Krätzel Function and Related Statistical Distributions

T. Princy

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Abstract The Krätzel function has many applications in applied analysis, so this function is used as a base to create a density function which will be called the Krätzel density. This density is applicable in chemical physics, Hartree–Fock energy, helium isoelectric series, statistical mechanics, nuclear energy generation, etc., and also connected to Bessel functions. The main properties of this new family are studied, showing in particular that it may be generated via mixtures of gamma random variables. Some basic statistical quantities associated with this density function such as moments, Mellin transform, and Laplace transform are obtained. Connection of Krätzel distribution to reaction rate probability integral in physics, inverse Gaussian density in stochastic processes, Tsallis statistics and superstatistics in non-extensive statistical mechanics, Mellin convolutions of products and ratios thereby to fractional integrals, synthetic aperture radar, and other areas are pointed out in this article. Finally, we extend the Krätzel density using the pathway model of Mathai, and some applications are also discussed. The new probability model is fitted to solar radiation data.

Keywords Laplace transform · Krätzel function · Mellin transform · H-function · Krätzel density · Pathway model

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T. Princy (✉)

Centre for Mathematical Sciences, Arunapuram P.O., Pala 686 574, Kerala, India

e-mail: princycms@gmail.com

URL: <http://www.cmsintl.org>

T. Princy

Department of Statistics, Banaras Hindu University (B.H.U), Varanasi 221005, India

1 Introduction

The Krätzel function was first introduced by Krätzel [9] as the kernel of an integral transform. Properties of Krätzel function were investigated by many authors. We note that the Krätzel function occurs in the study of astrophysical thermonuclear functions, which are derived on the basis of Boltzmann–Gibbs statistical mechanics [16]. The generalized Krätzel function was examined by Kilbas and Kumar [8]. Such investigations are of great interest in connection with applications. Due to the importance of the Krätzel function, we use this function as a base to create a density function.

This density will be shown to be connected to the following problems: Reaction rate probability integral and its generalizations in reaction rate theory in physics, Tsallis statistics and superstatistics in non-extensive statistical mechanics, Mellin convolutions of products and ratios in applied analysis, Krätzel transform, inverse Gaussian density in stochastic processes, Hartree–Fock energy theory, helium isoelectric series, microwave sea clutter, synthetic aperture radar, K-distribution model, neural networks, solar radiation statistics, etc.

The Krätzel function $Z_\rho^\nu(x)$ is defined for $x > 0$, $\rho \in \mathbb{R}$ and $\nu \in \mathbb{C}$ as

$$Z_\rho^\nu(x) = \int_0^\infty y^{\nu-1} e^{-y^\rho - \frac{x}{y}} dy. \quad (1.1)$$

In particular, when $\rho = 1$ and $x = \frac{y^2}{4}$, then it gives

$$Z_1^\nu\left(\frac{y^2}{4}\right) = 2\left(\frac{y}{2}\right)^\nu K_\nu(y),$$

where $K_\nu(y)$ is the modified Bessel function of the third kind or McDonald function as in [4]. We know that for all $u > 0$ and $\nu \in \mathbb{C}$

$$Z_1^\nu\left(\frac{u^2}{4}\right) = 2\left(\frac{u}{2}\right)^\nu K_\nu(u) = 2\left(\frac{u}{2}\right)^\nu K_{-\nu}(u),$$

and consequently

$$Z_1^\nu(u) = 2u^{\frac{\nu}{2}} K_\nu(2\sqrt{u}).$$

Note that this function is useful in chemical physics. More precisely, the function

$$u \mapsto 2^{\nu-1} Z_1^\nu(uq^2/4) = (q\sqrt{u})^\nu K_\nu(q\sqrt{u})$$

is related to Hartree–Fock energy and is used as a basis function for the helium isoelectronic series, see [2, 3] for more details. Moreover, the function

$$u \mapsto \sqrt{\frac{2}{\pi}} 2^{\nu-1} Z_1^\nu\left(\frac{u^2}{4}\right) = \sqrt{\frac{2}{\pi}} u^\nu K_\nu(u)$$

is called in the literature as reduced Bessel function and plays an important role in theoretical chemistry, see [18] and the references therein for more details.

This paper is organized as follows: First, we introduce a new class of continuous statistical distributions, which we name, Krätzel distributions. Some theoretical properties of the Krätzel density function are obtained in Sect. 2. In Sect. 3, the distributions of products and ratios of independent Krätzel random variables are mentioned. In Sect. 4, we extend the Krätzel density using the pathway model of Mathai and some applications are also discussed. In Sect. 5, we discuss an application of one of the above-mentioned probability models, which is shown by fitting the model to solar radiation data.

2 A Density Function

In this section, we introduce the Krätzel distribution and study the basic distributional properties.

A random variable X is said to have a Krätzel distribution with parameters α , ν and ρ if its density function is given by

$$f(x) = \begin{cases} c x^{\alpha-1} Z_{\rho}^{\nu}(x), & x \geq 0, \rho > 0, \alpha > 0, \nu > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (2.1)$$

where

$$Z_{\rho}^{\nu}(x) = \int_0^{\infty} y^{\nu-1} e^{-y^{\rho} - \frac{x}{y}} dy$$

and c is the normalizing constant, which can be evaluated using a gamma integral and it is given by $c = \frac{\rho}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+\nu}{\rho}\right)}$. A family of densities is available for various values of the parameters. The graphs of Krätzel density for various values of the parameters are given in the following figures.

Figure 1 illustrates the Krätzel density when the parameters ρ and ν are fixed and α is changing. Note that the parameter α is called the shape parameter in the Krätzel case because α throws light on the shape of the density curve. Figure 2 shows the density when α and ρ are fixed and ν is varying.

The Mellin transform technique is used to determine the moments and some properties of the proposed distribution.

2.1 Mellin Transform

The Mellin transform of the density function $f(x)$ is the $(s-1)^{\text{th}}$ moment of a positive random variable, where s is a complex parameter. The Mellin transform is given by

$$\begin{aligned} M_f(s) &= E(X^{s-1}) \\ &= \frac{1}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+\nu}{\rho}\right)} \Gamma(\alpha + s - 1) \Gamma\left(\frac{\alpha + \nu + s - 1}{\rho}\right), \end{aligned} \quad (2.2)$$

Fig. 1 Krätzel density functions: $\rho = 3$ and $\nu = 3$ fixed, α changing

Fig. 2 Krätzel density functions: $\rho = 3$ and $\alpha = 4$ fixed, ν changing

for $\Re(\alpha + s - 1) > 0, \nu > 0, \rho > 0$. From (2.2) we may also observe the following properties. If U is the Krätzel random variable, then we have

$$\begin{aligned} E(U^{s-1}) &= \frac{1}{\Gamma(\alpha)\Gamma\left(\frac{\alpha+\nu}{\rho}\right)} \Gamma(\alpha + s - 1) \Gamma\left(\frac{\alpha + \nu + s - 1}{\rho}\right) \\ &= E(Y^{s-1})E(Z^{s-1}), \end{aligned} \quad (2.3)$$

where

$$E(Y^{s-1}) = \frac{\Gamma(\alpha + s - 1)}{\Gamma(\alpha)} \quad \text{for } \Re(\alpha + s - 1) > 0 \quad (2.4)$$

and

$$E(Z^{s-1}) = \frac{\Gamma\left(\frac{\alpha+\nu+s-1}{\rho}\right)}{\Gamma\left(\frac{\alpha+\nu}{\rho}\right)} \quad \text{for } \Re(\alpha + \nu + s - 1) > 0, \rho > 0. \quad (2.5)$$

This suggests that the Krätzel random variable has the representation $U = YZ$, where Y is a gamma random variable with parameter $(\alpha, 1)$, Z is a generalized gamma random variable, U is a Krätzel random variable, and it is assumed that Y and Z are independently distributed. This will be stated as a theorem.

Theorem 2.1 *The Krätzel random variable X of (2.1) has the structural representation $X = YZ$, where Y and Z are independently distributed with Y having a gamma density with parameters $(\alpha, 1)$ and Z having a generalized gamma density with parameters (α, ν, ρ) .*

Using the Mellin transform of the density function, we give a representation of the Krätzel density in terms of H -function. Hence all the properties of H -function can now be made use of to study this density function further. The density function $f(x)$ of (2.1) in terms of H -function is given by

$$f(x) = \frac{1}{\Gamma(\alpha)\Gamma\left(\frac{\alpha+\nu}{\rho}\right)} H_{0,2}^{2,0} \left[x \middle|_{(\alpha-1,1),\left(\frac{\alpha+\nu-1}{\rho},\frac{1}{\rho}\right)} \right], \quad x \geq 0, \alpha > 0, \rho > 0, \nu > 0 \quad (2.6)$$

and $f(x) = 0$ elsewhere. Let us consider the special case for $\rho = 1$, then $f(x)$ can be written as

$$f(x) = \frac{2}{\Gamma(\alpha)\Gamma(\alpha + \nu)} x^{\alpha+\frac{\nu}{2}-1} K_\nu(2x^{\frac{1}{2}}), \quad x \geq 0, \alpha > 0, \nu > 0 \quad (2.7)$$

and $f(x) = 0$ elsewhere, where $K_\nu(2x^{\frac{1}{2}})$ is the modified Bessel function of the third kind or McDonald function. When ρ is real and rational, the H -function appearing in (2.6) can be reduced to G -function using the multiplication formula for the gamma function, namely,

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}} \Gamma(z) \Gamma\left(z + \frac{1}{m}\right) \dots \Gamma\left(z + \frac{m-1}{m}\right), \quad m = 1, 2, \dots$$

As a consequence, when $\rho = \frac{1}{m}$ we have

$$f(x) = \frac{1}{m^m \Gamma(\alpha) \Gamma(\alpha + \nu) \Gamma(\alpha + \nu + \frac{1}{m}) \dots \Gamma(\alpha + \nu + 1 - \frac{1}{m})} \\ \times G_{0,m+1}^{m+1,0} \left[\frac{x}{m^m} \middle| \alpha-1, \alpha+\nu-1, \alpha+\nu+\frac{1}{m}-1, \dots, \alpha+\nu-\frac{1}{m} \right], \quad (2.8)$$

$x \geq 0, \alpha > 0, \nu > 0, m = 1, 2, \dots$ and $f(x) = 0$ elsewhere. When $m = 2$, $f(x)$ becomes

$$f(x) = \frac{1}{4\Gamma(\alpha)\Gamma(\alpha+\nu)\Gamma(\alpha+\nu+\frac{1}{2})} G_{0,3}^{3,0} \left[\frac{x}{4} \middle| \alpha-1, \alpha+\nu-1, \alpha+\nu+\frac{1}{2} \right], \quad (2.9)$$

$x \geq 0, \alpha > 0, \nu > 0$. This is a real physical situation (reaction rate theory), see for example Mathai and Haubold [12].

2.1.1 Remark

The representation (2.6) of the Krätzel density function $f(x)$ as a Fox's $H_{0,2}^{2,0}$ -function reveals it as a special case ($m = 2$) of the Kernel function $H_{0,m}^{m,0}$, $m > 1$. When $\alpha = 1$, it is a very general H -integral transform generalizing the Laplace and Meijer transforms. For $\rho = 1$, the Krätzel function as mentioned before is the modified Bessel function of the third kind (McDonald function) representable as a Meijer's $G_{0,2}^{2,0}$ -function.

From (2.2), the mean and variance of the Krätzel density function can be derived as follows:

$$E(X) = \alpha \frac{\Gamma\left(\frac{\alpha+\nu+1}{\rho}\right)}{\Gamma\left(\frac{\alpha+\nu}{\rho}\right)}, \quad \rho > 0, \alpha + \nu > 0, \quad (2.10)$$

$$\text{Var}(X) = \alpha(\alpha+1) \frac{\Gamma\left(\frac{\alpha+\nu+2}{\rho}\right)}{\Gamma\left(\frac{\alpha+\nu}{\rho}\right)} - \left(\alpha \frac{\Gamma\left(\frac{\alpha+\nu+1}{\rho}\right)}{\Gamma\left(\frac{\alpha+\nu}{\rho}\right)} \right)^2, \quad \alpha + \nu > 0, \rho > 0. \quad (2.11)$$

Theorem 2.2 Let $Y \geq 0$ be a gamma random variable with parameters $(\alpha, 1)$ and $Z \geq 0$ be a generalized gamma random variable with parameters $(\alpha + \nu, 1, \rho)$ where ρ is the power and $\alpha + \nu$ is the shape parameter, and let Y and Z be independently distributed. Then $U = YZ$ is a Krätzel random variable with the Laplace transform

$$L_f(t) = \frac{1}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+\nu}{\rho}\right)} H_{2,1}^{1,2} \left[t \middle| \begin{matrix} (1-\alpha, 1), (1-\frac{\alpha+\nu}{\rho}, \frac{1}{\rho}) \\ (0, 1) \end{matrix} \right], \quad t > 0. \quad (2.12)$$

Proof The Laplace transform $L_f(t)$ of the density function (2.1) is given by

$$L_f(t) = \int_0^\infty e^{-tx} f(x) dx. \quad (2.13)$$

Using (2.1) and (2.13) and changing the order of integration, we have

$$L_f(t) = \frac{\rho}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+\nu}{\rho}\right)} \int_0^\infty u^{\nu-1} e^{-u\rho} \left\{ \int_0^\infty x^{\alpha-1} e^{-tx-\frac{x}{u}} dx \right\} du.$$

Making the substitution $v = \frac{x}{u}$ and using the gamma integral, we get

$$L_f(t) = \frac{\rho}{\Gamma\left(\frac{\alpha+\nu}{\rho}\right)} \int_0^\infty u^{\alpha+\nu-1} e^{-u\rho} (1+ut)^{-\alpha} du, \quad (1+ut) > 0. \quad (2.14)$$

We can evaluate this integral using Mellin convolution properties, for details see Appendix. Then the Laplace transform of the density function $f(x)$ can be obtained in terms of H -function and it is given by

$$L_f(t) = \frac{1}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+\nu}{\rho}\right)} H_{2,1}^{1,2} \left[t \left| \begin{matrix} (1-\alpha, 1), (1-\frac{\alpha+\nu}{\rho}, \frac{1}{\rho}) \\ (0, 1) \end{matrix} \right. \right], \quad t > 0. \quad (2.15)$$

□

2.2 Some Properties of the Krätzel Function

Log-concavity and log-convexity of functions and sequences in probability have been of interest to several authors, see for example Karlin [7]. Log-convexity is of interest in the study reliability and infinitely divisible random variables. In this section, we present log-convexity and complete monotonicity of the Krätzel density.

A function ϕ on $(0, \infty)$ is said to be completely monotone if ϕ has derivatives of all orders and satisfies

$$(-1)^n \phi^{(n)}(\lambda) \geq 0$$

for all $\lambda > 0$ and $n \in \{0, 1, \dots\}$, and as $\lambda \rightarrow 0$ the value of $\phi^{(n)}(\lambda)$ approaches a finite limit which we denote by $\phi^{(n)}(0)$. A function g on $(0, \infty)$ is said to be logarithmically convex, or simply log-convex, if its natural logarithm $\ln g$ is convex, that is, for all $x_1, x_2 > 0$ and $\alpha \in [0, 1]$, we have

$$g(\alpha x_1 + (1-\alpha)x_2) \leq [g(x_1)]^\alpha [g(x_2)]^{1-\alpha}.$$

Theorem 2.3 *If $\rho, \nu, \alpha \in (0, \infty)$ and $x \geq 0$, then the following assertions are true:*

a. *The Krätzel density $f(x; \alpha, \nu, \rho)$ satisfies the recurrence relation*

$$\nu f(x; \alpha, \nu, \rho) = (\alpha + \nu) f(x; \alpha, \rho + \nu, \rho) - \alpha f(x; \alpha + 1, \nu - 1, \rho). \quad (2.16)$$

- b. The density function $x \mapsto f(x)$ is completely monotonic on $(0, \infty)$ if $0 < \alpha \leq 1$.
 c. The function $x \mapsto f(x)$ is log-convex on $(0, \infty)$ if $0 < \alpha \leq 1$.

Proof (a) Using (2.1) and the recurrence relation of the Krätzel function [1],

$${}_vZ_\rho^v(x) = \rho Z_\rho^{v+\rho}(x) - x Z_\rho^{v-1}(x). \quad (2.17)$$

Now the rest of the derivation in this section is parallel to those in Princy [15]. For example, consider

$$\begin{aligned} {}_v f(x; \alpha, v, \rho) &= v \frac{\rho}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+v}{\rho}\right)} x^{\alpha-1} Z_\rho^v(x) \\ &= \frac{\rho x^{\alpha-1}}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+v}{\rho}\right)} [\rho Z_\rho^{v+\rho}(x) - x Z_\rho^{v-1}(x)] \\ &= \frac{\rho^2}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+v}{\rho}\right)} x^{\alpha-1} Z_\rho^{v+\rho}(x) - \frac{\rho}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+v}{\rho}\right)} x^\alpha Z_\rho^{v-1}(x) \\ &= \frac{\rho^2 \left(\frac{\alpha+v}{\rho}\right)}{\Gamma(\alpha) \left(\frac{\alpha+v}{\rho}\right) \Gamma\left(\frac{\alpha+v}{\rho}\right)} x^{\alpha-1} Z_\rho^{v+\rho}(x) - \frac{\rho \alpha}{\alpha \Gamma(\alpha) \Gamma\left(\frac{\alpha+v}{\rho}\right)} x^\alpha Z_\rho^{v-1}(x) \\ &= (\alpha + v) \frac{\rho}{\Gamma(\alpha) \Gamma\left(\frac{\alpha+v}{\rho} + 1\right)} x^{\alpha-1} Z_\rho^{v+\rho}(x) \\ &\quad - \alpha \frac{\rho}{\Gamma(\alpha + 1) \Gamma\left(\frac{\alpha+v}{\rho}\right)} x^\alpha Z_\rho^{v-1}(x) \\ &= (\alpha + v) f(x; \alpha, v + \rho, \rho) - \alpha f(x; \alpha + 1, v - 1, \rho). \end{aligned}$$

This implies that

$${}_v f(x; \alpha, v, \rho) = (\alpha + v) f(x; \alpha, v + \rho, \rho) - \alpha f(x; \alpha + 1, v - 1, \rho).$$

(b) Let

$$f(x) = g_1(x) g_2(x), \quad (2.18)$$

where $g_1(x) = cx^{\alpha-1}$, $g_2(x) = Z_\rho^v(x)$.

The function $g_1(x)$ is obviously completely monotonic because c is normalizing constant and $x^{\alpha-1}$ is completely monotonic on $0 < \alpha \leq 1$.

Next we consider

$$Z_\rho^v(x) = \int_0^\infty t^{v-1} e^{-t^\rho - \frac{x}{t}} dt. \quad (2.19)$$

The change of variable $\frac{1}{t} = s$ in (2.19) yields

$$Z_\rho^v(x) = \int_0^\infty (s^{-v-1} e^{-s^{-\rho}}) e^{-xs} ds.$$

That is, the Krätzel function $Z_{\rho}^{\nu}(x)$ is the Laplace transform of the function $s \mapsto s^{-\nu-1}e^{-s^{\rho}}$. Thus in the view of Bernstein–Widder theorem which implies that the function $x \mapsto Z_{\rho}^{\nu}(x)$ is completely monotonic, i.e., for all $n \in \{0, 1, 2, \dots\}$, $\nu, \rho \in (0, \infty)$ and $x > 0$, we have $(-1)^n [Z_{\rho}^{\nu}(x)]^{(n)} > 0$. This proves that $g_2(x)$ is completely monotonic.

Next we use the Leibnitz formula for the n^{th} derivative of a product, we have $(-1)^n f^{(n)}(x) \geq 0$. Therefore $x \mapsto f(x)$ is completely monotonic.

(c) Every completely monotonic functions is log-convex, see [[19], p. 167]. $\square$

2.3 Particular Cases of Interest

Some of the product distributions are special cases of Krätzel distribution as introduced in Sect. 1. The distributions of products of random variables are of great interest in many areas of science, especially in biology, physical sciences and econometrics. In traditional portfolio selection models, certain cases involve the product of random variables. The best example of this is in the case of investment in a number of different overseas markets. Some of the product distributions are special cases of Krätzel distribution. For instance, the following distributions arise as a particular case of (2.1):

1. When X and Y are independently distributed random variables having the gamma and Weibull distributions, respectively, then the density function of XY is obtained from (2.1) by letting $\alpha + \nu = \rho$. The product of gamma and Weibull random variables is studied in detail by Nadarajah and Kotz [14].
2. When X and Y are independently distributed random variables having the gamma and Rayleigh distributions, respectively, then the density function of XY is obtained from (2.1) by letting $\alpha + \nu = 2$ and $\rho = 2$. The product of gamma and Rayleigh random variables is studied in detail by Shakil and Kibria [17].
3. When X and Y are independently distributed random variables having one parameter gamma distributions, then the density function of XY is obtained from (2.1) by letting $\rho = 1$. The product of two gamma variables is studied in detail by Lorenzo Zaninetti [20]. This density function is used to describe the mass distribution of galaxies.

3 Evaluation of Densities of Products and Ratios

Theorem 3.1 *Let $X_i, i = 1, 2$ be two independently distributed random variables having Krätzel densities*

$$f(x_i) = \begin{cases} c_i x_i^{\alpha_i-1} Z_{\rho_i}^{\nu_i}(x_i), & x_i \geq 0, \rho_i > 0, \alpha_i > 0, \nu_i > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (3.1)$$

where

$$Z_{\rho_i}^{\nu_i}(x_i) = \int_0^{\infty} t^{\nu_i-1} e^{-t^{\rho_i} - \frac{x_i}{t}} dt$$

and $c_i = \frac{\rho_i}{\Gamma(\alpha_i) \Gamma(\frac{\alpha_i + v_i}{\rho_i})}$. Then the density of $Y = \frac{X_1}{X_2}$ is given by

$$g(y) = \left\{ \prod_{i=1}^2 \frac{1}{\Gamma(\alpha_i) \Gamma\left(\frac{\alpha_i + v_i}{\rho_i}\right)} \right\} H_{2,2}^{2,2} \left[y \middle| \begin{matrix} (-\alpha_2, 1), (1 - \frac{\alpha_2 + v_2 + 1}{\rho_2}, \frac{1}{\rho_2}) \\ (\alpha_1 - 1, 1), (\frac{\alpha_1 + v_1 - 1}{\rho_1}, \frac{1}{\rho_1}) \end{matrix} \right], \quad y \geq 0, \quad (3.2)$$

and the density of $Z = X_1 X_2$ is given by

$$h(z) = \left\{ \prod_{i=1}^2 \frac{1}{\Gamma(\alpha_i) \Gamma\left(\frac{\alpha_i + v_i}{\rho_i}\right)} \right\} H_{0,4}^{4,0} \left[z \middle| \begin{matrix} (\alpha_i - 1, 1), (\frac{\alpha_i + v_i - 1}{\rho_i}, \frac{1}{\rho_i}), i=1,2 \end{matrix} \right], \quad z \geq 0. \quad (3.3)$$

Proof Here we evaluate the density of $Y = \frac{X_1}{X_2}$ by the method of Mellin transform,

$$\begin{aligned} E(Y^{s-1}) &= E(X_1^{s-1}) E(X_2^{-(s-1)}) \\ &= \frac{\Gamma(\alpha_1 + s - 1) \Gamma(\alpha_2 - s + 1) \Gamma\left(\frac{\alpha_1 + v_1 + s - 1}{\rho_1}\right) \Gamma\left(\frac{\alpha_2 + v_2 - s + 1}{\rho_2}\right)}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma\left(\frac{\alpha_1 + v_1}{\rho_1}\right) \Gamma\left(\frac{\alpha_2 + v_2}{\rho_2}\right)}, \end{aligned} \quad (3.4)$$

$\Re(\alpha_i) > 0$, $\Re(\alpha_1 + s - 1) > 0$, $\Re(\alpha_2 - s + 1) > 0$, $\Re(\alpha_1 + v_1 + s - 1) > 0$, $\Re(\alpha_2 + v_2 - s + 1) > 0$, $\rho_i > 0$, $i = 1, 2$. By taking the inverse Mellin transform, we will get the density of $Y = \frac{X_1}{X_2}$ as in (3.2). Similarly, we can establish the density of $Z = X_1 X_2$. Note that for $\rho_i = 1$, $i = 1, 2$, is of the form $\rho_i = \frac{1}{m_i}$, $m_i = 1, 2, \dots$, then in (3.2) and (3.3) reduce to Meijer's G-functions $G_{2,2}^{2,2}$ and $G_{0,4}^{4,0}$ respectively. $\square$

4 Extension of the Krätzel Density Using Pathway Model

The scalar version of the pathway model of Mathai [13] is shown to be associated with a large number of probability models used in physics and statistics. The original pathway model of Mathai [13] is for the rectangular matrix case. The scalar version of the pathway model is as follows:

$$f_1(x) = c_1 |x|^\gamma [1 - a(1 - \alpha)|x|^\delta]^{-\frac{\eta}{1-\alpha}}, \quad -\infty < x < \infty \quad (4.1)$$

for $a > 0$, $\eta > 0$, $\delta > 0$, $\gamma > -1$ and c_1 is a constant. A statistical density can be created out of $f_1(x)$ under the additional conditions $1 - a(1 - \alpha)|x|^\delta > 0$ and then c_1 will act as a normalizing constant. Hence the model is applicable to random or non-random situations as well. For $\alpha < 1$, the model in (4.1) will stay in the generalized extended type-I beta family of functions. For $\alpha > 1$, writing $1 - \alpha = -(\alpha - 1)$ the model switches into the form

$$f_2(x) = c_2 |x|^\gamma [1 + a(\alpha - 1)|x|^\delta]^{-\frac{\eta}{\alpha-1}} \quad (4.2)$$

for $\alpha > 1$, $-\infty < x < \infty$, $a > 0$, $\eta > 0$, $\delta > 0$ which is a generalized extended type-II beta family of functions. Again taking c_2 as the normalizing constant, one can create a statistical density in $f_2(x)$. When α goes to 1 from the right or from the left, $f_1(x)$ and $f_2(x)$ will go into a generalized extended gamma family of functions, namely

$$f_3(x) = c_3 |x|^\nu e^{-a\eta|x|^\delta}, \quad (4.3)$$

where $a > 0$, $\eta > 0$, $\delta > 0$. The model in (27) covers almost all statistical densities in current use in statistics, physics, engineering and other areas. Here we extend the Krätzel density using the pathway model.

An extended form of the Krätzel density is given by

$$f_4(x) = \begin{cases} c_4 x^{\alpha-1} Z_{\rho,\delta}^{\nu,\beta}(x), & x \geq 0, \alpha > 0, \nu > 0, \rho > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (4.4)$$

where

$$Z_{\rho,\delta}^{\nu,\beta}(x) = \int_0^\infty t^{\nu-1} [1 + (\beta - 1)t^\rho]^{-\frac{1}{\beta-1}} e^{-\frac{x}{t^\delta}} dt \quad (4.5)$$

and c_4 is the normalizing constant.

More families are available when the variable is allowed to vary over the real line. So $f_4(x)$ is a more general class of the density function. When $\alpha = 1$, the density function $f_4(x)$ becomes

$$f_5(x) = \begin{cases} c_5 Z_{\rho,\delta}^{\nu,\beta}(x), & x \geq 0, \nu > 0, \rho > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (4.6)$$

where $Z_{\rho,\delta}^{\nu,\beta}(x)$ is given in (4.5).

The graph of the extended form of the Krätzel density for various values of parameters are given in the following figures (Fig. 3).

The integral involved in the density function (4.6) is connected to nuclear reaction rate theory in the non-resonant case, Tsallis statistics in non-extensive statistical mechanics, superstatistics in astrophysics, generalized type-1 and type-2 beta, gamma families of densities, density of a product of two real positive random variables, Krätzel integral in applied analysis, inverse Gaussian density in stochastic processes, etc. Special cases include a wide range of functions appearing in different disciplines.

When $\alpha = 1$ and $\beta \rightarrow 1$, the density function $f_4(x)$ becomes

$$f_6(x) = \begin{cases} c_6 Z_{\rho,\delta}^{\nu,1}(x), & x \geq 0, \nu > 0, \rho > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (4.7)$$

where

$$Z_{\rho,\delta}^{\nu,1}(x) = \int_0^\infty t^{\nu-1} e^{-t^\rho - \frac{x}{t^\delta}} dt. \quad (4.8)$$

Fig. 3 The graph of the extended form of the Krätzel density for various values of parameters

When $\delta = -\eta$, $\eta > 0$ in $f_6(x)$ the integral involved in the density function is denoted by I_2 and it is given by

$$I_2 = \int_0^\infty t^{\nu-1} e^{-t^\rho - xt^\eta} dt. \quad (4.9)$$

This integral can be evaluated using the convolution properties of Mellin transform, we get

$$I_2 = \int_0^\infty t^{\nu-1} e^{-t^\rho - xt^\eta} dt = \frac{1}{\rho\eta} H_{1,1}^{1,1} \left[x^{\frac{1}{\eta}} \left| \begin{matrix} (1-\frac{\nu}{\rho}, \frac{1}{\rho}) \\ (0, \frac{1}{\eta}) \end{matrix} \right. \right], \quad x > 0. \quad (4.10)$$

From this, we can make connection to Laplace transform of some related densities. Some of them are listed below:

- For $\eta = 1$ in (4.9), I_2 becomes the Laplace transform of the generalized gamma density with Laplace parameter x . When x is replaced by $-x$, we have the moment generating function there. Similar remarks apply to the following cases also.
- For $\eta = 1$, $\nu = \rho$ in (4.9), I_2 becomes the Laplace transform of Weibull density.
- For $\eta = 1$, $\rho = 1$ in (4.9), I_2 becomes the Laplace transform of the gamma density.
- For $\eta = 1$, $\nu = 3$, $\rho = 2$ in (4.9), I_2 becomes the Laplace transform of Maxwell-Boltzmann density.

4.1 Some Applications

- For $\alpha = 1$, $f(x)$ gives the Krätzel integral

$$f(x) = \int_0^\infty t^{v-1} e^{-t^\rho - \frac{x}{t}} dt, \quad x \geq 0, \quad (4.11)$$

which was studied in detail by Krätzel [9]. An additional property is that it can be written as H -function of the type $H_{0,2}^{2,0}(\cdot)$.

- Over the last decade, significant progress has been made toward the development of widely applicable radar clutter models. The K-distribution has proved to be an appropriate model for characterizing the amplitude of microwave sea clutter [6]. The K-distribution is a model for the statistics of synthetic aperture radar imagery. The K-distribution that appears in different areas such as radiation, ultrasonic backscatter, neural networks, etc was studied by many authors. One form of the K-distribution is as follows:

$$g_1(u) = \frac{2}{\Gamma(v+1)} \left(\frac{u}{2}\right)^v K_v(u), \quad u \geq 0. \quad (4.12)$$

For $\rho = 1$, $\alpha = 1$ and $x = \frac{u^2}{4}$ in (2.1), the Krätzel density function coincides with the K-distribution. Hence we can say that Krätzel distribution is a new generalization of the K-distribution.

- In a series of papers, Haubold and Mathai studied modifications of Maxwell Boltzmann theory of reaction rates. The basic reaction rate probability integral has the following form

$$I_3 = \int_0^\infty t^{\gamma-1} e^{-at-zt^{-\frac{1}{2}}} dt, \quad (4.13)$$

which is the non-resonant case of nuclear reactions. Compare integral (4.13) with (4.7). The reaction rate probability integral (4.13) is (4.7) for $\rho = 1$, $\delta = \frac{1}{2}$, $v = \gamma$ and $x = z$. The basic integral (4.13) is generalized in many different forms for various situations of resonant and non-resonant cases of reactions, depletion of high energy tail, cut-off of the high energy tail, and so on, see for example Mathai and Haubold [11, 12], Haubold and Kumar [5].

5 Application in Solar Radiation Data

The probability model in (2.1) has application in the solar modeling. In the following section, we discuss how well this model describes the solar radiation data.

5.1 Data Analysis

Data were collected from NREL-National Renewable Energy Laboratory, U.S. Department Energy. The data consist of wavelength and the corresponding solar radiance and they consist of 2000 observations. Here, mathematical software MAPLE and MATLAB are used for the data analysis. The summary of the statistics obtained from the data is as follows:

Fig. 4 The graph of the probability model fitted to the solar radiation data

$$\begin{aligned}\text{Mean value} &= 624.7485254, \\ \text{Variance} &= 746556.501.\end{aligned}$$

Fixing $\rho = 1$ and using the method of moments, the estimates of the parameters in the Krätzel density function are obtained as follows:

$$\begin{aligned}\hat{\alpha} &= 5.1718, \\ \hat{\nu} &= 1.1955.\end{aligned}$$

Figure 3 shows the histogram of the data embedded within the probability model.

Using χ^2 -test, we obtained the calculated χ^2 value as 5.451 and the tabulated value as 11.07. Hence, we conclude that the model is consistent for the data (Fig. 4).

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Appendix

H -function is defined as the following Mellin–Barnes integral:

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \phi(s) z^{-s} ds, \quad i = \sqrt{-1}, \quad (6.1)$$

where

$$\phi(s) = \frac{\{\prod_{j=1}^m \Gamma(b_j + \beta_j s)\} \{\prod_{j=1}^n \Gamma(1 - a_j - \alpha_j s)\}}{\{\prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)\} \{\prod_{j=n+1}^p \Gamma(a_j + \alpha_j s)\}}, \quad (6.2)$$

where a_j , $j = 1, 2, \dots, p$ and b_j , $j = 1, 2, \dots, q$ are complex numbers, α_j , $j = 1, 2, \dots, p$ and β_j , $j = 1, 2, \dots, q$ are positive real numbers, and L is a contour separating the poles of $\Gamma(b_j + \beta_j s)$, $j = 1, 2, \dots, m$ from those of $\Gamma(1 - a_j - \alpha_j s)$, $j = 1, 2, \dots, n$. Convergence conditions, properties and applications of H -function in various disciplines are available in the literature. Computable representations and series forms are also available. For example, see [10]. When $\alpha_j = 1$, $j = 1, 2, \dots, p$ and $\beta_j = 1$, $j = 1, 2, \dots, q$, the H -function in (6.1) reduces to Meijer's function $G_{p,q}^{m,n} \left[z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right]$.

Evaluation of (2.14). Let

$$I_1 = \int_0^\infty u^{\alpha+\nu-1} e^{-u^\rho} (1+ut)^{-\alpha} du. \quad (6.3)$$

We can evaluate this integral using Mellin convolution properties, by taking it as a statistical distribution problem for the sake or novelty. Let X_1 and X_2 be independently distributed real scalar positive random variables. Let the densities be $h_1(x_1)$ and $h_2(x_2)$,

respectively. Consider the transformation $W = \frac{X_1}{X_2}$ and $V = X_2$. Then the density of W is denoted by

$$g(w) = \int_v v h_1(wv) h_2(v) dv. \quad (6.4)$$

Also,

$$E(W^{s-1}) = E(X_1^{s-1}) E(X_2^{1-s}) \quad (6.5)$$

due to independence of X_1 and X_2 . But when $h_1(x_1)$ and $h_2(x_2)$ are densities and when X_1 and X_2 are real positive variables, then $E(X_1^{s-1}) = M_{h_1}(s)$ and $E(X_2^{1-s}) = M_{h_2}(2-s)$. Let

$$h_1(x_1) = \begin{cases} c_1 (1+x_1)^{-\alpha}, & x_1 \geq 0, \alpha > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (6.6)$$

and

$$h_2(x_2) = \begin{cases} c_2 x_2^{\alpha+\nu-2} e^{-x_2^\rho}, & x_2 \geq 0, \rho > 0, \alpha > 0, \nu > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (6.7)$$

where c_1 and c_2 are the normalizing constants. Then the density of $W = \frac{X_1}{X_2}$ is given by

$$g(w) = c_1 c_2 \int_0^\infty v v^{\alpha+\nu-2} e^{-v^\rho} (1+wv)^{-\alpha} dv. \quad (6.8)$$

Comparing (6.3) and (6.8), we see that the integral I_1 is available from (6.8) for $w = t$. Now let us evaluate the density using Mellin transform. Therefore the density of u is given by

$$g(w) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_{h_1}(s) M_{h_2}(2-s) w^{-s} ds. \quad (6.9)$$

But

$$E(X_1^{s-1}) = c_1 \frac{\Gamma(s)\Gamma(\alpha-s)}{\Gamma(\alpha)}, \quad \text{for } \Re(s) > 0, \Re(\alpha-s) > 0, \quad (6.10)$$

$$E(X_2^{1-s}) = \frac{c_2}{\rho} \Gamma\left(\frac{\alpha+\nu-s}{\rho}\right), \quad \text{for } \Re(\alpha+\nu-s) > 0, \rho > 0. \quad (6.11)$$

Hence from (6.9)–(6.11),

$$\int_0^\infty v v^{\alpha+\nu-2} e^{-v^\rho} (1+wv)^{-\alpha} dv = \frac{1}{\rho \Gamma(\alpha)} H_{2,1}^{1,2} \left[w \left| \begin{matrix} (1-\alpha, 1), (1-\frac{\alpha+\nu}{\rho}, \frac{1}{\rho}) \\ (0, 1) \end{matrix} \right. \right]. \quad (6.12)$$

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