The 2D Euler–Boussinesq Equations in Planar Polygonal Domains with Yudovich’s Type Data

Aimin Huang

Received: 24 November 2014 / Accepted: 9 February 2015 / Published online: 6 March 2015
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Abstract We address the well-posedness of the 2D (Euler)–Boussinesq equations with zero viscosity and positive diffusivity in the polygonal-like domains with Yudovich’s type data, which gives a positive answer to part of the questions raised in Lai (Arch Ration Mech Anal 199(3):739–760, 2011). Our analysis on the the polygonal-like domains essentially relies on the recent elliptic regularity results for such domains proved in Bardos et al. (J Math Anal Appl 407(1):69–89, 2013) and Di Plinio (SIAM J Math Anal 47(1):159–178, 2015).

Keywords Boussinesq system · Euler equations · Existence and uniqueness · Yudovich’s type data · Initial-boundary value problem

Mathematics Subject Classification (2010) 35Q30 · 35Q31 · 34A12 · 76N10

1 Introduction

Motivated by the well-posedness results for the 2D Euler equations in non-smooth domains in [2, 3] and the questions about the Boussinesq system over non-smooth domains raised in [1, Sect. 4], we aim in this article to address the global well-posedness of the 2D Euler–Boussinesq equations in a non-smooth domain $\Omega \subset \mathbb{R}^2$ of polygonal type. The 2D Euler–Boussinesq equations describing the evolution of mass and heat flow of an inviscid incompressible fluid read in the non-dimensional form:

A. Huang (✉)
The Institute for Scientific Computing and Applied Mathematics, Indiana University,
831 East Third Street, Rawles Hall, Bloomington, IN 47405, USA
e-mail: huangepn@gmail.com

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = T \mathbf{e}_2, & \mathbf{e}_2 = (0, 1), \\ \operatorname{div} \mathbf{u} = 0, \\ \partial_t T - \kappa \Delta T + \mathbf{u} \cdot \nabla T = 0, \end{cases} \quad (1.1)$$

where $(x, y) \in \Omega$, $t \in (0, t_1)$, $\mathbf{u} = (u_1, u_2)$ and T denote the velocity field and the temperature of the fluid, respectively, π stands for the pressure, and $\kappa > 0$ is the thermal diffusivity. We associate with (1.1) the following initial and boundary conditions:

$$\begin{cases} \mathbf{u}(0, x, y) = \mathbf{u}_0(x, y), & T(0, x, y) = T_0(x, y), \\ \mathbf{u}(t, x, y) \cdot \mathbf{n} = 0, & (x, y) \in \partial\Omega, \\ T = \eta, & (x, y) \in \partial\Omega, \end{cases} \quad (1.2)$$

where $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$ and $\mathbf{u}_0$, T_0 and η are the given initial and boundary data. We also denote by $\boldsymbol{\tau}$ the unit tangent vector to $\partial\Omega$.

The general 2D Boussinesq system with full viscosity ν and diffusivity κ reads

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = T \mathbf{e}_2, & \mathbf{e}_2 = (0, 1), \\ \operatorname{div} \mathbf{u} = 0, \\ \partial_t T - \kappa \Delta T + \mathbf{u} \cdot \nabla T = 0. \end{cases}$$

From the mathematical point of view, the global well-posedness and global regularity of the Boussinesq system as well as the existence of the global attractor in presence of viscosity have been widely studied, see for example [4–15]. Recently, there are many works devoted to the study of the 2D Boussinesq system with partial viscosity, see for example [16–21] in the whole space $\mathbb{R}^2$ and [1, 22, 23] in bounded smooth domains. There are also many works which considered the case when only the horizontal viscosity or vertical viscosity is present, see for example [24–27]. However, the global regularity for the 2D Boussinesq system when $\nu = \kappa = 0$ is still an outstanding open problem, and to the best of our knowledge, the well-posedness issue regarding the 2D Euler–Boussinesq system (1.1) in non-smooth domains has not yet been addressed in the literature, which is the goal of this article. In some realistic applications, the variation of the fluid viscosity and thermal diffusivity with the temperature may not be disregarded (see for example [28] and references therein) and there are many works on this direction too, see for example [28–32] where the existence of weak solutions, global regularity, and existence of global attractor have been studied.

The 2D Boussinesq system also has close connection to the fundamental fluid models, for example, the Euler equations. It is well known that the standard 2D Euler equations are globally well posed if the initial data satisfy the Yudovich’s type condition, see [33–35]. Roughly speaking, if the initial vorticity is bounded or unbounded but with small growth rate of the L^p -norm, then the 2D Euler equations possess a global unique solution and recently this result has been extended to non-smooth domains in [2, 3]. Note that the global well-posedness for the 2D Euler–Boussinesq system has been studied in [17] with Yudovich’s type data for the whole space $\mathbb{R}^2$ and also studied in [23] with H^3 -regular data for bounded smooth domains. Here, we would like to establish the global well-posedness result for the 2D Euler–Boussinesq system in non-

smooth domains with Yudovich’s type data, which generalizes the results in [17, 23] and gives a definite answer to part of the questions asked in [1]. We also remark that the author in [23] only studied the case when the boundary data are constant, while here we will consider arbitrary boundary data for the 2D Euler–Boussinesq system.

In this article, we are interested in the polygonal-like (non-smooth) domains with maximum aperture $\max \alpha_j \leq \pi/2$ because the elliptic regularity results are only available for such domains (see (2.7) below). Here, a domain $\Omega \subset \mathbb{R}^2$ is said to be a polygonal-like domain if it is a bounded simply connected open set and the boundary $\partial\Omega$ is enclosed by piecewise $\mathcal{C}^{1,1}$ planar curves, with finitely many points $\{O_j\}_{j=1}^N$ of discontinuity for the tangent vector, and such that, in some neighborhood of each point O_j , Ω coincides with the cone of vertex O_j and aperture $\alpha_j \in (0, 2\pi)$.

In order to deal with the non-homogeneous boundary conditions on $\partial\Omega$ for the temperature T , we need to use classical lifting results (see for example [36, Theorem 1.5.2.3, Theorem 1.5.2.8]). But in order to avoid the technical conditions for η at the corner points of the domain Ω , we assume that the boundary data η are inferred from a function S defined on Ω , that is

$$S = \eta, \quad \text{on } \partial\Omega.$$

For the sake of simplicity, we also assume that the boundary data η and hence the function S are independent of time t .

Following a traditional approach (see for example [14, 37]), we recast the 2D Euler–Boussinesq system in terms of the perturbative variable (perturbation away from the stationary state $(0, S)$); namely we set

$$(\mathbf{u}, \theta) = (\mathbf{u}, T - S).$$

In the perturbative variables, the 2D Euler–Boussinesq system (1.1)–(1.2) reads

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \theta \mathbf{e}_2 + S \mathbf{e}_2, & \mathbf{e}_2 = (0, 1), \\ \operatorname{div} \mathbf{u} = 0, \\ \partial_t \theta - \Delta \theta + \mathbf{u} \cdot \nabla \theta + \mathbf{u} \cdot \nabla S = \Delta S, \end{cases} \quad (1.3)$$

with the initial and boundary conditions

$$\begin{cases} \mathbf{u}(0, x, y) = \mathbf{u}_0(x, y), & \theta(0, x, y) = \theta_0(x, y) := T_0(x, y) - S(x, y), \\ \mathbf{u}(t, x, y) \cdot \mathbf{n} = 0, & \theta(t, x, y) = 0, \quad (x, y) \in \partial\Omega. \end{cases} \quad (1.4)$$

Note that we have set the diffusivity $\kappa = 1$ in (1.3) for simplicity.

The rest of the article is organized as follows. At the end of this introduction, we introduce the notion of quasi-strong solution for the 2D Euler–Boussinesq system (1.3)–(1.4) and state our main result. We prove the existence of the quasi-strong solution by the vanishing viscosity method, which was used by Bardos in [38] to study the 2D Euler equations. In Sect. 2, we collect the necessary tools for the analysis of the Boussinesq system in the polygonal-like domains. Section 3 is devoted to prove the

uniform estimates for the approximated solutions constructed by the vanishing viscosity method. Finally in Sect. 4, we prove the main result, Theorem 1.1 below, that is the existence of the quasi-strong solution and also the regularity and uniqueness of the solutions for the 2D Euler–Boussinesq system (1.3)–(1.4). The proof of the uniqueness follows Yudovich’s energy method and relies on the endpoint $L^\infty(\Omega) \rightarrow L^{\gamma_{\text{exp}}}(\Omega)$ regularity result for the solution to the Dirichlet problem in the polygonal-like domains. In Appendix 1, we recast the standard L^p -estimate for the 2D Euler equations using the vorticity formulation.

1.1 Definition of the Quasi-Strong Solution and the Main Result

In order to set up the framework of how to study the 2D Euler–Boussinesq system (1.3)–(1.4), we recall the classical space

$$V = \left\{ \mathbf{u} \in H^1(\Omega) : \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \right\},$$

and we say that a couple $(\mathbf{u}, \theta)$ satisfying

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, t_1; V), \quad \partial_t \mathbf{u} \in L^2(0, t_1; L^{3/2}(\Omega)); \\ \theta &\in \mathcal{C}([0, t_1]; H_0^1(\Omega)) \cap L^2(0, t_1; H^2(\Omega)), \quad \partial_t \theta \in L^2(0, t_1; L^2(\Omega)), \end{aligned} \quad (1.5)$$

is a quasi-strong solution of the problem (1.3)–(1.4) if

$$\begin{aligned} & - \int_0^{t_1} \langle \mathbf{u}(t), \tilde{\mathbf{u}} \rangle_{L^2} \psi'(t) dt + \int_0^{t_1} \langle \mathbf{u}(t) \cdot \nabla \mathbf{u}(t), \tilde{\mathbf{u}} \rangle_{L^2} \psi(t) dt \\ & = \langle \mathbf{u}_0, \tilde{\mathbf{u}} \rangle \psi(0) + \int_0^{t_1} \langle \theta \mathbf{e}_2 + S \mathbf{e}_2, \tilde{\mathbf{u}} \rangle_{L^2} \psi(t) dt, \end{aligned}$$

for all $\tilde{\mathbf{u}} \in L_\tau^3(\Omega)$ and $\psi \in \mathcal{C}^1([0, t_1])$ with $\psi(t_1) = 0$, and

$$\begin{aligned} & - \int_0^{t_1} \langle \theta, \tilde{\theta} \rangle_{L^2} \varphi'(t) dt - \int_0^{t_1} \langle \Delta \theta, \tilde{\theta} \rangle_{L^2} \varphi(t) dt + \int_0^{t_1} \langle \mathbf{u} \cdot \nabla (\theta + S), \tilde{\theta} \rangle_{L^2} \varphi(t) dt \\ & = \langle \theta_0, \tilde{\theta} \rangle \varphi(0) + \int_0^{t_1} \langle \Delta S, \tilde{\theta} \rangle_{L^2} \varphi(t) dt, \end{aligned}$$

for all $\tilde{\theta} \in L^2(\Omega)$ and $\varphi \in \mathcal{C}^1([0, t_1])$ with $\varphi(t_1) = 0$. For the meaning of the notation $L_\tau^p(\Omega)$ ($1 < p < \infty$), see Sect. 2.

The existence of a global weak solution when the boundary data η and hence S are constants is obtained via the fixed point theory in [23]. It seems that the fixed point arguments could not be adapted to the case of arbitrary boundary data. Here, we are going to utilize the vanishing viscosity method to prove the existence of a global quasi-strong solution and furthermore prove the global well-posedness of the 2D Euler–Boussinesq (1.3)–(1.4) with Yudovich’s type data. We now state the main result of this article, with the proof presented in the Sects. 3–4.

Theorem 1.1 *Let Ω be a polygonal-like domain (piecewise $C^{1,1}$ -boundary) with maximum aperture $\alpha_j \leq \pi/2$ and let there be given $S \in H^2(\Omega)$, $\mathbf{u}_0 \in V$, $\theta_0 \in H_0^1(\Omega)$, and $t_1 > 0$. Then there exists a global quasi-strong solution $(\mathbf{u}, \theta) \in \mathcal{C}([0, t_1]; L_\tau^2(\Omega)) \times \mathcal{C}([0, t_1]; H_0^1(\Omega))$ of the 2D Euler–Boussinesq system (1.3)–(1.4) such that the following estimates hold:*

$$\begin{cases} \|\mathbf{u}\|_{L^\infty(0,t_1;V)} + \|\theta\|_{L^\infty(0,t_1;H_0^1(\Omega))} + \|\theta\|_{L^2(0,t_1;H^2(\Omega))} \leq \mathcal{Q}_2, \\ \|\mathbf{u}_t\|_{L^2(0,t_1;L^{3/2}(\Omega))} + \|\theta_t\|_{L^2(0,t_1;L^2(\Omega))} \leq \mathcal{Q}_2, \end{cases} \quad (1.6)$$

where $\mathcal{Q}_2$ is a positive function defined by

$$\mathcal{Q}_2 := \mathcal{Q}_2(t_1, \|\mathbf{u}_0\|_{H^1}, \|\theta_0\|_{H^1}, \|S\|_{H^2}),$$

which is increasing in all its arguments.

Furthermore, if we additionally assume $\omega_0 = \operatorname{curl} \mathbf{u}_0 \in L^\infty(\Omega)$, $\theta_0 \in H^2(\Omega)$, and $S \in H^3(\Omega)$, then there exists a unique solution $(\mathbf{u}, \theta)$ of the 2D Euler–Boussinesq system (1.3)–(1.4) satisfying

$$\begin{aligned} \omega &= \operatorname{curl} \mathbf{u} \in L^\infty(0, t_1; L^\infty(\Omega)), \quad \theta \in \mathcal{C}([0, t_1]; H^2(\Omega)) \cap L^2(0, t_1; H^3(\Omega)), \\ \theta_t &\in L^\infty(0, t_1; L^2(\Omega)) \cap L^2(0, t_1; H^1(\Omega)), \end{aligned}$$

and the estimates

$$\begin{aligned} \|\theta\|_{L^\infty(0,t_1;H^2(\Omega))} + \|\theta\|_{L^2(0,t_1;H^3(\Omega))} &\leq \mathcal{Q}_3, \\ \|\theta_t\|_{L^\infty(0,t_1;L^2(\Omega))} + \|\theta_t\|_{L^2(0,t_1;H^1(\Omega))} &\leq \mathcal{Q}_3, \\ \|\omega\|_{L^\infty(\Omega \times (0,t_1))} &\leq \mathcal{Q}_4, \end{aligned} \quad (1.7)$$

where $\mathcal{Q}_3$ and $\mathcal{Q}_4$ are positive functions defined by

$$\mathcal{Q}_3 := \mathcal{Q}_3(t_1, \|\omega_0\|_{L^4}, \|\theta_0\|_{H^2}, \|S\|_{H^3}), \quad \mathcal{Q}_4 := \mathcal{Q}_4(t_1, \|\omega_0\|_{L^\infty(\Omega)}, \|\theta_0\|_{H^2}, \|S\|_{H^3}),$$

which are increasing in all their arguments.

Remark 1.1 We first note that the regularity of θ in Theorem 1.1 only depends on the L^4 -norm of the initial vorticity ω_0 and hence, as in [35], the estimate (2.3) can be used to show the uniqueness part of Theorem 1.1 under an assumption weaker than $\omega_0 \in L^\infty(\Omega)$, including unbounded initial vorticity with controlled growth rate of the L^p -norm of ω_0 as $p \rightarrow \infty$. For instance, one can take

$$\omega_0 \in \bigcap_{1 < p < \infty} L^p(\Omega), \quad \text{with} \quad \sup_{p > e^e} \frac{\|\omega_0\|_{L^p}}{\log \log p} < \infty;$$

see [35, Sect. 5] for a precise definition of the class of allowed data.

2 Notations and Preliminaries

Here and throughout this article, we will not distinguish the notations for vector and scalar function spaces whenever they are self-evident from the context. For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by $W^{s,p}(\Omega)$ (resp. $H^s(\Omega)$) the classical Sobolev space of order s on Ω with norm $\|\cdot\|_{W^{s,p}}$ (resp. $\|\cdot\|_{H^s}$), by $W_0^{s,p}(\Omega)$ (resp. $H_0^s(\Omega)$) the closure of $\mathcal{D}(\Omega)$ in the space $W^{s,p}(\Omega)$ (resp. $H^s(\Omega)$) when $s > 0$, and by $L^p(\Omega)$ the classical L^p -Lebesgue space with norm $\|\cdot\|_{L^p}$. For simplicity, we reserve the notation $\|\cdot\|$ for the L^2 -norm.

In this article, we denote by $Q_i(\cdot)$ ($i = 1, 2, \dots$) the positive increasing functions in all their arguments, which may vary from line to line. The symbol C denotes a generic positive constant, which may depend on the domain Ω , but is independent of the data $\mathbf{u}_0$, θ_0 , and S and of the time t_1 .

2.1 L^p -Tangential Vector Fields and Helmholtz Decomposition

We also introduce the L^p -tangential vector fields space as in [2, Sect. 2.2.1]:

$$L_{\tau}^p(\Omega) = \{\mathbf{u} \in L^p(\Omega) : \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega\}, \quad 1 < p < \infty,$$

and the space of smooth functions

$$\mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega) : \operatorname{div} \mathbf{u} = 0\}.$$

It is well known that for general Lipschitz domains (see for example [39, Theorem I.1.4]), the space $\mathcal{V}$ is dense in $L_{\tau}^2(\Omega)$ and

$$L_{\tau}^2(\Omega)^{\perp} = \{\nabla \pi : \pi \in H^1(\Omega)\}.$$

Let us denote by $P_{\Omega} : L^2(\Omega) \rightarrow L_{\tau}^2(\Omega)$ the corresponding orthogonal projection operator. Recently, P_{Ω} has been shown to extend to a bounded linear operator on $L^p(\Omega)$ for bounded convex domains (see [40, Theorem 1.3]) with $1 < p < \infty$ and for general Lipschitz domains with the range $p \in (3/2 - \epsilon, 3 + \epsilon)$ (see [41]). Here, we collect those results as follows.

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^2$ be a polygonal-like domain (piecewise $C^{1,1}$ -boundary¹). Then there holds*

- (i) *For $p \in (1, \infty)$, the space $\mathcal{V}$ is dense in $L_{\tau}^p(\Omega)$;*
- (ii) *For $p \in [3/2, 3]$, the operator P_{Ω} is extended to be a bounded linear projection operator from $L^p(\Omega)$ to $L_{\tau}^p(\Omega)$ with the operator norm only depending on p and the domain Ω ;*

¹ In Proposition 2.1, the aperture α_j for the domain Ω could be arbitrary (i.e., any number between 0 and 2π).

- (iii) For $p \in [3/2, 3]$ and for each $\mathbf{v} \in L^p(\Omega)$, there exists π belonging to the space $W^{1,p}(\Omega)$, unique up to an additive constant such that

$$\mathbf{P}_\Omega^\perp \mathbf{v} := (1 - \mathbf{P}_\Omega)\mathbf{v} = \nabla \pi, \quad (2.1)$$

and with the estimate

$$\max \{ \|\mathbf{P}_\Omega \mathbf{v}\|_{L^p}, \|\nabla \pi\|_{L^p} \} \leq C_{p,\Omega} \|\mathbf{v}\|_{L^p},$$

where $C_{p,\Omega} > 0$ depends only on p and on the domain Ω .

In Proposition 2.1, item (i) is contained in [40, Lemma 6.1], and items (ii) and (iii) are proved in [41].

2.2 The Dirichlet Problem and the Biot–Savart Law

Let $F = \mathbf{G}_\Omega f$ be the solution of the Dirichlet problem

$$-\Delta F = f, \quad F|_{\partial\Omega} = 0. \quad (2.2)$$

The Lax–Milgram lemma tells us that, if $f \in H^{-1}(\Omega)$, then there exists a unique $F \in H_0^1(\Omega)$ denoted by $\mathbf{G}_\Omega f$ satisfying (2.2) in the distributional sense. If we further assume $f \in L^p(\Omega)$ with $p \geq 2$, then the elliptic regularity result in [36], which is improved in [3, Theorem 1], for the polygonal-like domains with maximum aperture $\max \alpha_j \leq \pi/2$ guarantees that $\mathbf{G}_\Omega f$ still has two derivatives in $L^p(\Omega)$ and the following estimate holds:

$$\mathbf{G}_\Omega f \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad \|\mathbf{G}_\Omega f\|_{W^{2,p}} \leq C_\Omega p \|f\|_{L^p}, \quad 2 \leq p < \infty, \quad (2.3)$$

where C_Ω only depends on Ω . We also infer from [36, Theorem 5.1.1.4] that

$$\|\mathbf{G}_\Omega f\|_{H^3} \leq C_\Omega \|f\|_{H^1}, \quad \forall f \in H_0^1(\Omega). \quad (2.4)$$

We now set

$$\mathbf{K}_\Omega := \nabla^\perp(\mathbf{G}_\Omega f), \quad f \in H^{-1}(\Omega),$$

where $\nabla^\perp = (\partial_y, -\partial_x)$. Then the Biot–Savart law reads that for all $2 \leq p < \infty$, there holds

$$\begin{cases} \mathbf{K}_\Omega \in \mathcal{L}(H^{-1}(\Omega), L_\tau^2(\Omega)), \\ \mathbf{K}_\Omega \in \mathcal{L}(L^p(\Omega), W^{1,p}(\Omega) \cap L_\tau^2(\Omega)), \\ \mathbf{K}_\Omega \in \mathcal{L}(H_0^1(\Omega), H^2(\Omega) \cap L_\tau^2(\Omega)). \end{cases} \quad (2.5)$$

To prove (2.5), due to the regularity estimates (2.3)–(2.4), we only need to verify that

$$\operatorname{div} \mathbf{K}_\Omega f = 0, \quad \mathbf{K}_\Omega f \cdot \mathbf{n} = 0, \quad \forall f \in H^{-1}(\Omega),$$

which follows from the fact that $\mathcal{V}$ is dense in $L^2_\tau(\Omega)$ and the following identity:

$$\begin{aligned}\langle \mathbf{K}_\Omega f, \nabla \varphi \rangle_\Omega &= \langle \nabla(\mathbf{G}_\Omega f), \nabla^\perp \varphi \rangle_\Omega \\ &= -\langle \mathbf{G}_\Omega f, \operatorname{div} \nabla^\perp \varphi \rangle_\Omega + \int_{\partial\Omega} (\mathbf{G}_\Omega f) \nabla^\perp \varphi \cdot \mathbf{n} = 0, \quad \forall \varphi \in \mathcal{D}(\overline{\Omega}).\end{aligned}$$

2.3 Elliptic Regularity at $p \rightarrow \infty$

In order to extend the elliptic regularity (2.3) to the end point when $p \rightarrow \infty$, one needs to work with the Orlicz spaces, where the elliptic regularity result in these spaces was recently proved in [3] for the polygonal-like domains.

2.3.1 The Orlicz Spaces

Here, we briefly recall some preliminaries on the Orlicz spaces; see [42, 43] for more details. A function $\gamma : [0, \infty) \mapsto [0, \infty)$ is said to be a Young function if

- (1) γ is increasing and $\gamma(0) = 0$, $\lim_{s \rightarrow \infty} \gamma(s) = \infty$;
- (2) γ is a convex lower-semicontinuous $[0, \infty]$ -valued function on $\mathbb{R}$;
- (3) γ is non-trivial, that is there exists a number $0 < s_0 < \infty$ such that $0 < \gamma(s_0) < \infty$.

The convex conjugate γ^* of a γ is defined by

$$\gamma^*(t) := \sup\{st - \gamma(s), \quad s \geq 0\},$$

and one can show that γ is a Young function if and only if γ^* is a Young function. The convex conjugacy allows us to obtain the Orlicz space version of Hölder's inequality. The typical examples of Young functions are

$$\gamma_p(s) = s^p, \quad p > 1, \quad \gamma_{\exp}(s) = e^s - 1$$

and their corresponding convex conjugates:

$$\gamma_p(t) = t^{p'}, \quad p' = \frac{p}{p-1}, \quad \gamma_{\exp}^*(t) = \begin{cases} t \ln t - t + 1, & \forall t \geq 1, \\ 0, & 0 \leq t \leq 1. \end{cases}$$

We now define the Orlicz spaces $L^\gamma(\Omega)$ to be the set of all measurable functions such that the Luxemburg norm is finite, that is

$$L^\gamma(\Omega) = \{f \text{ is a measurable function on } \Omega : \|f\|_{L^\gamma} < \infty\},$$

where the Luxemburg norm $\|f\|_{L^\gamma}$ is defined by

$$\|f\|_{L^\gamma} := \inf\{\lambda > 0 : \int_\Omega \gamma(|f|/\lambda) dx dy \leq 1\}.$$

One can easily verify that $L^{\gamma p}(\Omega) = L^p(\Omega)$ for all $p > 1$ and we also have the following Hölder’s inequality for the Orlicz spaces.

Proposition 2.2 *Let $\gamma = \gamma(s)$ be a Young function. Then the space $L^\gamma(\Omega)$ with the norm $\|\cdot\|_\gamma$ is a Banach space and for all $f \in L^\gamma(\Omega)$ and $g \in L^{\gamma^*}(\Omega)$, there holds*

$$\int_{\Omega} |f| |g| dx dy \leq 2 \|f\|_{L^\gamma} \|g\|_{L^{\gamma^*}}.$$

In this article, we are interested in the Young function γ_{exp} and its convex conjugate γ_{exp}^* . Direct calculation shows that

$$\gamma_{\text{exp}}^*(t) \leq \frac{t^{1+\epsilon}}{\epsilon}, \quad \forall t \geq 0, \quad \forall 0 < \epsilon \leq 1,$$

which permits us to conclude the following:

$$\|f\|_{L^{\gamma_{\text{exp}}^*}} \leq \epsilon^{-1/(1+\epsilon)} \|f\|_{L^{1+\epsilon}}, \quad \forall 0 < \epsilon \leq 1. \quad (2.6)$$

2.3.2 Elliptic Regularity

The following result, which we borrow from [3, Theorem 1], gives an analog to the elliptic regularity (2.3) at the end point $p \rightarrow \infty$. In our case, when Ω is a polygonal-like domain (piecewise $C^{1,1}$ -boundary) with maximum aperture $\max \alpha_j \leq \pi/2$, there holds

$$\|D^2 G_\Omega f\|_{L^{\gamma_{\text{exp}}}} \leq C_\Omega \|f\|_{L^\infty}, \quad (2.7)$$

where the constant $C_\Omega > 0$ depends only on the domain Ω .

Remark 2.1 In the case, when Ω is a polygonal-like domain with the aperture α_j of the form $\frac{\pi}{k}$ for some integer $k \geq 2$, the elliptic regularity results in [2, Proposition 3.1 and Remark 5.2] tell us that

$$\|D^2 G_\Omega f\|_{\text{bmo}_r(\Omega)} \leq C_\Omega \|f\|_{\text{bmo}_z(\Omega)}, \quad (2.8)$$

which is a stronger inequality than (2.7). For a definition of the local $\text{bmo}_\star(\Omega)$ ($\star = z, r$) spaces, see [44] or [2, Sect. 3.1]. The extension of (2.8) to general polygonal-like domains is still an open problem, to the best of our knowledge (see also [3, Remark 1.1]).

3 Approximate Solutions

Inspired by [38] where the vanishing viscosity method is applied to the 2D Euler equations in a bounded smooth domain, we here utilize the same method to study the

2D Euler–Boussinesq system. Hence, we introduce the 2D Boussinesq system with full viscosity $0 < \nu \leq 1$ and diffusivity $\kappa = 1$:

$$\begin{cases} \partial_t \mathbf{u}_\nu - \nu \Delta \mathbf{u}_\nu + \mathbf{u}_\nu \cdot \nabla \mathbf{u}_\nu + \nabla \pi_\nu = \theta_\nu \mathbf{e}_2 + S \mathbf{e}_2, & \mathbf{e}_2 = (0, 1), \\ \operatorname{div} \mathbf{u}_\nu = 0, \\ \partial_t \theta_\nu - \Delta \theta_\nu + \mathbf{u}_\nu \cdot \nabla \theta_\nu + \mathbf{u}_\nu \cdot \nabla S = \Delta S, \end{cases} \quad (3.1)$$

with the initial and boundary conditions

$$\begin{aligned} \mathbf{u}_\nu(0) &= \mathbf{u}_0, & \theta_\nu(0) &= \theta_0, & \text{in } \Omega, \\ \mathbf{u}_\nu \cdot \mathbf{n} &= 0, & \frac{\partial(\mathbf{u}_\nu \cdot \boldsymbol{\tau})}{\partial \mathbf{n}} &= 0, & \theta_\nu = 0, & \text{on } \partial\Omega. \end{aligned} \quad (3.2)$$

The existence and uniqueness of a global strong solution $(\mathbf{u}_\nu, \theta_\nu)$ of the 2D Boussinesq system (3.1)–(3.2) in the polygonal-like domain Ω are classically obtained using the Galerkin procedure, see for example [9, 37]. Here, we only need to prove some uniform estimates independent of ν .

Lemma 3.1 *Assume that $S \in H^2(\Omega)$, $\mathbf{u}_0 \in V$, and $\theta_0 \in H_0^1(\Omega)$. Then the solutions $(\mathbf{u}_\nu, \theta_\nu)$ of (3.1)–(3.2) satisfy the following estimates:*

$$\begin{aligned} \sup_{t \in [0, t_1]} (\|\mathbf{u}_\nu(t)\|_{H^1}^2 + \|\theta_\nu(t)\|_{H^1}^2) + \int_0^{t_1} \|\Delta \theta_\nu(t)\|^2 dt + \nu \int_0^{t_1} \|\mathbf{u}_\nu(t)\|_{H^2}^2 dt &\leq \mathcal{Q}_2, \\ \|\partial_t \mathbf{u}_\nu\|_{L^2(0, t_1; L^{3/2}(\Omega))} + \|\partial_t \theta_\nu\|_{L^2(0, t_1; L^2(\Omega))} &\leq \mathcal{Q}_2, \end{aligned} \quad (3.3)$$

where $\mathcal{Q}_2$ is a positive function independent of ν ($0 < \nu \leq 1$) defined by

$$\mathcal{Q}_2 := \mathcal{Q}_2(t_1, \|\mathbf{u}_0\|_{H^1}, \|\theta_0\|_{H^1}, \|S\|_{H^2}),$$

which is increasing in all its arguments.

In the sequel, the symbol C denotes a generic positive constant, which may depend on the domain Ω and vary from line to line.

Proof of Lemma 3.1 For the sake of simplicity, we write $(\mathbf{u}, \theta)$ instead of $(\mathbf{u}_\nu, \theta_\nu)$ by dropping the subscript ν in the following proof. Multiplying (3.1)₁ with $\mathbf{u}$, integrating in $L^2(\Omega)$, and using the Cauchy–Schwarz inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 + \nu \|\nabla \mathbf{u}\|^2 \leq \|\mathbf{u}\|^2 + \frac{1}{2} \|\theta\|^2 + \frac{1}{2} \|S\|^2. \quad (3.4)$$

Taking the inner product of (3.1)₃ with θ in $L^2(\Omega)$ and using Hölder’s inequality and the Sobolev embedding, we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \|\nabla \theta\|^2 &\leq \|\mathbf{u}\| \|\nabla S\|_{L^4} \|\theta\|_{L^4} + \|\Delta S\| \|\theta\| \\ &\leq C \|\mathbf{u}\| \|S\|_{H^2} \|\nabla \theta\| + \|S\|_{H^2} \|\theta\|, \end{aligned}$$

which, by Young's inequality, yields

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \|\nabla \theta\|^2 \leq C \|S\|_{H^2}^2 \|\mathbf{u}\|^2 + \frac{1}{2} \|S\|_{H^2}^2 + \frac{1}{2} \|\theta\|^2 + \frac{1}{2} \|\nabla \theta\|^2, \quad (3.5)$$

where the constant $C > 0$ only depends on the domain Ω .

Summing (3.4) and (3.5) together, we arrive at

$$\frac{d}{dt} (\|\mathbf{u}\|^2 + \|\theta\|^2) + 2\nu \|\nabla \mathbf{u}\|^2 + \|\nabla \theta\|^2 \leq C \|S\|_{H^2}^2 + C (\|S\|_{H^2}^2 + 1) (\|\mathbf{u}\|^2 + \|\theta\|^2).$$

Applying the Gronwall lemma, we obtain

$$\begin{aligned} & \sup_{t \in [0, t_1]} (\|\mathbf{u}(t)\|^2 + \|\theta(t)\|^2) + 2\nu \int_0^{t_1} \|\nabla \mathbf{u}(t)\|^2 dt + \int_0^{t_1} \|\nabla \theta(t)\|^2 dt \\ & \leq e^{C t_1 (\|S\|_{H^2}^2 + 1)} (\|\mathbf{u}_0\|^2 + \|\theta_0\|^2 + C t_1 \|S\|_{H^2}^2). \end{aligned} \quad (3.6)$$

In order to find the uniform H^1 -estimate, we need to use the vorticity formulation together with the Biot–Savart law. Let $\omega = \operatorname{curl} \mathbf{u} = \partial_x u_2 - \partial_y u_1$, then the vorticity ω satisfies

$$\partial_t \omega - \nu \Delta \omega + \mathbf{u} \cdot \nabla \omega = \partial_x \theta + \partial_x S, \quad (3.7)$$

with the Dirichlet boundary condition

$$\omega = 0, \quad \text{on } \partial\Omega.$$

That ω satisfies the homogeneous Dirichlet boundary condition is from the boundary conditions (3.2)₂ and the calculation:

$$\omega = \operatorname{curl} \mathbf{u} = \operatorname{curl} ((\mathbf{u} \cdot \boldsymbol{\tau}) \boldsymbol{\tau} + (\mathbf{u} \cdot \mathbf{n}) \mathbf{n}) = \frac{\partial(\mathbf{u} \cdot \mathbf{n})}{\partial \boldsymbol{\tau}} - \frac{\partial(\mathbf{u} \cdot \boldsymbol{\tau})}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega.$$

By the Biot–Savart law (2.5) (see also [38, 45]), we have

$$\|\mathbf{u}\|_{H^1}^2 \leq C \|\omega\|_{L^2}^2, \quad \|\mathbf{u}\|_{H^2}^2 \leq C \|\omega\|_{H^1}^2 \leq C \|\nabla \omega\|^2, \quad (3.8)$$

where the Poincaré inequality is employed for the last inequality.

Taking the inner product of (3.7) with ω in $L^2(\Omega)$ and using the Cauchy–Schwarz inequality gives

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 \leq \|\omega\|^2 + \frac{1}{2} \|\partial_x \theta\|^2 + \frac{1}{2} \|\partial_x S\|^2. \quad (3.9)$$

Taking the inner product of (3.1)₃ with $-\Delta\theta$ in $L^2(\Omega)$ and using Hölder's and Ladyzhenskaya's inequalities, we arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla\theta\|^2 + \|\Delta\theta\|^2 &\leq \|\mathbf{u}\|_{L^4} \|\nabla\theta\|_{L^4} \|\Delta\theta\| + \|\mathbf{u}\|_{L^4} \|\nabla S\|_{L^4} \|\Delta\theta\| + \|\Delta S\| \|\Delta\theta\| \\ &\leq C \|\mathbf{u}\|^{1/2} \|\mathbf{u}\|_{H^1}^{1/2} \|\nabla\theta\|^{1/2} \|\Delta\theta\|^{3/2} \\ &\quad + C \|\mathbf{u}\|_{H^1} \|S\|_{H^2} \|\Delta\theta\| + \|S\|_{H^2} \|\Delta\theta\|, \end{aligned}$$

which, by Young's inequality, yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla\theta\|^2 + \|\Delta\theta\|^2 &\leq C \|\mathbf{u}\|^2 \|\nabla\theta\|^2 + \|\mathbf{u}\|_{H^1}^2 \\ &\quad + C \|S\|_{H^2}^2 \|\mathbf{u}\|_{H^1}^2 + C \|S\|_{H^2}^2 + \frac{1}{2} \|\Delta\theta\|^2. \end{aligned} \quad (3.10)$$

Combining the estimates (3.9) and (3.10) and using (3.8), we see that

$$\begin{aligned} \frac{d}{dt} (\|\omega\|^2 + \|\nabla\theta\|^2) + 2\nu \|\nabla\omega\|^2 + \|\Delta\theta\|^2 \\ \leq C (\|\mathbf{u}\|^2 \|\nabla\theta\|^2 + \|S\|_{H^2}^2 + 1) (\|\omega\|^2 + \|\nabla\theta\|^2) + C \|S\|_{H^2}^2. \end{aligned}$$

Applying the Gronwall lemma, we obtain

$$\begin{aligned} \sup_{t \in [0, t_1]} (\|\omega(t)\|^2 + \|\nabla\theta(t)\|^2) + 2\nu \int_0^{t_1} \|\nabla\omega(t)\|^2 dt + \int_0^{t_1} \|\Delta\theta(t)\|^2 dt \\ \leq (\|\omega_0\|^2 + \|\nabla\theta_0\|^2 + C t_1 \|S\|_{H^2}^2) \exp \left\{ C \left(\sup_{t \in [0, t_1]} \|\mathbf{u}(t)\|^2 \int_0^{t_1} \|\nabla\theta\|^2 dt \right. \right. \\ \left. \left. + t_1 \|S\|_{H^2}^2 + t_1 \right) \right\}, \end{aligned} \quad (3.11)$$

which implies the first inequality in (3.3) by taking the estimate (3.6) and the Biot-Savart law (3.8) into consideration.

We now turn to the second inequality in (3.3) on the time derivatives of $(\mathbf{u}, \theta)$. Applying the projection operator P_Ω to (3.1)₁ gives the identity

$$\partial_t \mathbf{u} = P_\Omega (\nu \Delta \mathbf{u} - \mathbf{u} \nabla \mathbf{u} + \mathbf{f}),$$

where $\mathbf{f} = \theta \mathbf{e}_2 + S \mathbf{e}_2$. Noticing that, by (3.8) and (3.11), $\nu \|\Delta \mathbf{u}\|_{L^2(0, t_1; L^2(\Omega))}$ is uniformly bounded independently of ν ($0 < \nu \leq 1$) and using the estimate (3.11) again, the arguments for (4.25) in the case when $p = 2$ and $q = 2$ tell that

$$\partial_t \mathbf{u} \in L^2(0, t_1; L^s(\Omega)), \quad \forall \frac{3}{2} \leq s < 2. \quad (3.12)$$

Hence,

$$\partial_t \mathbf{u} \in L^2(0, t_1; L^{3/2}(\Omega)).$$

Regarding $\partial_t \theta$, we take a test function $\tilde{\theta} \in L^2(0, t_1; L^2(\Omega))$ with norm at most 1 and find from (3.1)₃ that

$$|\langle \partial_t \theta, \tilde{\theta} \rangle| \leq \|\Delta \theta\| \|\tilde{\theta}\| + \|\mathbf{u}\|_{L^4} \|\nabla \theta\|_{L^4} \|\tilde{\theta}\| + \|\mathbf{u}\|_{L^4} \|\nabla S\|_{L^4} \|\tilde{\theta}\| + \|\Delta S\| \|\tilde{\theta}\|.$$

Thanks to the uniform estimate (3.3)₁ again, we obtain

$$\partial_t \theta \in L^2(0, t_1; L^2(\Omega)).$$

Therefore, we finished proving the inequality (3.3). This ends the proof of Lemma 3.1. $\square$

4 Proof of Theorem 1.1

The goal here is to prove the main result of this article and we divide it to three parts. We first prove the existence of quasi-strong solution for the 2D Euler–Boussinesq system (1.3)–(1.4), then improve the regularity of the solution, and finally show the uniqueness of the solution.

4.1 Existence of a Quasi-Strong Solution

Thanks to the fact that the estimate (3.3) in Lemma 3.1 is independent of ν , we infer the existence of a couple $(\mathbf{u}, \theta)$ such that

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, t_1; V), \quad \partial_t \mathbf{u} \in L^2(0, t_1; L^{3/2}(\Omega)), \\ \theta &\in L^\infty(0, t_1; H_0^1(\Omega)) \cap L^2(0, t_1; H^2(\Omega)), \quad \partial_t \theta \in L^2(0, t_1; L^2(\Omega)), \end{aligned}$$

for which the following convergences up to not relabeled subsequences are true.

- $\mathbf{u}_\nu \rightharpoonup \mathbf{u}$ weak-* in $L^\infty(0, t_1; V)$ and $\partial_t \mathbf{u}_\nu \rightharpoonup \partial_t \mathbf{u}$ weakly in $L^2(0, t_1; L^{3/2}(\Omega))$. As a consequence (see e.g., [46]), $\mathbf{u}_\nu \rightarrow \mathbf{u}$ strongly in $L^2(0, t_1; L^6(\Omega))$.
- $\theta_\nu \rightharpoonup \theta$ weak-* in $L^\infty(0, t_1; H_0^1(\Omega))$ and weakly in $L^2(0, t_1; H^2(\Omega))$, and $\partial_t \theta_\nu \rightharpoonup \partial_t \theta$ weakly in $L^2(0, t_1; L^2(\Omega))$. Therefore, $\theta_\nu \rightarrow \theta$ strongly in $L^2(0, t_1; H_0^1(\Omega))$.

By interpolation (see e.g., [47]), we also have $\mathbf{u} \in \mathcal{C}([0, t_1]; L_\tau^2(\Omega))$ and $\theta \in \mathcal{C}([0, t_1]; H_0^1(\Omega))$. The estimate (1.6) in Theorem 1.1 directly follows from the uniform estimate (3.3) which is independent of ν .

Let $\tilde{\mathbf{u}} \in L_\tau^3(\Omega)$, $\tilde{\theta} \in L^2(\Omega)$ and $\psi, \varphi \in \mathcal{C}^1([0, t_1])$ with $\psi(t_1) = \varphi(t_1) = 0$, we then take the L^2 -inner product of (3.1) with $(\tilde{\mathbf{u}}\psi(t), \tilde{\theta}\varphi(t))$, integrate in time from 0

to t_1 , and integrate by parts for the first term; we arrive at

$$\begin{aligned} & - \int_0^{t_1} \langle \mathbf{u}_v(t), \tilde{\mathbf{u}} \rangle \psi'(t) dt - \nu \int_0^{t_1} \langle \Delta \mathbf{u}_v(t), \tilde{\mathbf{u}} \rangle \psi(t) dt + \int_0^{t_1} \langle \mathbf{u}_v(t) \cdot \nabla \mathbf{u}_v(t), \tilde{\mathbf{u}} \rangle \psi(t) dt \\ & = \langle \mathbf{u}_0, \tilde{\mathbf{u}} \rangle \psi(0) + \int_0^{t_1} \langle \boldsymbol{\theta}_v(t) \mathbf{e}_2 + S \mathbf{e}_2, \tilde{\mathbf{u}} \rangle \psi(t) dt, \end{aligned} \quad (4.1)$$

$$\begin{aligned} & - \int_0^{t_1} \langle \theta_v(t), \tilde{\theta} \rangle \varphi(t) dt - \int_0^{t_1} \langle \Delta \theta_v(t), \tilde{\theta} \rangle \varphi(t) dt + \int_0^{t_1} \langle \mathbf{u}_v(t) \cdot \nabla \theta_v(t), \tilde{\theta} \rangle \varphi(t) dt \\ & + \int_0^{t_1} \langle \mathbf{u}_v(t) \cdot \nabla S, \tilde{\theta} \rangle \varphi(t) dt = \langle \theta_0, \tilde{\theta} \rangle \varphi(0) + \int_0^{t_1} \langle \Delta S, \tilde{\theta} \rangle \varphi(t) dt. \end{aligned} \quad (4.2)$$

Thanks to the uniform estimate (3.3) in Lemma 3.1, the second term in (4.1) converges to zero, that is

$$\nu \int_0^{t_1} \langle \Delta \mathbf{u}_v(t), \tilde{\mathbf{u}} \rangle \psi(t) dt \rightarrow 0, \quad \text{as } \nu \rightarrow 0.$$

The other linear terms in (4.1)–(4.2) converge to their corresponding limits in a straightforward manner due to the above convergences. The nonlinear term in (4.1) can be written as

$$\int_0^{t_1} \langle (\mathbf{u}_v(t) - \mathbf{u}(t)) \cdot \nabla \mathbf{u}_v(t), \tilde{\mathbf{u}} \rangle \psi(t) dt + \int_0^{t_1} \langle \mathbf{u}(t) \cdot \nabla \mathbf{u}_v(t), \tilde{\mathbf{u}} \rangle \psi(t) dt,$$

and the first term above converges to zero due to the strong convergence of $\mathbf{u}_v \rightarrow \mathbf{u}$ in $L^2(0, t_1; L^6(\Omega))$ and the uniform boundedness of $\mathbf{u}_v$ in $L^\infty(0, t_1; V)$, and the second term above converges to

$$\int_0^{t_1} \langle \mathbf{u}(t) \cdot \nabla \mathbf{u}(t), \tilde{\mathbf{u}} \rangle \psi(t) dt,$$

because of the weak-* convergence of $\mathbf{u}_v \rightarrow \mathbf{u}$ in $L^\infty(0, t_1; V)$. The convergence of the nonlinear term in (4.2) is similar and simpler since we have better convergence results for θ_v . Therefore, we completed the proof of existence part of Theorem 1.1.

4.2 Regularity

Now, if we assume additionally $\omega_0 = \text{curl } \mathbf{u}_0 \in L^\infty(\Omega)$ and $\theta_0 \in H^2(\Omega)$, $S \in H^3(\Omega)$, then we are able to prove L^∞ -estimate for the vorticity ω and hence the L^p -estimate for the velocity $\mathbf{u}$ for any $1 < p < \infty$ and the uniform H^2 and the time average of $H^3(\Omega)$ -estimate for θ .

For proving the L^∞ -estimate of the vorticity ω , we require the $L^2(0, t_1; W^{1,\infty}(\Omega))$ -estimate of the forcing term $\theta + S$ for the Euler equations. Hence, we first need an $L^2(0, t_1; W^{1,\infty}(\Omega))$ -regularity of θ , which turns to be the $L^2(0, t_1; H^3(\Omega))$ -regularity for θ . To obtain the time average of H^3 -regularity for θ , we at least need the uniform

$W^{1,4}(\Omega)$ -estimate for the velocity $\mathbf{u}$. In conclusion, the plan for this subsection is as follows. We first derive the uniform $W^{1,4}$ -estimate for $\mathbf{u}$, then show the uniform H^2 and the time average of H^3 -estimates for θ , and finally prove the L^∞ -estimate for the vorticity ω .

From the Ladyzhenskaya's inequality

$$\|f\|_{L^4}^4 \leq \|f\|_{L^2}^2 \|f\|_{H^1}^2,$$

we deduce that $\theta \in L^4(0, t_1; W^{1,4}(\Omega))$ and by the Sobolev embedding, $S \in W^{1,4}(\Omega)$. Currently, if we only assume $\omega_0 = \operatorname{curl} \mathbf{u}_0 \in L^4(\Omega)$, then applying Proposition 4.1 with $\mathbf{f} = \theta \mathbf{e}_2 + S \mathbf{e}_2 \in L^4(0, t_1; W^{1,4}(\Omega))$ and $p = 4$ shows that

$$\mathbf{u} \in L^\infty(0, t_1; W^{1,4}(\Omega)), \quad \|\mathbf{u}\|_{L^\infty(0, t_1; W^{1,4}(\Omega))} \leq \mathcal{Q}_3(t_1, \|\omega_0\|_{L^4}, \|\theta_0\|_{H^1}, \|S\|_{H^2}). \quad (4.3)$$

Furthermore, choosing $p = 3$ and $q = 4$ in (4.25) gives

$$\partial_t \mathbf{u} \in L^4(0, t_1; L^3(\Omega)), \quad \|\partial_t \mathbf{u}\|_{L^4(0, t_1; L^3(\Omega))} \leq \mathcal{Q}_3(t_1, \|\omega_0\|_{L^4}, \|\theta_0\|_{H^1}, \|S\|_{H^2}). \quad (4.4)$$

To obtain the H^2 -regularity of θ , we differentiate (1.3) in time t to find

$$\partial_t \theta_t - \Delta \theta_t + \mathbf{u} \cdot \nabla \theta_t + \mathbf{u}_t \cdot \nabla \theta + \mathbf{u}_t \cdot \nabla S = 0.$$

Applying the standard energy estimate, we arrive at

$$\frac{1}{2} \frac{d}{dt} \|\theta_t\|^2 + \|\nabla \theta_t\|^2 \leq \|\mathbf{u}_t\| \|\nabla \theta\|_{L^4} \|\theta_t\|_{L^4} + \|\mathbf{u}_t\| \|\nabla S\|_{L^4} \|\theta_t\|_{L^4}, \quad (4.5)$$

and, by Ladyzhenskaya's inequality and the Sobolev embedding, the right-hand side is bounded by

$$C \|\mathbf{u}_t\| \|\nabla \theta\|^{1/2} \|\Delta \theta\|^{1/2} \|\theta_t\|^{1/2} \|\nabla \theta_t\|^{1/2} + C \|\mathbf{u}_t\| \|S\|_{H^2} \|\theta_t\|^{1/2} \|\nabla \theta_t\|^{1/2},$$

which is further bounded by

$$C \|\mathbf{u}_t\|^4 + C \|\nabla \theta\|^2 \|\Delta \theta\|^2 + C \|S\|_{H^2}^4 + \frac{1}{2} \|\theta_t\|^2 + \frac{1}{2} \|\nabla \theta_t\|^2,$$

where we used Young's inequality.

Now, we derive from (4.5) that

$$\frac{d}{dt} \|\theta_t\|^2 + \|\nabla \theta_t\|^2 \leq C \|\mathbf{u}_t\|^4 + C \|\nabla \theta\|^2 \|\Delta \theta\|^2 + C \|S\|_{H^2}^4 + \|\theta_t\|^2.$$

Thus, the Gronwall lemma implies

$$\sup_{t \in [0, t_1]} \|\theta_t(t)\|^2 + \int_0^{t_1} \|\nabla \theta_t(t)\|^2 dt \leq e^{t_1} (\|\theta_t(0)\|^2 + C \int_0^{t_1} [\|\mathbf{u}_t\|^4 + \|\nabla \theta\|^2 \|\Delta \theta\|^2 + \|S\|_{H^2}^4] dt),$$

and, from equation (1.3)₃, one has

$$\|\theta_t(0)\| \leq \|\Delta \theta_0\| + \|\mathbf{u}_0\|_{L^4} \|\nabla \theta_0\|_{L^4} + \|\mathbf{u}_0\|_{L^4} \|\nabla S\|_{L^4} + \|\Delta S\|.$$

Therefore, together with (4.4), we find

$$\sup_{t \in [0, t_1]} \|\theta_t(t)\|^2 + \int_0^{t_1} \|\nabla \theta_t(t)\|^2 dt \leq \mathcal{Q}_3(t_1, \|\omega_0\|_{L^4}, \|\theta_0\|_{H^2}, \|S\|_{H^2}). \quad (4.6)$$

From equation (1.3)₃ again, we obtain

$$\|\Delta \theta\| \leq \|\theta_t\| + \|\mathbf{u}\|_{L^4} \|\nabla \theta\|_{L^4} + \|\mathbf{u}\|_{L^4} \|\nabla S\|_{L^4} + \|\Delta S\|,$$

which, together with the estimates (4.3) and (4.6), immediately gives

$$\sup_{t \in [0, t_1]} \|\Delta \theta(t)\| \leq \mathcal{Q}_3(t_1, \|\omega_0\|_{L^4}, \|\theta_0\|_{H^2}, \|S\|_{H^2}). \quad (4.7)$$

Taking the gradient ∇ on (1.3)₃, we similarly have

$$\begin{aligned} \|\nabla \Delta \theta\| &\leq \|\nabla \theta_t\| + \|\nabla(\mathbf{u} \cdot \nabla \theta)\| + \|\nabla(\mathbf{u} \cdot \nabla S)\| + \|\nabla \Delta S\| \\ &\leq \|\nabla \theta_t\| + \|\nabla \mathbf{u}\|_{L^4} \|\nabla \theta\|_{L^4} + \|\mathbf{u}\|_{L^4} \|\Delta \theta\|_{L^4} \\ &\quad + \|\nabla \mathbf{u}\|_{L^4} \|\nabla S\|_{L^4} + \|\mathbf{u}\|_{L^4} \|\Delta S\|_{L^4} + \|S\|_{H^3}. \end{aligned} \quad (4.8)$$

The troublesome term in (4.8) is $\|\mathbf{u}\|_{L^4} \|\Delta \theta\|_{L^4}$, which can be estimated by Ladyzhenskaya's and Young's inequalities:

$$\|\mathbf{u}\|_{L^4} \|\Delta \theta\|_{L^4} \leq C \|\mathbf{u}\|_{L^4} \|\Delta \theta\|^{1/2} \|\nabla \Delta \theta\|^{1/2} \leq C \|\mathbf{u}\|_{L^4}^2 \|\Delta \theta\| + \frac{1}{4} \|\nabla \Delta \theta\|.$$

Hence, by the Sobolev embedding, the inequality (4.8) becomes

$$\begin{aligned} \|\nabla \Delta \theta\| &\leq C (\|\nabla \theta_t\| + \|\nabla \mathbf{u}\|_{L^4} \|\theta\|_{H^1} + \|\mathbf{u}\|_{L^4}^2 \|\Delta \theta\| \\ &\quad + \|\nabla \mathbf{u}\|_{L^4} \|S\|_{H^2} + \|\mathbf{u}\|_{L^4} \|S\|_{H^2} + \|S\|_{H^3}), \end{aligned}$$

which, by utilizing the estimates (4.3) and (4.6)–(4.7), shows

$$\int_0^{t_1} \|\nabla \Delta \theta(t)\|^2 ds \leq \mathcal{Q}_3(t_1, \|\omega_0\|_{L^4}, \|\theta_0\|_{H^2}, \|S\|_{H^3}). \quad (4.9)$$

We thus proved the first two estimates in (1.7) and we now turn to the L^∞ -estimate of the vorticity ω .

By the Sobolev embedding, we have

$$\|\theta \mathbf{e}_2 + S \mathbf{e}_2\|_{L^2(0,t_1;W^{1,\infty}(\Omega))} \leq \mathcal{Q}_3(t_1, \|\omega_0\|_{L^4}, \|\theta_0\|_{H^2}, \|S\|_{H^3}). \quad (4.10)$$

At this point, applying Proposition (4.19) again, we read from (4.22) that

$$\|\omega\|_{L^\infty(0,t_1;L^p(\Omega))} \leq \|\operatorname{curl} \mathbf{u}_0\|_{L^p} + \int_0^{t_1} \|\theta + S\|_{W^{1,p}} ds, \quad p \geq 2,$$

and letting $p \rightarrow \infty$ and using (4.10) yield the last estimate in (1.7). This completes the proof of regularity part of Theorem 1.1.

4.3 Uniqueness

Let $(\mathbf{u}_1, \theta_1)$ and $(\mathbf{u}_2, \theta_2)$ be two solutions of the 2D Euler–Boussinesq system (1.3)–(1.4) satisfying (1.7). Observe that, from (2.7) and (1.7),

$$\|\nabla \mathbf{u}_j(t)\|_{L^{\gamma_{\exp}}} \leq C_\Omega \|\omega_j\|_{L^\infty} \leq \mathcal{Q}_4, \quad \omega_j = \operatorname{curl} \mathbf{u}_j, \quad \forall t \in [0, t_1]. \quad (4.11)$$

The differences $\mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1$ and $\theta = \theta_2 - \theta_1$ then satisfy the equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}_2 \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_1 + \nabla \pi = \theta \mathbf{e}_2, & \mathbf{e}_2 = (0, 1), \\ \partial_t \theta - \Delta \theta + \mathbf{u}_2 \cdot \nabla \theta + \mathbf{u} \cdot \nabla \theta_1 + \mathbf{u} \cdot \nabla S = 0, \end{cases} \quad (4.12)$$

for some pressure function $\pi = \pi(t, x, y)$.

Taking the inner product of (4.12)₁ with $\mathbf{u}$ in $L^2(\Omega)$, (legitimately) integrating by parts, and applying the Orlicz space version of Hölder's inequality (see Proposition 2.2), we arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 &\leq \|\theta\| \|\mathbf{u}\| + \|\nabla \mathbf{u}_1\|_{L^{\gamma_{\exp}}} \|\mathbf{u}\|^2_{L^{\gamma_{\exp}^*}}, \\ &\leq \frac{1}{4} \|\Delta \theta\|^2 + C \|\mathbf{u}\|^2 + \mathcal{Q}_4 \epsilon^{-1/(1+\epsilon)} \|\mathbf{u}\|^2_{L^{1+\epsilon}}, \quad \forall 0 < \epsilon \leq 1, \end{aligned}$$

where we used the inequalities (2.6) and (4.11) and the Poincaré inequality for θ . Furthermore, by the interpolation inequality,

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 \leq \frac{1}{4} \|\Delta \theta\|^2 + C \|\mathbf{u}\|^2 + \mathcal{Q}_4 \epsilon^{-1/(1+\epsilon)} \|\mathbf{u}\|^{2/(1+\epsilon)} \|\mathbf{u}\|^{\epsilon/(1+\epsilon)}_{L^\infty}, \quad \forall 0 < \epsilon \leq 1. \quad (4.13)$$

Multiplying (4.12)₂ by $-\Delta\theta$ and integrating in Ω , we deduce from Ladyzhenskaya's and Young's inequalities and the Sobolev embedding that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla\theta\|^2 + \|\Delta\theta\|^2 &\leq C\|\mathbf{u}_2\|_{L^4} \|\nabla\theta\|^{1/2} \|\Delta\theta\|^{3/2} \\ &\quad + (\|\nabla\theta_1\|_{L^\infty} + \|\nabla S\|_{L^\infty}) \|\mathbf{u}\| \|\Delta\theta\| \\ &\leq C\|\mathbf{u}_2\|_{L^4}^4 \|\nabla\theta\|^2 + C(\|\theta_1\|_{H^3}^2 + \|S\|_{H^3}^2) \|\mathbf{u}\|^2 + \frac{1}{4} \|\Delta\theta\|^2. \end{aligned} \quad (4.14)$$

Adding the inequalities (4.13)–(4.14) together and using

$$\|\mathbf{u}\|_{L^\infty(\Omega \times (0, t_1))}^{\epsilon/(1+\epsilon)} \leq \|\mathbf{u}\|_{L^\infty(\Omega \times (0, t_1))} + 1,$$

we obtain

$$\begin{aligned} \frac{d}{dt} (\|\mathbf{u}\|^2 + \|\nabla\theta\|^2) + \|\Delta\theta\|^2 &\leq C\|\mathbf{u}\|^2 + Q_4 \epsilon^{-1/(1+\epsilon)} \|\mathbf{u}\|^{2/(1+\epsilon)} \|\mathbf{u}\|_{L^\infty}^{\epsilon/(1+\epsilon)} \\ &\quad + C\|\mathbf{u}_2\|_{L^4}^4 \|\nabla\theta\|^2 + C(\|\theta_1\|_{H^3}^2 + \|S\|_{H^3}^2) \|\mathbf{u}\|^2 \\ &\leq \kappa_1 \epsilon^{-1/(1+\epsilon)} (\|\mathbf{u}\|^2 + \|\nabla\theta\|^2)^{1/(1+\epsilon)} + g(t) (\|\mathbf{u}\|^2 \\ &\quad + \|\nabla\theta\|^2), \end{aligned} \quad (4.15)$$

where

$$\kappa_1 := Q_4 (\|\mathbf{u}\|_{L^\infty(\Omega \times (0, t_1))} + 1), \quad g(t) := C(1 + \|\mathbf{u}_2\|_{L^4}^4 + \|\theta_1\|_{H^3}^2 + \|S\|_{H^3}^2).$$

From the estimate (1.7), we have

$$\kappa_1 < \infty, \quad g(t) \in L^1(0, t_1).$$

We denote by $Y(t)$ the sum $\|\mathbf{u}(t)\|^2 + \|\nabla\theta(t)\|^2$, then the differential inequality (4.15) yields

$$\frac{d}{dt} Y(t) \leq \kappa_1 \epsilon^{-1/(1+\epsilon)} Y(t)^{1/(1+\epsilon)} + g(t) Y(t),$$

which, by letting $\tilde{Y}(t) = e^{-\int_0^t g(s) ds} Y(t)$, implies

$$\frac{d}{dt} \tilde{Y}(t) \leq \kappa_1 \epsilon^{-1/(1+\epsilon)} e^{-\int_0^t g(s) ds} Y(t)^{1/(1+\epsilon)} \leq \kappa_1 \epsilon^{-1/(1+\epsilon)} \tilde{Y}(t)^{1/(1+\epsilon)}, \quad \forall 0 < \epsilon \leq 1. \quad (4.16)$$

We compute

$$\frac{d}{dt} \tilde{Y}(t) = \frac{d}{dt} \left[(\tilde{Y}(t)^{\epsilon/(1+\epsilon)})^{(1+\epsilon)/\epsilon} \right] = \frac{1+\epsilon}{\epsilon} \tilde{Y}(t)^{1/(1+\epsilon)} \frac{d}{dt} (\tilde{Y}(t)^{\epsilon/(1+\epsilon)}),$$

and deduce from (4.16) that

$$\frac{d}{dt} \left(\tilde{Y}(t)^{\epsilon/(1+\epsilon)} \right) \leq \epsilon^{\epsilon/(1+\epsilon)} \frac{\kappa_1}{1+\epsilon}, \quad \forall 0 < \epsilon \leq 1. \quad (4.17)$$

Due to the continuity $(\mathbf{u}, \theta) \in \mathcal{C}([0, t_1]; L^2_\tau(\Omega)) \times \mathcal{C}([0, t_1]; H^1_0(\Omega))$, the functional $\tilde{Y}$ is continuous on $[0, t_1]$. Noting that $\tilde{Y}(0) = 0$ and integrating (4.17) in time on $(0, t)$ gives

$$\tilde{Y}(t) \leq \epsilon \left(\frac{\kappa_1 t}{1+\epsilon} \right)^{(1+\epsilon)/\epsilon}, \quad \forall t > 0, \quad \forall 0 < \epsilon \leq 1. \quad (4.18)$$

Choosing $t^* > 0$ small enough such that $\kappa_1 t^*/(1+\epsilon) \leq \kappa_1 t^*/2 < 1$ and letting ϵ tend to 0 in (4.18) entails that $\tilde{Y}(t) \equiv 0$ on $[0, t^*]$. By the induction method, we can conclude that $\tilde{Y}(t) \equiv 0$ and hence $Y(t) \equiv 0$ on $[0, t_1]$. This completes the proof of uniqueness part of Theorem 1.1.

Remark 4.1 We note that the proof of uniqueness shows the continuity of the solution semigroup in the topology of $L^2_\tau(\Omega) \times H^1_0(\Omega)$ within the set $\{\mathbf{u} \in L^2_\tau(\Omega) : \text{curl } \mathbf{u} \in L^\infty(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega)\}$. It is not clear whether the semigroup is continuous on $L^p_\tau(\Omega) \times H^2(\Omega)$ for some $p > 2$.

Acknowledgments The author would like to thank Professor Roger Temam for providing insightful comments and invaluable advices on this work. The author would also like to thank Aseel Farhat and Francesco Di Plinio for very useful discussions during this research. This work was partially supported by the National Science Foundation [Grant No. NSF DMS-1206438] and by the Research Fund of Indiana University.

Appendix 1: A Preliminary Result for the 2D Euler Equations

In this appendix, we consider the standard 2D Euler equations in the polygonal-like domain Ω which read

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathbf{f}, \\ \text{div } \mathbf{u} = 0, \end{cases} \quad (4.19)$$

with the initial and boundary conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \text{ in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega. \quad (4.20)$$

The 2D Euler equations can be expressed in terms of vorticity which allows to underline the conservation of the vorticity. This fact will turn out to be of great relevance in the analysis of the two-dimensional flow as we will see below. The vorticity formulation of the Euler equations also has the advantage of having eliminated the pressure term $\nabla \pi$ and it reads

$$\begin{cases} \partial_t \omega + \mathbf{u} \cdot \nabla \omega = \text{curl } \mathbf{f}, \\ \text{div } \mathbf{u} = 0, \quad \mathbf{u} = \mathbf{K}_\Omega \omega = \nabla^\perp \mathbf{G}_\Omega \omega, \\ \omega(0) = \text{curl } \mathbf{u}_0, \end{cases} \quad (4.21)$$

where $\omega = \text{curl } \mathbf{u} = \partial_x u_2 - \partial_y u_1$.

We recall the following standard L^p a priori estimates for the 2D Euler equations (see for example [2, Lemmas 4.3, 4.5]).

Proposition 4.1 *Let Ω be a polygonal-like domain (piecewise $C^{1,1}$ -boundary) with maximum aperture $\max \alpha_j \leq \pi/2$ and let $2 \leq p < \infty$. Assume that $\omega_0 = \operatorname{curl} \mathbf{u}_0$ belongs to $L^p(\Omega)$ and $\mathbf{f}$ belongs to $L^2(0, t_1; W^{1,p}(\Omega))$. If $\mathbf{u}$ is a solution of (4.19), then*

$$\begin{aligned} \omega = \operatorname{curl} \mathbf{u} \in L^\infty(0, t_1; L^p(\Omega)), \quad \|\omega\|_{L^\infty(0, t_1; L^p(\Omega))} &\leq \|\operatorname{curl} \mathbf{u}_0\|_{L^p} \\ &+ \int_0^{t_1} \|\operatorname{curl} \mathbf{f}\|_{L^p} ds, \end{aligned} \quad (4.22)$$

and for all $2 \leq p < \infty$, there holds

$$\mathbf{u} \in L^\infty(0, t_1; W^{1,p}(\Omega)), \quad \|\mathbf{u}\|_{L^\infty(0, t_1; W^{1,p}(\Omega))} \leq \mathcal{Q}_8, \quad (4.23)$$

where $\mathcal{Q}_8$ is a positive function defined by

$$\mathcal{Q}_8 := \mathcal{Q}_8\left(p, \|\omega_0\|_{L^p}, \|\mathbf{f}\|_{L^2(0, t_1; W^{1,p}(\Omega))}\right), \quad (4.24)$$

which is increasing on its arguments.

Furthermore, if we additionally suppose that $2 \leq p \leq 3$ and $\mathbf{f} \in L^q(0, t_1; W^{1,p}(\Omega))$ for some $q \geq 2$, then we actually have

$$\begin{aligned} \partial_t \mathbf{u} &\in L^q(0, t_1; L^{s(p)}(\Omega)), \\ \|\partial_t \mathbf{u}\|_{L^q(0, t_1; L^{s(p)}(\Omega))} &\leq \mathcal{Q}_9\left(p, \|\omega_0\|_{L^p}, \|\mathbf{f}\|_{L^q(0, t_1; W^{1,p}(\Omega))}\right), \end{aligned} \quad (4.25)$$

where $s(p)$ is defined by

$$s(p) = \begin{cases} \text{any } s \in [\frac{3}{2}, 2), & p = 2, \\ p, & 2 < p \leq 3. \end{cases}$$

We present the proof of Proposition 4.1 for the sake of completeness.

Proof of Proposition 4.1 Multiplying (4.21) by $p|\omega|^{p-2}\omega$, integrating on Ω , and using the fact that

$$p\langle \mathbf{u} \cdot \nabla \omega, |\omega|^{p-2}\omega \rangle = \int_{\Omega} \mathbf{u} \cdot \nabla |\omega|^p dx dy = 0,$$

which stems from that $\mathbf{u}$ is divergence free and has zero normal component on $\partial\Omega$, we obtain

$$\frac{d}{dt} \|\omega\|_{L^p}^p = p\langle \operatorname{curl} \mathbf{f}, |\omega|^{p-2}\omega \rangle \leq p\|\operatorname{curl} \mathbf{f}\|_{L^p} \|\omega\|_{L^p}^{p-1},$$

which implies

$$\frac{d}{dt} \|\omega\|_{L^p} \leq \|\operatorname{curl} \mathbf{f}\|_{L^p}.$$

Integrating in time on $(0, t)$ and then taking the sup over $[0, t_1]$ show the desired estimate (4.22). Now, the Biot–Savart law (2.5) yields

$$\mathbf{u} = \mathbf{K}_\Omega \omega \in L^\infty(0, t_1; W^{1,p}(\Omega) \cap L^2_\tau(\Omega)), \quad \|\mathbf{u}\|_{L^\infty(0,t_1;W^{1,p}(\Omega))} \leq \mathcal{Q}_8, \quad (4.26)$$

where $\mathcal{Q}_8$ is defined by (4.24).

We are now left to show the estimate (4.25) for $\mathbf{u}_t$. Applying the projection operator P_Ω on (4.19) gives the identity

$$\partial_t \mathbf{u} = P_\Omega(-\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{f}). \quad (4.27)$$

Since for the range $s(p) \in [\frac{3}{2}, 3]$, Proposition 2.1 guarantees that P_Ω is a linear bounded operator on $L^{s(p)}$ and hence, we can find from (4.27) that

$$\begin{aligned} \|\partial_t \mathbf{u}\|_{L^q(0,t_1;L^{s(p)}(\Omega))} &\leq C_{p,\Omega} \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^q(0,t_1;L^{s(p)}(\Omega))} + C_{p,\Omega} \|\mathbf{f}\|_{L^q(0,t_1;L^{s(p)}(\Omega))} \\ &\leq \max(t_1, 1) C_{p,\Omega} \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^\infty(0,t_1;L^{s(p)}(\Omega))} \\ &\quad + C_{p,\Omega} \|\mathbf{f}\|_{L^q(0,t_1;L^{s(p)}(\Omega))}, \end{aligned} \quad (4.28)$$

where the constant $C_{p,\Omega}$ is the operator norm of P_Ω , only depending on p and the domain Ω . Next, by the Sobolev embedding and (4.26), we obtain

$$\|\mathbf{u}\|_{L^\infty(0,t_1;L^r(\Omega))} \lesssim \|\mathbf{u}\|_{L^\infty(0,t_1;W^{1,p}(\Omega))} \leq \mathcal{Q}_8, \quad r = \begin{cases} s(p)^*, & p = 2, \\ \infty, & p > 2, \end{cases}$$

where the $s(p)^*$ is the Sobolev conjugate exponent of $s(p)$, that is

$$s(p)^* = \frac{2s(p)}{2-s(p)}, \quad \frac{3}{2} \leq s(p) < 2,$$

and we then deduce from Hölder's inequality that

$$\|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^\infty(0,t_1;L^{s(p)}(\Omega))} \leq \|\mathbf{u}\|_{L^\infty(0,t_1;L^r(\Omega))} \|\nabla \mathbf{u}\|_{L^\infty(0,t_1;L^p(\Omega))} \leq \mathcal{Q}_8,$$

which, together with (4.28), implies (4.25). This ends the proof of Proposition 4.1. $\square$

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