are to minimize the total completion time of all jobs and the total load on all machines. They showed that the problems are polynomially solvable. Kononov [4] established the ordinary NP-completeness of $P2|p_j = b_j t, r_j = t_0|C_{\max}$ and $P2|p_j = b_j t, r_j = t_0|\sum C_j$. Mosheiov [5] studied multi-machine makespan minimization and total load minimization with linear deterioration, he proved that the two problems are NP-hard even for two machines. Liu et al. [6] considered m uniform machine scheduling with linear deterioration to minimize the makespan, they proposed a fully polynomial-time approximation scheme for this problem. An extensive survey of different models and problems was provided by Alidaee and Womer [7]. Cheng et al. [8] presented an updated survey of the results on scheduling problems with time-dependent processing times. In addition, the concept of the learning effect and deteriorating jobs has been extensively studied. Lee [9] considered single-machine scheduling problem with deteriorating jobs and the learning effect, the objective was to minimize the makespan and the total completion time. He introduced the polynomial solutions for scheduling problems with the effects of learning and deterioration. But in the literature, there are only a few studies dealing with uniform parallel-machine scheduling problems under linear deterioration.

In this paper, we consider uniform-machine scheduling problems in which the actual processing time of job J_j is given by $p_j = a_j + b_j t_j$. The objectives are to minimize the total completion time of all jobs and the total load on all machines. Following the three-field notation introduced by Graham et al. [10], the corresponding problems are denoted by $Qm|p_j = a_j + b_j t_j|\sum C_j$ and $Qm|p_j = a_j + b_j t_j|\sum C_{\max}^i$. We show that the two problems are polynomially solvable. For the standard linear deteriorating function, we denote the problem by $Qm|p_j = a_j + b_j t_j|\sum C_{\max}^i$. We propose a fully polynomial-time approximation scheme for this problem. We also consider the scheduling problem with simple linear deteriorating jobs on uniform parallel machine, each job J_j has a delivery time q_j . The objective is to find a schedule which minimizes the time by which all jobs are delivered. Let C_j denote the completion time of job J_j , thus the delivery completion time of a job J_j is $C_j + q_j$, we will continue to

use L_j to represent $C_j + q_j$. The problem is denoted by $Qm|p_j = b_j t_j, r_j = t_0|L_{\max}$, we present a fully polynomial-time approximation scheme for it.

2 Problem Description

There are n independent jobs $\mathcal{J} = \{J_1, J_2, \dots, J_n\}$ to be processed on m uniform parallel machines and each machine can process at most one job at a time. Preemption is not allowed. For the problems $Qm|p_j = a_j + b t_j | \sum C_j$, $Qm|p_j = a_j + b t_j | \sum C_{\max}^i$ and $Qm|p_j = a_j + b_j t_j | \sum C_{\max}^i$, we assume that all jobs are simultaneously available at time 0 and each job J_j has a positive deteriorating rate b or b_j . For the problem $Qm|p_j = b_j t_j, r_j = t_0|L_{\max}$, all the jobs are simultaneously available at time $t_0 > 0$. Each job J_j has a positive deterioration rate b_j and has a subsequent nonnegative delivery time q_j .

Let M_i and s_i denote the i th machine in the system and its speed factor, respectively. Given a schedule, we use n_i to denote the number of jobs scheduled on machine M_i , $J_{[i,r]}$ denotes a job when it is scheduled in position r on machine M_i in a sequence. Let $C_{[i,r]}$, $C_{\max}^i$, and $L_{[i,r]}$ denote the completion time of job $J_{[i,r]}$, the total load on machine M_i , and the delivery completion time of job $J_{[i,r]}$, respectively, for $i = 1, 2, \dots, m$, $r = 1, 2, \dots, n_i$, and $\sum_{i=1}^m n_i = n$.

For $Qm|p_j = a_j + b t_j | \sum C_j$ and $Qm|p_j = a_j + b t_j | \sum C_{\max}^i$, the completion time of each job scheduled on M_i can be expressed as follows:

$$\begin{aligned} C_{[i,1]} &= \frac{a_{[i,1]}}{s_i}, \\ C_{[i,2]} &= C_{[i,1]} + \frac{a_{[i,2]} + b C_{[i,1]}}{s_i} = \frac{a_{[i,2]}}{s_i} + \left(1 + \frac{b}{s_i}\right) \frac{a_{[i,1]}}{s_i}, \\ C_{[i,3]} &= C_{[i,2]} + \frac{a_{[i,3]} + b C_{[i,2]}}{s_i} = \frac{a_{[i,3]}}{s_i} + \left(1 + \frac{b}{s_i}\right) \frac{a_{[i,2]}}{s_i} + \left(1 + \frac{b}{s_i}\right)^2 \frac{a_{[i,1]}}{s_i}, \\ &\vdots \\ C_{[i,r]} &= C_{[i,r-1]} + \frac{a_{[i,r]} + b C_{[i,r-1]}}{s_i} \\ &= \frac{a_{[i,r]}}{s_i} + \left(1 + \frac{b}{s_i}\right) \frac{a_{[i,r-1]}}{s_i} + \dots + \left(1 + \frac{b}{s_i}\right)^{r-1} \frac{a_{[i,1]}}{s_i}, \\ C_{\max}^i &= C_{[i,n_i]} = \frac{a_{[i,n_i]}}{s_i} + \left(1 + \frac{b}{s_i}\right) \frac{a_{[i,n_i-1]}}{s_i} + \dots + \left(1 + \frac{b}{s_i}\right)^{n_i-1} \frac{a_{[i,1]}}{s_i}. \end{aligned}$$

The total completion time and the total load can be written as follows:

$$\sum_{i=1}^m \sum_{r=1}^{n_i} C_{[i,r]} = \sum_{i=1}^m \sum_{r=1}^{n_i} \frac{1}{s_i} \left(1 + \left(1 + \frac{b}{s_i}\right) + \left(1 + \frac{b}{s_i}\right)^2 + \dots + \left(1 + \frac{b}{s_i}\right)^{n_i-r}\right) a_{[i,r]},$$

$$\sum_{i=1}^m C_{\max}^i = \sum_{i=1}^m \sum_{r=1}^{n_i} \frac{1}{s_i} \left(1 + \frac{b}{s_i}\right)^{n_i-r} a_{[i,r]}.$$

For the problem $Qm|p_j = a_j + b_j t_j| \sum C_{\max}^i$, the total load on M_i can be expressed as follows:

$$C_{\max}^i = C_{[i,n_i]} = \frac{a_{[i,n_i]}}{s_i} + \sum_{j=1}^{n_i-1} \left(1 + \frac{b_{[i,n_i]}}{s_i}\right) \left(1 + \frac{b_{[i,n_i-1]}}{s_i}\right) \cdots \left(1 + \frac{b_{[i,j+1]}}{s_i}\right) \frac{a_{[i,j]}}{s_i}.$$

For the problem $Qm|p_j = b_j t_j, r_j = t_0|L_{\max}$, the delivery completion time of job $J_{[i,r]}$ can be denoted by $L_{[i,r]} = C_{[i,r]} + q_{[i,r]} = t_0 \prod_{j=1}^r \left(1 + \frac{b_{[i,j]}}{s_i}\right) + q_{[i,r]}$.

3 Minimizing Total Completion Time and Minimizing Total Load

3.1 Polynomial Algorithms

Lemma 1 Let x_i and y_i be two sequences of numbers, then the sum $\sum_i x_i y_i$ of products of the corresponding elements is the least if the sequences are monotonic in the opposite sense.

Proof The proof is obtained from [11]. □

Then, according to Lemma 1 and the expression form of the total completion time, we can construct a polynomial algorithm for the problem $Qm|p_j = a_j + b_j t_j| \sum C_j$.

Algorithm 1

Step 1 Select n smallest numbers from the set $\{\frac{1}{s_i}, \frac{1}{s_i} + \frac{1}{s_i}(1 + \frac{b}{s_i}), \frac{1}{s_i} + \frac{1}{s_i}(1 + \frac{b}{s_i}) + \frac{1}{s_i}(1 + \frac{b}{s_i})^2, \frac{1}{s_i} + \frac{1}{s_i}(1 + \frac{b}{s_i}) + \frac{1}{s_i}(1 + \frac{b}{s_i})^2 + \frac{1}{s_i}(1 + \frac{b}{s_i})^3, \dots | i = 1, 2, \dots, m\}$ and place these numbers in a nondecreasing sequence.

Step 2 Sort the jobs in the nonincreasing order of normal processing a_j (i.e., $a_n \geq a_{n-1} \geq \dots \geq a_1$).

Step 3 Form a correspondence between these numbers and the jobs, if J_j corresponds to $\frac{1}{s_i} + \frac{1}{s_i}(1 + \frac{b}{s_i}) + \dots + \frac{1}{s_i}(1 + \frac{b}{s_i})^{k-1}$, then schedule job J_j on machine M_i as the k th last position.

Theorem 1 Algorithm 1 gives an optimal schedule for the problem $Qm|p_j = a_j + b_j t_j| \sum C_j$ in $O(n \log n)$ time.

Proof Suppose that σ^* is an optimal schedule which is different from the schedule σ obtained by Algorithm 1. Let k be the largest index such that job J_k is scheduled on different processors in σ^* and σ . Let job J_k be scheduled on machine M_i in σ^* and r_i denote the number of jobs scheduled to be processed after and including job J_k on machine M_i in σ^* . Let job J_k be scheduled on machine M_j in σ and r_j denote

the number of jobs scheduled to be processed after and including job J_k on machine M_j in σ . Since σ^* and σ are identical for job indices greater than k and Algorithm 1 assigned job J_k to machine M_j , it follows that

$$\frac{1}{s_j} + \frac{1}{s_j} \left(1 + \frac{b}{s_j}\right) + \cdots + \frac{1}{s_j} \left(1 + \frac{b}{s_j}\right)^{r_j-1} \leq \frac{1}{s_i} + \frac{1}{s_i} \left(1 + \frac{b}{s_i}\right) + \cdots + \frac{1}{s_i} \left(1 + \frac{b}{s_i}\right)^{r_i-1}.$$

Let J_l be the job at position r_j in σ^* , by interchanging the position of job J_k and J_l in σ^* , the objective function of σ^* changes by

$$\begin{aligned} & \left(\sum_{r=0}^{r_j-1} \frac{1}{s_j} \left(1 + \frac{b}{s_j}\right)^r \right) a_k + \left(\sum_{r=0}^{r_i-1} \frac{1}{s_i} \left(1 + \frac{b}{s_i}\right)^r \right) a_l \\ & - \left(\sum_{r=0}^{r_i-1} \frac{1}{s_j} \left(1 + \frac{b}{s_j}\right)^r \right) a_l - \left(\sum_{r=0}^{r_j-1} \frac{1}{s_i} \left(1 + \frac{b}{s_i}\right)^r \right) a_k \\ & = \left(\sum_{r=0}^{r_j-1} \frac{1}{s_j} \left(1 + \frac{b}{s_j}\right)^r - \sum_{r=0}^{r_i-1} \frac{1}{s_i} \left(1 + \frac{b}{s_i}\right)^r \right) (a_k - a_l) \leq 0. \end{aligned}$$

Based on the definition of k and the ordering of job's normal processing time, it follows that $a_k - a_l \geq 0$. Then, a finite number of repetitions of this argument establishes that there exists an optimal schedule in which the jobs are sequenced by Algorithm 1. $\square$

Similarly, we give an optimal algorithm for the problem $Qm|p_j = a_j + bt_j|\sum C_{\max}^i$.

Algorithm 2

Step 1 Select n smallest numbers from the set $\{\frac{1}{s_i}, \frac{1}{s_i}(1 + \frac{b}{s_i}), \frac{1}{s_i}(1 + \frac{b}{s_i})^2, \dots | i = 1, 2, \dots, m\}$ and place these numbers in a nondecreasing sequence.

Step 2 Sort the jobs in the nonincreasing order of normal processing a_j .

Step 3 Form a correspondence between these numbers and the jobs, if J_j corresponds to $\frac{1}{s_i}(1 + \frac{b}{s_i})^{k-1}$, then schedule job J_j on machine M_i as the k th last position.

Theorem 2 Algorithm 2 gives an optimal schedule for the problem $Qm|p_j = a_j + bt_j|\sum C_{\max}^i$, which can be solved in $O(n \log n)$ time.

Proof The proof is similar to that of Theorem 1. $\square$

3.2 An FPTAS

An algorithm $\mathcal{A}$ is a ρ -approximation ($\rho \geq 1$) algorithm for a minimization problem if it produces a solution that is at most ρ times that of the optimal one (ρ is also referred to as the worst-case ratio). A family of algorithms $\{\mathcal{A}_\epsilon : \epsilon > 0\}$ is called a fully polynomial-time approximation scheme (FPTAS) if for each $\epsilon > 0$ the algorithm

$\mathcal{A}_\epsilon$ is a $(1 + \epsilon)$ -approximation algorithm running in polynomial time in the input size and $1/\epsilon$. In the sequel, we assume $0 < \epsilon \leq 1$.

Since the problem $P2|p_j = b_j t_j, r_j = t_0| \sum C_{\max}^i$ has been proved to be ordinarily NP-hard by Mosheiov [5], $Qm|p_j = a_j + b_j t_j| \sum C_{\max}^i$ is at least NP-hard in the ordinary sense. In this section, we propose a fully polynomial-time approximation scheme for the scheduling problem $Qm|p_j = a_j + b_j t_j| \sum C_{\max}^i$. In order to use the procedure Partition proposed in Kovalyov and Kubiak [12] which requires that a function used within it be a nonnegative integer function, we first modify the original objective function $\sum C_{\max}^i$ to satisfy this restriction without affecting the sequence. The transformation method from the original objective function to a nonnegative integer function is given by the following scale procedure:

For any $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, define $\theta_1 = \min\{\frac{a_j}{s_i}\}$, $\theta_2 = \min\{1 + \frac{b_j}{s_i}\}$. For simplicity, we suppose that θ_1 and θ_2 are finite decimals. Indeed, data are always obtained within some error range in industrial production. If the error range is set to be infinitesimal, the values will be real numbers. Find integers $\{l_1, l_2\} \in \mathbb{N}^+$ such that $10^{l_1} \theta_1 \in \mathbb{N}^+$ and $10^{l_2} \theta_2 \in \mathbb{N}^+$. Since

$$C_{[i,r]} \frac{a_{[i,r]}}{s_i} + \sum_{j=1}^{r-1} \left(1 + \frac{b_{[i,r]}}{s_i}\right) \left(1 + \frac{b_{[i,r-1]}}{s_i}\right) \cdots \left(1 + \frac{b_{[i,j+1]}}{s_i}\right) \frac{a_{[i,j]}}{s_i},$$

$10^{l_1+(r-1)l_2} C_{[i,r]}$ can be verified as an integer. Then define $L = 10^{l_1+nl_2}$, where n is the total number of jobs, the transformed objective function can be expressed as $L \sum C_{\max}^i$. We use the new objective function instead of the original one in the following. Now our concerned problem can be denoted by $Qm|p_j = a_j + b_j t_j|L \sum C_{\max}^i$.

Gupta and Gupta [13] proved that the problem $1|p_j = a_j + b_j t_j|C_{\max}$ is solved by sequencing the jobs in nondecreasing order of a_j/b_j , which leads to the following lemma.

Lemma 2 For $Qm|p_j = a_j + b_j t_j|L \sum C_{\max}^i$, on each machine M_i ($i \in \{1, \dots, m\}$) in an optimal solution, all the jobs are sequenced in the non-decreasing order of $\frac{a_j}{b_j}$.

Based on Lemma 2, it is natural to consider the jobs in nondecreasing order of a_j/b_j . So we index the jobs such that $a_1/b_1 \leq a_2/b_2 \leq \dots \leq a_n/b_n$. We introduce variables x_j , $j = 1, 2, \dots, n$, where $x_j = i$ if job J_j is scheduled on machine M_i , $i = 1, 2, \dots, m$. Let X be the set of all vectors $x = (x_1, x_2, \dots, x_n)$ with $x_j \in \{1, 2, \dots, m\}$, and $j = 1, 2, \dots, n$. We define the following initial and recursive functions on X :

Initial function: $F_0^i(x) = 0, i = 1, \dots, m$.

Recursive function, for $j = 1, \dots, n$,

$$F_j^i(x) = \frac{a_j}{s_i} 10^{l_1+jl_2} + F_{j-1}^i(x) \left(1 + \frac{b_j}{s_i}\right) \cdot 10^{l_2}, \quad i = x_j,$$

$$F_j^i(x) = F_{j-1}^i(x) \cdot 10^{l_2}, \quad i \neq x_j,$$

$$G_j(x) = \sum_{i=1}^m F_j^i(x),$$

where $F_j^i(x)$ is the magnified workload of machine M_i for the jobs among $J_1, J_2, \dots, J_j$.

Note that in the recursion functions, the workload of each machine is enlarged by a scale factor exactly $10^{l_1+nl_2}$. Therefore, $\sum_{i=1}^m F_n^i(x)$ is our desired value for a given x ; the problem $Q_m | p_j = a_j + b_j t_j | L \sum C_{\max}^i$ reduces to the following minimization problem:

$$\min G_n(x), \quad x \in X.$$

First, we present the procedure $\text{Partition}(A, h, \delta)$ proposed by Kovalyov and Kubiak [12], where $A \subseteq X$, h is a nonnegative integer function on X , and $0 < \delta \leq 1$. This procedure partitions A into disjoint subsets $A_1^h, A_2^h, \dots, A_{k_h}^h$ such that $|h(x) - h(x')| \leq \delta \min\{h(x), h(x')\}$ for any x, x' from the same subset A_j^h $j = 1, 2, \dots, k_h$. The following description gives the details of $\text{Partition}(A, h, \delta)$.

Procedure $\text{Partition}(A, h, \delta)$

Arrange vectors $x \in A$ in the order $x^{(1)}, x^{(2)}, \dots, x^{(|A|)}$ such that $0 \leq h(x^{(1)}) \leq h(x^{(2)}) \leq \dots \leq h(x^{(|A|)})$.

Assign vectors $x^{(1)}, x^{(2)}, \dots, x^{(i_1)}$ to the set A_1^h until a certain i_1 is found such that $h(x^{(i_1)}) \leq (1 + \delta)h(x^{(1)})$ and $h(x^{(i_1+1)}) > (1 + \delta)h(x^{(1)})$. If such an i_1 does not exist, then set $A_1^h = A$ and stop.

Assign vectors $x^{(i_1+1)}, x^{(i_1+2)}, \dots, x^{(i_2)}$ to the set A_2^h until a certain i_2 is found such that $h(x^{(i_2)}) \leq (1 + \delta)h(x^{(i_1+1)})$ and $h(x^{(i_2+1)}) > (1 + \delta)h(x^{(i_1+1)})$. If such an i_2 does not exist, then set $A_2^h = A \setminus A_1^h$ and stop.

Continue the above construction until $x^{(|A|)}$ is included in $A_{k_h}^h$ for some k_h .

Obviously, procedure $\text{Partition}(A, h, \delta)$ requires $O(|A| \log |A|)$ operations to arrange the vectors of A in nondecreasing order of $h(x)$ and $O(|A|)$ operations to provide a partition. The two properties of $\text{Partition}(A, h, \delta)$ that will be used in the development of our FPTAS were presented in [12] as follows:

Lemma 3 $|h(x) - h(x')| \leq \delta \min\{h(x), h(x')\}$ for any $x, x' \in A_j^h$, $j = 1, 2, \dots, k_h$.

Lemma 4 $k_h \leq \frac{\log h(x^{(|A|)})}{\delta} + 2$ if $h(x^{(|A|)}) \geq 1$.

Furthermore, we need the following lemma which was presented in [14].

Lemma 5 For any $0 < x \leq 1$ and for any integer $n \geq 1$, $(1 + \frac{x}{n})^n \leq 1 + 2x$ holds.

Motivated by the idea of Kovalyov and Kubiak [12], a formal description of the FPTAS $\mathcal{A}_\epsilon$ for problem $Q_m | p_j = a_j + b_j t_j | L \sum C_{\max}^i$ is given below.

Algorithm $\mathcal{A}_\epsilon$

Step 1 (Initialization) Number the jobs so that $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \dots \leq \frac{a_n}{b_n}$. Set $Y_0 = \{(0, 0, \dots, 0)\}$, $F_0^i = 0$, $i = 1, 2, \dots, m$ and $j = 1$.

Step 2 (Generation of $Y_1, Y_2, \dots, Y_n$) For the set Y_{j-1} , we generate the set Y'_j by adding k , $k = 1, 2, \dots, m$, in position j of each vector from Y_{j-1} . Calculate the following for any $x \in Y'_j$:

$$F_j^i(x) = \frac{a_j}{s_i} 10^{k_1+jk_2} + F_{j-1}^i(x) \left(1 + \frac{b_j}{s_i}\right) \cdot 10^{k_2}, \quad i = x_j,$$

$$F_j^i(x) = F_{j-1}^i(x) \cdot 10^{k_2}, \quad i \neq x_j,$$

$$G_j(x) = \sum_{i=1}^m F_j^i(x).$$

If $j = n$, then set $Y_n = Y'_n$ and go to Step 3.

If $j < n$, then set $\delta = \frac{\epsilon}{2(n+1)}$ and perform the following computations:

Call Partition(Y'_j, F_j^i, δ), ($i = 1, 2, \dots, m$) to partition the set Y'_j into disjoint subsets $Y_1^{F^i}, Y_2^{F^i}, \dots, Y_{k_{F^i}}^{F^i}$.

Divide the set Y'_j into disjoint subsets $Y_{c_1, \dots, c_m} = Y_{c_1}^{F^1} \cap \dots \cap Y_{c_m}^{F^m}$, $c_1 = 1, 2, \dots, k_{F^1}; \dots; c_m = 1, 2, \dots, k_{F^m}$. For each nonempty subset $Y_{c_1, \dots, c_m}$, choose a vector $x^{(c_1, \dots, c_m)}$ such that

$$G_j(x^{(c_1, \dots, c_m)}) = \min\{G_j(x) | x \in Y_{c_1, \dots, c_m}\}.$$

Set $Y_j := \{x^{(c_1, \dots, c_m)} : Y_{c_1}^{F^1} \cap \dots \cap Y_{c_m}^{F^m} \neq \emptyset, \text{ where } c_1 = 1, 2, \dots, k_{F^1}; \dots; c_m = 1, 2, \dots, k_{F^m}\}$ and set $j := j + 1$. Repeat Step 2.

Step 3 (Solution) Select a vector $x^0 \in Y_n$ such that

$$G_n(x^0) = \min\{G_n(x) | x \in Y_n\}.$$

Let $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ be an optimal solution for the problem $Q_m | p_j = a_j + b_j t_j | L \sum C_{\max}^i$, write $x_{[j]}^* = (x_1^*, \dots, x_j^*, 0, \dots, 0)$ for $1 \leq j \leq n$.

We have the following theorem.

Theorem 3 For the problem $Q_m | p_j = a_j + b_j t_j | L \sum C_{\max}^i$, Algorithm $\mathcal{A}_\epsilon$ finds a solution x^0 such that $Q_n(x^0) \leq (1 + \epsilon) Q_n(x^*)$.

Proof Suppose that $x_{[j]}^* = (x_1^*, \dots, x_j^*, 0, \dots, 0) \in Y_{c_1, \dots, c_m} \subseteq Y'_j$ for some j and $c_1, \dots, c_m$. By the definition of Algorithm $\mathcal{A}_\epsilon$, such j always exists, for instance, $j = 1$. Algorithm $\mathcal{A}_\epsilon$ may not choose $x_{[j]}^*$ for further construction, however, a vector $x^{(c_1, \dots, c_m)}$ is chosen instead of it. By Lemma 3, we have

$$|F_j^i(x_{[j]}^*) - F_j^i(x^{(c_1, \dots, c_m)})| \leq \delta F_j^i(x_{[j]}^*), \quad i = 1, 2, \dots, m.$$

Setting $\delta_1 = \delta$, we consider the vectors $x_{[j+1]}^* = (x_1^*, \dots, x_j^*, x_{j+1}^*, 0, \dots, 0)$ and $\tilde{x}^{(c_1, \dots, c_m)} = (x_1^{(c_1, \dots, c_m)}, \dots, x_j^{(c_1, \dots, c_m)}, x_{j+1}^*, 0, \dots, 0)$.

For $x_{j+1}^* = i, i \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned} & |F_{j+1}^i(x_{[j+1]}^*) - F_{j+1}^i(\tilde{x}^{(c_1, \dots, c_m)})| \\ &= \left| \frac{a_{j+1}}{s_i} 10^{l_1 + (j+1)l_2} + F_j^i(x_{[j]}^*) \left(1 + \frac{b_{j+1}}{s_i}\right) 10^{l_2} \right. \\ &\quad \left. - \frac{a_{j+1}}{s_i} 10^{l_1 + (j+1)l_2} - F_j^i(x^{(c_1, \dots, c_m)}) \left(1 + \frac{b_{j+1}}{s_i}\right) 10^{l_2} \right| \\ &= \left(1 + \frac{b_{j+1}}{s_i}\right) 10^{l_2} |F_j^i(x_{[j]}^*) - F_j^i(x^{(c_1, \dots, c_m)})| \\ &\leq \left(1 + \frac{b_{j+1}}{s_i}\right) 10^{l_2} \delta F_j^i(x_{[j]}^*) \\ &\leq \delta_1 F_{j+1}^i(x_{[j+1]}^*). \end{aligned}$$

Similarly, for $x_{j+1}^* \neq i$, we have

$$|F_{j+1}^i(x_{[j+1]}^*) - F_{j+1}^i(\tilde{x}^{(c_1, \dots, c_m)})| \leq \delta_1 F_{j+1}^i(x_{[j+1]}^*).$$

As a consequence, we have

$$|F_{j+1}^i(x_{[j+1]}^*) - F_{j+1}^i(\tilde{x}^{(c_1, \dots, c_m)})| \leq \delta_1 F_{j+1}^i(x_{[j+1]}^*), \quad i = 1, 2, \dots, m. \quad (1)$$

This leads to

$$F_{j+1}^i(\tilde{x}^{(c_1, \dots, c_m)}) \leq (1 + \delta_1) F_{j+1}^i(x_{[j+1]}^*), \quad i = 1, 2, \dots, m. \quad (2)$$

Assume that $\tilde{x}^{(c_1, \dots, c_m)} \in Y_{b_1, \dots, b_m} \subseteq Y'_{j+1}$ and Algorithm $\mathcal{A}_\epsilon$ chooses $x^{b_1, \dots, b_m} \in Y_{b_1, \dots, b_m}$ instead of $\tilde{x}^{(c_1, \dots, c_m)}$ in the $(j+1)$ th iteration. From (2) and by Lemma 3, for $i = 1, 2, \dots, m$, we have

$$\begin{aligned} & |F_{j+1}^i(\tilde{x}^{(c_1, \dots, c_m)}) - F_{j+1}^i(x^{b_1, \dots, b_m})| \leq \delta F_{j+1}^i(\tilde{x}^{(c_1, \dots, c_m)}) \\ & \leq \delta(1 + \delta_1) F_{j+1}^i(x_{[j+1]}^*). \end{aligned} \quad (3)$$

From (1) and (3), for $i = 1, 2, \dots, m$, we obtain

$$\begin{aligned} & |F_{j+1}^i(x_{[j+1]}^*) - F_{j+1}^i(x^{b_1, \dots, b_m})| \\ & \leq |F_{j+1}^i(x_{[j+1]}^*) - F_{j+1}^i(\tilde{x}^{(c_1, \dots, c_m)})| + |F_{j+1}^i(\tilde{x}^{(c_1, \dots, c_m)}) - F_{j+1}^i(x^{b_1, \dots, b_m})| \\ & \leq (\delta_1 + \delta(1 + \delta_1)) F_{j+1}^i(x_{[j+1]}^*) \\ & = (\delta + \delta_1(1 + \delta)) F_{j+1}^i(x_{[j+1]}^*). \end{aligned} \quad (4)$$

Set $\delta_l = \delta + \delta_{l-1}(1 + \delta)$, $l = 2, 3, \dots, n - j + 1$. From (4), we have

$$|F_{j+1}^i(x_{[j+1]}^*) - F_{j+1}^i(x^{b_1, \dots, b_m})| \leq \delta_2 F_{j+1}^i(x_{[j+1]}^*), \quad i = 1, 2, \dots, m.$$

By repeating the above argument for $j + 2, \dots, n$, we eventually show that there exists an $x' \in Y_n$ such that

$$|F_n^i(x^*) - F_n^i(x')| \leq \delta_{n-j+1} F_n^i(x^*), \quad i = 1, 2, \dots, m.$$

By Lemma 5, we have

$$\delta_{n-j+1} \leq \delta \sum_{j=0}^n (1 + \delta)^j = (1 + \delta)^{n+1} - 1 \leq \left(1 + \frac{\epsilon}{2(n+1)}\right)^{n+1} - 1 \leq \epsilon.$$

Therefore,

$$F_n^i(x') \leq (1 + \delta_{n-j+1}) F_n^i(x^*) \leq (1 + \epsilon) F_n^i(x^*), \quad i = 1, 2, \dots, m. \quad (5)$$

From (5), we have

$$G_n(x') = \sum_{i=1}^m F_n^i(x') \leq (1 + \epsilon) \sum_{i=1}^m F_n^i(x^*) = (1 + \epsilon) G_n(x^*).$$

By Step 3 of Algorithm $\mathcal{A}_\epsilon$, a vector x^0 will be chosen such that

$$G_n(x^0) \leq G_n(x') \leq (1 + \epsilon) G_n(x^*).$$

This completes the proof. $\square$

Define $L = \log \max\{n, \frac{1}{\epsilon}, A_{\max}, 1 + B_{\max}, 10^{l_1}, 10^{l_2}\}$, where $A_{\max} = \max\{\frac{a_j}{s_i}\}$, $B_{\max} = \max\{\frac{b_j}{s_i}\}$, $i = 1, \dots, m$; $j = 1, \dots, n$. We have the following theorem.

Theorem 4 Algorithm $\mathcal{A}_\epsilon$ requires $O(\frac{n^{2m+1} L^{m+1}}{\epsilon^m})$ time.

Proof The time complexity of Algorithm $\mathcal{A}_\epsilon$ can be established by noting that the most time-consuming operation of iteration j of Step 2 is a call of procedure Partition, which requires $O(|Y'_j| \log |Y'_j|)$ time to complete. To estimate $|Y'_j|$, recall that

$$|Y'_{j+1}| \leq m|Y_j| \leq m k_{F^1} k_{F^2} \cdots k_{F^m}.$$

By Lemma 4, we have

$$\begin{aligned} k_{F^i} &\leq \frac{2(n+1) \log(10^{l_1+n l_2} n A_{\max} (1 + B_{\max})^n)}{\epsilon} + 2 \\ &\leq \frac{2(n+1)(2n+3)L}{\epsilon} + 2, \quad i = 1, 2, \dots, m. \end{aligned}$$

Thus $|Y'_j| = O(\frac{n^{2m} L^m}{\epsilon^m})$ and $|Y'_j| \log |Y'_j| = O(\frac{n^{2m} L^{m+1}}{\epsilon^m})$. There are at most n iterations; therefore, Algorithm $\mathcal{A}_\epsilon$ runs in $O(\frac{n^{2m+1} L^{m+1}}{\epsilon^m})$ time. $\square$

4 Minimizing the Maximum Delivery-Completion Time

4.1 NP-Hardness

Theorem 5 *The problem $Qm|p_j = b_j t_j, r_j = t_0|L_{\max}$ is NP-hard.*

Proof If all delivery times are zero, the problem $Pm|p_j = b_j t_j, r_j = t_0|L_{\max}$ is equivalent to the problem $Pm|p_j = b_j t_j, r_j = t_0|C_{\max}$. Since $Pm|p_j = b_j t_j, r_j = t_0|C_{\max}$ is ordinarily NP-hard according to Kononov [4], $Qm|p_j = b_j t_j, r_j = t_0|L_{\max}$ is at least NP-hard in the ordinary sense. $\square$

4.2 An FPTAS

In this section, we derive a fully polynomial-time approximation scheme for the NP-hard problem $Qm|p_j = b_j t_j, r_j = t_0|L_{\max}$. This is done by applying the “rounding-the-input-data” technique of Sahni [15] to a given problem instance. The definition of the modified deteriorating rates involves a geometric rounding technique developed by Sengupta [16]. And the rounding technique is stated as follows:

For any $\epsilon' > 0$ and $x \geq 1$, if $(1 + \epsilon')^{k-1} < x < (1 + \epsilon')^k$, then we define $\lceil x \rceil_{\epsilon'} = (1 + \epsilon')^k$, $\lfloor x \rfloor_{\epsilon'} = (1 + \epsilon')^{k-1}$. If x is an exact power of $(1 + \epsilon')$, then $\lceil x \rceil_{\epsilon'} = \lfloor x \rfloor_{\epsilon'} = x$. Note that $\lceil x \rceil_{\epsilon'} \leq (1 + \epsilon')x$ for any $x \geq 1$.

For any $0 < \epsilon \leq 1$, we define the modified deteriorating rate of job J_j as $b'_j = s_i(\lceil 1 + \frac{b_j}{s_i} \rceil_{\epsilon'} - 1)$, where $\epsilon' = \frac{\epsilon}{2n}$, for $i = 1, \dots, m$ and $j = 1, \dots, n$. Let β_{ji} denote the exponent of $1 + \frac{b'_j}{s_i}$, i.e., $1 + \frac{b'_j}{s_i} = \lceil 1 + \frac{b_j}{s_i} \rceil_{\epsilon'} = (1 + \epsilon')^{\beta_{ji}}$, then $\beta_{ji} = \frac{\log \lceil 1 + \frac{b_j}{s_i} \rceil_{\epsilon'}}{\log(1 + \epsilon')} = O(\frac{n \log(1 + \frac{b_j}{s_i})}{\epsilon})$.

Theorem 6 *For any $0 < \epsilon \leq 1$, the optimal objective function value for $Qm|p_j = b_j t_j, r_j = t_0|L_{\max}$ under the modified deteriorating rates is at most $(1 + \epsilon)$ times the optimal value for the same problem under the original deteriorating rates.*

Proof Let σ be an arbitrary feasible schedule. Consider an arbitrary job J_j scheduled in position r on machine M_i , $1 \leq i \leq m$. We denote by C_j the completion time of J_j under the original deteriorating rates, and by C'_j ($C'_{[i,r]}$) the completion time of J_j under the modified deteriorating rates in σ . By the definition of the modified deteriorating rate b'_j , we have

$$1 + \frac{b'_j}{s_i} = \left\lceil 1 + \frac{b_j}{s_i} \right\rceil_{\epsilon'} \leq (1 + \epsilon') \left(1 + \frac{b_j}{s_i} \right).$$

This yields

$$C'_j = C'_{[i,r]} = t_0 \prod_{l=1}^r \left(1 + \frac{b'_{[i,l]}}{s_i} \right) \leq t_0 (1 + \epsilon')^r \prod_{l=1}^r \left(1 + \frac{b_{[i,l]}}{s_i} \right) \leq (1 + \epsilon')^n C_j.$$

Since $(1 + \epsilon')^n = (1 + \frac{\epsilon}{2n})^n \leq 1 + \epsilon$ for $0 < \epsilon \leq 2$,

$$L'_j = C'_j + q_j \leq (1 + \epsilon')^n (C_j + q_j) \leq (1 + \epsilon)(C_j + q_j) = (1 + \epsilon)L_j.$$

As a consequence, we have

$$L'_{\max} = \max\{L'_j\} \leq (1 + \epsilon) \max\{L_j\} = (1 + \epsilon)L_{\max}(\sigma).$$

Since the above inequality is valid for any feasible schedule σ , the results holds. $\square$

Lemma 6 *For a single machine problem $1|p_j = b_j t, r_j = t_0|L_{\max}$, sorting the jobs by nonincreasing delivery times yields an optimum schedule.*

Proof Consider an optimal schedule σ^* . If σ^* contains jobs that are not sequenced in nonincreasing order of delivery times, then there exists a pair of adjacent jobs J_i, J_j such that job J_i is followed by job J_j but $q_i < q_j$. Assuming that job J_i starts at time S , we have

$$L_i(\sigma^*) = S + b_i S + q_i, \quad L_j(\sigma^*) = S + b_i S + b_j(S + b_i S) + q_j.$$

Consider a schedule σ which is obtained from σ^* by interchanging jobs J_i and J_j . Under σ , we have

$$L_j(\sigma) = S + b_j S + q_j, \quad L_i(\sigma) = S + b_j S + b_i(S + b_j S) + q_i,$$

then we obtain that

$$L_j(\sigma^*) > L_j(\sigma), \quad L_j(\sigma^*) > L_i(\sigma).$$

This contradicts the optimality of σ^* since all other completion times are unchanged. It follows that sorting the jobs by nonincreasing delivery times yields an optimum schedule for a single machine. $\square$

Corollary 1 *There exists an optimal job sequence for $Q_m|p_j = b_j t, r_j = t_0|L_{\max}$ such that on each machine the jobs are sequenced in nonincreasing order of delivery times.*

As a result of Theorem 6, with $(1 + \epsilon)$ loss, we can design a dynamic program for the problem $Q_m|p_j = b_j t, r_j = t_0|L_{\max}$ under the modified deteriorating rates. Based on Corollary 1, it is natural to consider the jobs in nonincreasing order of q_j . Let us index the jobs such that $q_1 \geq q_2 \geq \dots \geq q_n$. The dynamic programming algorithm is stated as follows:

Algorithm DP Let $\phi_j(\gamma_1, \gamma_2, \dots, \gamma_m)$ denote the optimal value when the jobs in consideration are $J_1, J_2, \dots, J_j$, and the maximum completion time of jobs on machine M_k is $t_0(1 + \epsilon')^{\gamma_k}$ for $k = 1, 2, \dots, m$.

Step 1 (Initialization)

$$\phi_0(\gamma_1, \gamma_2, \dots, \gamma_m) = \begin{cases} t_0, & \text{if } \gamma_1 = \dots = \gamma_m = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Step 2 (Recursion)

$$\begin{aligned} & \phi_j(\gamma_1, \gamma_2, \dots, \gamma_m) \\ &= \min_{1 \leq k \leq m} \left\{ \max \left\{ \phi_{j-1}(\gamma_1, \dots, \gamma_k - \beta_{jk}, \dots, \gamma_m), t_0(1 + \epsilon')^{\gamma_k} + q_j \right\} \right\}. \end{aligned}$$

If $j = n$, go to Step 3. Otherwise, set $j := j + 1$ and repeat Step 2.

Step 3 (Output) The optimal value is determined by

$$\min \left\{ \phi_n(\gamma_1, \gamma_2, \dots, \gamma_m) \mid 0 \leq \gamma_k \leq \sum_{j=1}^n \beta_{jk}, 1 \leq k \leq m \right\}.$$

The corresponding optimal schedule can be found by backtracking.

Some remarks should be made about Algorithm DP. In Step 2, we assume that J_j is scheduled on machine M_k , $1 \leq k \leq m$. For the jobs $J_1, J_2, \dots, J_{j-1}$, the maximum completion time of jobs on machine M_i must be $t_0(1 + \epsilon')^{\gamma_i}$, for $i = 1, \dots, k-1, k+1, \dots, m$, and be $t_0(1 + \epsilon')^{\gamma_k - \beta_{jk}}$ on machine M_k . Hence the delivery completion time of J_j is compared with the maximum delivery completion times of the jobs that have been scheduled earlier.

Let $L = \log(1 + \frac{b_{\max}}{s_{\min}})$, where $\frac{b_{\max}}{s_{\min}} = \max\{\frac{b_j}{s_i} : 1 \leq j \leq n, 1 \leq i \leq m\}$. We have the following theorem.

Theorem 7 *There exists an FPTAS for the problem $Qm|p_j = b_j t, r_j = t_0|L_{\max}$ which runs in $O(n^{2m-1} L^{m-1} / \epsilon^{m-1})$ time.*

Proof In the dynamic programming recursion, we have n states for j , and at most $\sum_{j=1}^n \beta_{jk}$ states for each γ_k , $1 \leq k \leq m$, where only $m-1$ of the values $\gamma_1, \gamma_2, \dots, \gamma_m$ are independent. So the total complexity is $O(n(\sum_{j=1}^n \beta_{jk})^{m-1}) = O(n^{2m-1} L^{m-1} / \epsilon^{m-1})$. $\square$

5 Conclusion

In this paper, we focus on uniform parallel machine scheduling problems with time dependent processing times. The actual processing time of a job is a linear increasing function of its starting time. The objectives are to minimize the total completion time of all jobs and the total load on all machines. When all jobs are associated with a common deteriorating rate b , we provide an optimal $O(n \log n)$ time algorithm

for them. We also presented fully polynomial-time approximation schemes for the problems $Qm|p_j = a_j + b_j t_j|\sum C_{\max}^i$ and $Qm|p_j = b_j t_j, r_j = t_0|L_{\max}$. For future research, it would be interesting to focus on scheduling deteriorating jobs with other objectives.

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