The Second Variation of the Functional L of Symplectic Critical Surfaces in Kähler Surfaces

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Abstract Let M be a complete Kähler surface and Σ be a symplectic surface which is smoothly immersed in M . Let α be the Kähler angle of Σ in M . In the previous paper Han and Li (JEMS 12:505–527, 2010) 2010, we study the symplectic critical surfaces, which are critical points of the functional $L = \int_{\Sigma} \frac{1}{\cos \alpha} d\mu$ in the class of symplectic surfaces. In this paper, we calculate the second variation of the functional L and derive some consequences. In particular, we show that, if the scalar curvature of M is positive, Σ is a stable symplectic critical surface with $\cos \alpha \geq \delta > 0$, whose normal bundle admits a holomorphic section $X \in L^2(\Sigma)$, then Σ is holomorphic. We construct symplectic critical surfaces in $\mathbb{C}^2$. We also prove a Liouville theorem for symplectic critical surfaces in $\mathbb{C}^2$.

Keywords Symplectic critical surface · Holomorphic curve · Kähler surface

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1 Introduction

Let M be a complete Kähler surface and Σ be a surface immersed in M . One defines a metric on Σ induced from M . The Kähler angle [2] α of Σ in M is defined by

$$\omega|_{\Sigma} = \cos \alpha d\mu_{\Sigma}, \quad (1.1)$$

where $d\mu_{\Sigma}$ is the area element of Σ and ω is the Kähler form on M . We say that a surface Σ is symplectic if $\cos \alpha > 0$ on Σ . In [3], we introduce the functional

$$L = \int_{\Sigma} \frac{1}{\cos \alpha} d\mu_{\Sigma}$$

in the class of symplectic surfaces in M . It is clear that holomorphic curves minimize the functional. We call a critical point of the functional L a symplectic critical surface. In [3], we calculate the first variation formula of the functional L and derive the Euler–Lagrange equation, it is

$$\cos^3 \alpha H - (J(J\nabla \cos \alpha)^{\top})^{\perp} = 0,$$

where H is the mean curvature vector of Σ in M , and $()^{\top}$ denotes tangential components of $()$ and $()^{\perp}$ denotes the normal components of $()$. We checked that it is an elliptic equation. We showed that [3], if M is a Kähler–Einstein surface with nonnegative scalar curvature, each compact symplectic critical surface is holomorphic.

The main purpose of this paper is to compute the second variation formula of the functional L of a symplectic critical surface. In particular, let X be a normal variation with compact support, and let $Y = -J_{\nu}X$, then the second variation of the functional L is

$$\begin{aligned} II(X) + II(Y) = & -4 \int_{\Sigma} \frac{R|X|^2 \sin^2 \alpha}{\cos \alpha} d\mu_{\Sigma} + 2 \int_{\Sigma} \frac{|\bar{\partial}X|^2(1 + \cos^2 \alpha)}{\cos^3 \alpha} d\mu_{\Sigma} \\ & - 2 \int_{\Sigma} \frac{(1 + \cos^2 \alpha)}{\cos^5 \alpha} |X|^2 |\nabla \alpha|^2 d\mu_{\Sigma}. \end{aligned}$$

It follows from the formula clearly that if the scalar curvature of M is positive, Σ is a stable symplectic critical surface in M with $\cos \alpha \geq \delta > 0$, whose normal bundle admits a holomorphic section $X \in L^2(\Sigma)$, then Σ is holomorphic. In the computation, we used techniques in [4].

In the last section, we study symplectic critical surfaces in $\mathbf{C}^2$. We construct symplectic critical surfaces in $\mathbf{C}^2$. The first example is a rotational symmetric surface $z = -\frac{1}{2} \log(x^2 + y^2)$ in $\mathbf{R}^3$ which we see as a surface in $\mathbf{C}^2$. It is a simply connected complete symplectic critical surface with Gauss curvature close to -1 at one end and close to 0 at the other end, which is not holomorphic. We also prove a Liouville theorem for symplectic critical surfaces in $\mathbf{C}^2$. If Σ is a complete symplectic critical surface in $\mathbf{C}^2$ with area quadratic growth, and $\cos \alpha \geq \delta > 0$, then it is a holomorphic curve.

2 Preliminaries

In this section, we recall the geometry of the normal bundle ν of Σ in M (see [4]) and fix our notation. We choose a local orthonormal frames $\{e_1, e_2, e_3, e_4\}$ in M such that $\{e_1, e_2\}$ is a local orthonormal frames of Σ and $\{e_3, e_4\}$ is a local orthonormal frames of the normal bundle ν of Σ in M . We shall use the following conventions on the range of the indices,

$$\begin{aligned} 1 &\leq i, j, k \leq 2 \\ 1 &\leq A, B, C, D \leq 4 \\ 3 &\leq \lambda, \mu \leq 4. \end{aligned}$$

Let $\{\theta^A\}$ be the dual frame of $\{e_A\}$, then the structure equations of M are

$$de_A = \theta_{AB} \otimes e_B, \quad \theta_{AB} + \theta_{BA} = 0, \quad (2.1)$$

$$d\theta_{AB} = \theta_{AC} \wedge \theta_{CB} + \Phi_{AB}, \quad \Phi_{AB} = \frac{1}{2} R_{ABCD} \theta_C \wedge \theta_D. \quad (2.2)$$

The induced connection one form θ_{ij} of Σ is determined by

$$\nabla e_i = \theta_{ij} \otimes e_j. \quad (2.3)$$

The induced connection one form $\theta_{\lambda\mu}$ of ν is determined by

$$\nabla e_\lambda = \theta_{\lambda\mu} \otimes e_\mu. \quad (2.4)$$

Note that (θ_{ij}) defines a connection in the tangent bundle $T\Sigma$, hence a connection in the cotangent bundle $T^*\Sigma$ and $(\theta_{\lambda\mu})$ defines a connection in the normal bundle ν , we have covariant differential ∇ which maps a section of $\nu \otimes (T^*\Sigma)^k$ into a section of $\nu \otimes (T^*\Sigma)^{k+1}$. In particular, for a normal section of Σ , $x = x_\mu e_\mu$, we define

$$\nabla x = x_{\mu j} \theta_j \otimes e_\mu, \quad (2.5)$$

where $x_{\mu j}$ is the covariant derivative of x_μ and is determined by

$$x_{\mu j} \theta_j = dx_\mu + x_\lambda \theta_{\lambda\mu}. \quad (2.6)$$

Similarly, $x_{\mu jk}$ is the second covariant derivative of x_μ and is determined by

$$x_{\mu jk} \theta_k = dx_{\mu j} + x_{\mu k} \theta_{kj} - x_{\lambda j} \theta_{\mu\lambda}. \quad (2.7)$$

Taking the exterior derivative of (2.6), using (2.3), (2.7) and the fact that $\theta_3 = \theta_4 = 0$ on Σ , we have

$$x_\lambda d\theta_{\lambda\mu} = x_{\mu jk} \theta_k \wedge \theta_j. \quad (2.8)$$

By (2.2), we see that the curvature of the normal bundle K_{3412} satisfies

$$d\theta_{34} = -K_{3412}\theta_1 \wedge \theta_2. \quad (2.9)$$

By (2.8) and (2.9), we obtain the following identities

$$-x_3 K_{3412} = x_{421} - x_{412} \quad (2.10)$$

and

$$x_4 K_{3412} = x_{321} - x_{312}. \quad (2.11)$$

Denote the complexification of ν by $\nu_{\mathbb{C}} = \nu \otimes \mathbb{C} = \{\nu + i\nu\}$. Extend the covariant differential ∇ to $\nu_{\mathbb{C}}$ complex linearly, i.e., $\nabla(x + iy) = \nabla x + i\nabla y$, x and y are sections of ν . The surface Σ admits a complex structure defined by the $(1, 0)$ form $\phi = \theta_1 + i\theta_2$ and the $(0, 1)$ form $\bar{\phi} = \theta_1 - i\theta_2$. We can decompose its $(1, 0)$ and $(0, 1)$ parts and write it as

$$\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}.$$

Using the Newlander–Nirenberg theorem, $\nu_{\mathbb{C}}$ can be made into a holomorphic bundle whose $\bar{\partial}$ -operator is $\nabla^{(0,1)}$. Let $x = x_{\mu}e_{\mu}$ and $y = y_{\mu}e_{\mu}$ be sections of the normal bundle so that $x + iy$ is a section of $\nu_{\mathbb{C}}$, we have

$$\begin{aligned} \nabla^{(1,0)}(x + iy) &= \frac{1}{2}\{(x_{31} + y_{32}) + i(y_{31} - x_{32})\}\phi \otimes e_3 \\ &\quad + \frac{1}{2}\{(x_{41} + y_{42}) + i(y_{41} - x_{42})\}\phi \otimes e_4, \\ \nabla^{(0,1)}(x + iy) &= \frac{1}{2}\{(x_{31} - y_{32}) + i(y_{31} + x_{32})\}\bar{\phi} \otimes e_3 \\ &\quad + \frac{1}{2}\{(x_{41} - y_{42}) + i(y_{41} + x_{42})\}\bar{\phi} \otimes e_4 \\ &= \bar{\partial}(x + iy). \end{aligned} \quad (2.12)$$

It is clear that $x + iy$ is a holomorphic section of $\nu_{\mathbb{C}}$ if $\nabla^{(0,1)}(x + iy) = 0$.

There exists a natural almost complex structure on ν , defined by

$$J_{\nu}(e_3) = e_4, \quad J_{\nu}(e_4) = -e_3.$$

Extend J_{ν} to $\nu_{\mathbb{C}}$ complex linearly. Let $e' = \frac{1}{2}(e_3 - ie_4)$ and $e'' = \frac{1}{2}(e_3 + ie_4)$, then

$$\nu_{\mathbb{C}} = \nu' \oplus \nu'',$$

where ν' is spanned by e' , ν'' is spanned by e'' and they are complex line bundles. Notice that

$$\nabla e' = i\theta_{34} \otimes e', \quad \nabla e'' = -i\theta_{34} \otimes e'',$$

v' and v'' are parallel and hence holomorphic sub-bundles of $v_{\mathbb{C}}$. By (2.12), we know that a section $x'e'$ of v' is holomorphic if

$$\nabla x' \equiv 0 \pmod{\phi}.$$

A normal vector $x = x_3e_3 + x_4e_4$ can be expressed by e', e'' uniquely as

$$\begin{aligned} x &= (x_3 + ix_4)e' + (x_3 - ix_4)e'' \\ &= x'e' + x''e''. \end{aligned}$$

We identify x with $x'e'$ and this isomorphism imposes a holomorphic structure on v . More precisely, if $x = x_3e_3 + x_4e_4$ we have

$$\begin{aligned} &d(x_3 + ix_4) + (x_3 + ix_4)i\theta_{34} \\ &= \frac{1}{2}[(x_{31} + x_{42}) + i(x_{41} - x_{32})]\phi + \frac{1}{2}[(x_{31} - x_{42}) + i(x_{41} + x_{32})]\bar{\phi}. \end{aligned} \quad (2.13)$$

Set

$$\bar{\partial}x = (x_{31} - x_{42})\bar{\phi} \otimes e_3 + (x_{41} + x_{32})\bar{\phi} \otimes e_4, \quad (2.14)$$

then x is a holomorphic section of v if and only if $\bar{\partial}x = 0$.

3 Second Variation Formulae

In this section, we deduce the second variation of the L -functional of a symplectic critical surface. Recall that

$$L(\phi) = \int_{\Sigma} \frac{1}{\cos \alpha} d\mu_{\Sigma},$$

where $\phi : \Sigma \rightarrow M$ is a smooth symplectic immersion. Let $\phi(\Sigma)$ be a symplectic critical surface of M . Let

$$\phi_{t,\varepsilon} : \Sigma \times (-\delta, \delta) \times (-a, a) \rightarrow M$$

be one family of smooth immersions with

$$\phi_{0,0} = \phi.$$

Since symplectic is an open condition, we can assume that $\phi_{t,\varepsilon}(\Sigma)$ is symplectic for every $t \in (-\delta, \delta)$ and every $\varepsilon \in (-a, a)$, which means $\phi_{t,\varepsilon}$ is a symplectic variation of ϕ . For simplicity, we write $\phi_{0,0}(\Sigma) = \phi(\Sigma) = \Sigma$. Let

$$\left. \frac{\partial \phi_{t,0}}{\partial t} \right|_{t=0} = X, \quad \left. \frac{\partial \phi_{0,\varepsilon}}{\partial \varepsilon} \right|_{\varepsilon=0} = Y \quad \text{and} \quad \left. \frac{\partial^2 \phi_{t,\varepsilon}}{\partial t \partial \varepsilon} \right|_{t=0, \varepsilon=0} = Z.$$

We assume in this section that X , Y and Z are of compact supports.

Lemma 3.1 *We have the general second variation formula:*

$$\begin{aligned} \left. \frac{\partial^2}{\partial t \partial \varepsilon} \right|_{t=0, \varepsilon=0} L(\phi_{t,\varepsilon}) &= 4 \int_{\Sigma} \frac{\operatorname{div}_{\Sigma} X \operatorname{div}_{\Sigma} Y}{\cos \alpha} - 2 \int_{\Sigma} \frac{R(X, e_i, Y, e_i)}{\cos \alpha} d\mu_{\Sigma} \\ &+ 2 \int_{\Sigma} \frac{\langle \bar{\nabla}_{e_i}^{\perp} X, \bar{\nabla}_{e_i}^{\perp} Y \rangle}{\cos \alpha} d\mu_{\Sigma} \\ &- \int_{\Sigma} \frac{\langle e_i, \bar{\nabla}_{e_j} X \rangle \langle e_j, \bar{\nabla}_{e_i} Y \rangle + \langle e_j, \bar{\nabla}_{e_i} X \rangle \langle e_i, \bar{\nabla}_{e_j} Y \rangle}{\cos \alpha} d\mu_{\Sigma} \\ &- 2 \int_{\Sigma} \frac{\operatorname{div}_{\Sigma} X (\omega(\bar{\nabla}_{e_1} Y, e_2) + \omega(e_1, \bar{\nabla}_{e_2} Y))}{\cos^2 \alpha} d\mu_{\Sigma} \\ &- 2 \int_{\Sigma} \frac{\operatorname{div}_{\Sigma} Y (\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X))}{\cos^2 \alpha} d\mu_{\Sigma} \\ &- \int_{\Sigma} \frac{\omega(R(Y, e_1)X, e_2) + \omega(e_1, R(Y, e_2)X)}{\cos^2 \alpha} d\mu_{\Sigma} \\ &- \int_{\Sigma} \frac{\omega(\bar{\nabla}_{e_1} X, \bar{\nabla}_{e_2} Y) + \omega(\bar{\nabla}_{e_1} Y, \bar{\nabla}_{e_2} X)}{\cos^2 \alpha} d\mu_{\Sigma} \\ &+ 2 \int_{\Sigma} \frac{(\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X))(\omega(\bar{\nabla}_{e_1} Y, e_2) + \omega(e_1, \bar{\nabla}_{e_2} Y))}{\cos^3 \alpha} d\mu_{\Sigma}, \end{aligned} \quad (3.1)$$

where $\{e_1, e_2\}$ is a base of Σ , $\bar{\nabla}$ is the covariant differential of M , and $\operatorname{div}_{\Sigma} X = \langle \bar{\nabla}_{e_i} X, e_i \rangle$, R is the curvature operator on M .

Proof Let (x_1, x_2) be a local normal coordinates in Σ . The induced metric on $\phi_{t,\varepsilon}(\Sigma)$ is

$$g_{ij}(t, \varepsilon) = \left\langle \frac{\partial \phi_{t,\varepsilon}}{\partial x_i}, \frac{\partial \phi_{t,\varepsilon}}{\partial x_j} \right\rangle.$$

For simplicity, we denote $\frac{\partial \phi_{0,0}}{\partial x_i}$ by e_i , $g_{ij}(t, \varepsilon)$ by g_{ij} , and $\phi_{t,\varepsilon}$ by ϕ .

A simple calculation gives

$$\left. \frac{\partial}{\partial t} \right|_{t=0, \varepsilon=0} g_{ij} = \langle \bar{\nabla}_{e_i} X, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} X \rangle, \quad (3.2)$$

$$\left. \frac{\partial}{\partial \varepsilon} \right|_{t=0, \varepsilon=0} g_{ij} = \langle \bar{\nabla}_{e_i} Y, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} Y \rangle \quad (3.3)$$

and

$$\begin{aligned} \frac{\partial^2}{\partial t \partial \varepsilon} \Big|_{t=0, \varepsilon=0} g_{ij} &= \langle \bar{\nabla}_{e_i} Z + R(Y, e_i)X, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} Z + R(Y, e_j)X \rangle \\ &\quad + \langle \bar{\nabla}_{e_i} X, \bar{\nabla}_{e_j} Y \rangle + \langle \bar{\nabla}_{e_i} Y, \bar{\nabla}_{e_j} X \rangle. \end{aligned} \quad (3.4)$$

Since

$$\cos \alpha(\varepsilon, t) = \frac{\omega \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)}{\sqrt{\det(g_{ij})}},$$

we have

$$L = \int_{\Sigma} \frac{\det(g_{ij})}{\omega \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)} dx_1 \wedge dx_2. \quad (3.5)$$

We write the integrand of the functional L as

$$I = \frac{\det(g_{ij})}{\omega \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)},$$

then

$$\begin{aligned} \frac{\partial I}{\partial t} \Big|_{t=0} &= \frac{\frac{\partial}{\partial t} \Big|_{t=0} g_{ij} g^{ij}}{\omega \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)} \det(g_{ij}) \\ &\quad - \frac{\omega \left(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial t} \Big|_{t=0}, \frac{\partial \phi}{\partial x_2} \right) + \omega \left(\frac{\partial \phi}{\partial x_1}, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial t} \Big|_{t=0} \right)}{\omega^2 \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)} \det(g_{ij}). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial^2 I}{\partial t \partial \varepsilon} \Big|_{t=0, \varepsilon=0} &= \frac{\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} g_{kl} g^{kl} \frac{\partial}{\partial t} \Big|_{t=0} g_{ij} g^{ij}}{\omega(e_1, e_2)} + \frac{\frac{\partial^2}{\partial \varepsilon \partial t} \Big|_{t=0, \varepsilon=0} g_{ij} g^{ij}}{\omega(e_1, e_2)} + \frac{\frac{\partial}{\partial t} \Big|_{t=0} g_{ij} \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} g^{ij}}{\omega(e_1, e_2)} \\ &\quad - \frac{\frac{\partial}{\partial t} \Big|_{t=0} g_{ij} g^{ij} \left[\omega \left(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial \varepsilon} \Big|_{\varepsilon=0}, e_2 \right) + \omega \left(e_1, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial \varepsilon} \Big|_{\varepsilon=0} \right) \right]}{\omega^2(e_1, e_2)} \\ &\quad - \frac{\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} g_{ij} g^{ij} \left[\omega \left(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial t} \Big|_{t=0}, e_2 \right) + \omega \left(e_1, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial t} \Big|_{t=0} \right) \right]}{\omega^2(e_1, e_2)} \\ &\quad - \frac{\omega \left(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial t} \Big|_{t=0}, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial \varepsilon} \Big|_{\varepsilon=0} \right) + \omega \left(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial \varepsilon} \Big|_{\varepsilon=0}, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial t} \Big|_{t=0} \right)}{\omega^2(e_1, e_2)} \\ &\quad - \frac{\omega \left(\bar{\nabla}_{e_1} \frac{\partial^2 \phi}{\partial t \partial \varepsilon} \Big|_{t=0, \varepsilon=0} + R(Y, e_1) \frac{\partial \phi}{\partial t} \Big|_{t=0}, e_2 \right)}{\omega^2(e_1, e_2)} \end{aligned}$$

$$\begin{aligned}
& - \frac{\omega \left(e_1, \bar{\nabla}_{e_2} \frac{\partial^2 \phi}{\partial t \partial \varepsilon} \Big|_{t=0, \varepsilon=0} + R(Y, e_2) \frac{\partial \phi}{\partial t} \Big|_{t=0} \right)}{\omega^2(e_1, e_2)} \\
& + \frac{2}{\omega^3(e_1, e_2)} \left[\omega \left(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial t} \Big|_{t=0}, \frac{\partial \phi}{\partial x_2} \right) + \omega \left(\frac{\partial \phi}{\partial x_1}, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial t} \Big|_{t=0} \right) \right] \\
& \cdot \left[\omega \left(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial \varepsilon} \Big|_{\varepsilon=0}, \frac{\partial \phi}{\partial x_2} \right) + \omega \left(\frac{\partial \phi}{\partial x_1}, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial \varepsilon} \Big|_{\varepsilon=0} \right) \right].
\end{aligned}$$

By (3.2)–(3.4), we get that

$$\begin{aligned}
\frac{\partial^2 I}{\partial t \partial \varepsilon} \Big|_{t=0, \varepsilon=0} &= 4 \frac{\langle \bar{\nabla}_{e_j} X, e_j \rangle \langle \bar{\nabla}_{e_i} Y, e_i \rangle}{\omega(e_1, e_2)} + 2 \frac{\langle \bar{\nabla}_{e_i} Z, e_i \rangle}{\omega(e_1, e_2)} \\
&+ 2 \frac{\langle \bar{\nabla}_{e_i} X, \bar{\nabla}_{e_i} Y \rangle}{\omega(e_1, e_2)} - 2 \frac{R(X, e_i, Y, e_i)}{\omega(e_1, e_2)} \\
&- \frac{(\langle \bar{\nabla}_{e_i} X, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} X \rangle)(\langle \bar{\nabla}_{e_i} Y, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} Y \rangle)}{\omega(e_1, e_2)} \\
&- 2 \frac{\langle \bar{\nabla}_{e_i} X, e_i \rangle [\omega(\bar{\nabla}_{e_1} Y, e_2) + \omega(e_1, \bar{\nabla}_{e_2} Y)]}{\omega^2(e_1, e_2)} \\
&- 2 \frac{\langle \bar{\nabla}_{e_i} Y, e_i \rangle [\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X)]}{\omega^2(e_1, e_2)} \\
&- \frac{\omega(\bar{\nabla}_{e_1} Z, e_2) + \omega(e_1, \bar{\nabla}_{e_2} Z)}{\omega^2(e_1, e_2)} \\
&- \frac{\omega(R(Y, e_1)X, e_2) + \omega(e_1, R(Y, e_2)X)}{\omega^2(e_1, e_2)} \\
&- \frac{\omega(\bar{\nabla}_{e_1} X, \bar{\nabla}_{e_2} Y) + \omega(\bar{\nabla}_{e_1} Y, \bar{\nabla}_{e_2} X)}{\omega^2(e_1, e_2)} \\
&+ 2 \frac{[\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X)][\omega(\bar{\nabla}_{e_1} Y, e_2) + \omega(e_1, \bar{\nabla}_{e_2} Y)]}{\omega^3(e_1, e_2)}.
\end{aligned} \tag{3.6}$$

Since Σ is a symplectic critical surface, by the first variation formula [3], we have

$$\int_{\Sigma} \left(\frac{2 \langle \bar{\nabla}_{e_i} Z, e_i \rangle}{\cos \alpha} - \frac{\omega(\bar{\nabla}_{e_1} Z, e_2) + \omega(e_1, \bar{\nabla}_{e_2} Z)}{\cos^2 \alpha} \right) d\mu_{\Sigma} = 0. \tag{3.7}$$

Notice that

$$\begin{aligned}
& 2 \langle \bar{\nabla}_{e_i} X, \bar{\nabla}_{e_i} Y \rangle - (\langle \bar{\nabla}_{e_i} X, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} X \rangle)(\langle \bar{\nabla}_{e_i} Y, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} Y \rangle) \\
&= 2 \langle \bar{\nabla}_{e_i}^{\perp} X, \bar{\nabla}_{e_i}^{\perp} Y \rangle - (\langle e_i, \bar{\nabla}_{e_j} X \rangle \langle e_j, \bar{\nabla}_{e_i} Y \rangle + \langle e_i, \bar{\nabla}_{e_j} Y \rangle \langle e_j, \bar{\nabla}_{e_i} X \rangle) \tag{3.8}
\end{aligned}$$

and

$$\langle \bar{\nabla}_{e_i} X, e_i \rangle = \operatorname{div}_{\Sigma} X. \quad (3.9)$$

Putting (3.7)–(3.9) into (3.6), we obtain the identity in the lemma. $\square$

If $X = Y$ is a normal variation, we have

$$\operatorname{div}_{\Sigma} X = -X \cdot H, \quad (3.10)$$

$$\langle e_i, \bar{\nabla}_{e_j} X \rangle \langle e_j, \bar{\nabla}_{e_i} X \rangle = (X \cdot A(e_i, e_j))^2 \quad (3.11)$$

and

$$\begin{aligned} & - \int_{\Sigma} \frac{\omega(R(X, e_1)X, e_2) + \omega(e_1, R(X, e_2)X)}{\cos^2 \alpha} d\mu_{\Sigma} - 2 \int_{\Sigma} \frac{\omega(\bar{\nabla}_{e_1} X, \bar{\nabla}_{e_2} X)}{\cos^2 \alpha} d\mu_{\Sigma} \\ &= \int_{\Sigma} \frac{\langle J e_2, R(X, e_1)X \rangle - \langle J e_1, R(X, e_2)X \rangle}{\cos^2 \alpha} d\mu_{\Sigma} \\ & \quad - \int_{\Sigma} \frac{\bar{\nabla}_{e_1}(\omega(X, \bar{\nabla}_{e_2} X)) - \omega(X, \bar{\nabla}_{e_1} \bar{\nabla}_{e_2} X)}{\cos^2 \alpha} d\mu_{\Sigma} \\ & \quad - \int_{\Sigma} \frac{\bar{\nabla}_{e_2}(\omega(\bar{\nabla}_{e_1} X, X)) - \omega(\bar{\nabla}_{e_2} \bar{\nabla}_{e_1} X, X)}{\cos^2 \alpha} d\mu_{\Sigma} \\ &= \int_{\Sigma} \frac{R(X, e_1, J e_2, X) + R(X, e_2, X, J e_1) + R(e_1, e_2, J X, X)}{\cos^2 \alpha} d\mu_{\Sigma} \\ & \quad - 2 \int_{\Sigma} \frac{\omega(X, \bar{\nabla}_{e_2} X) \nabla_{e_1} \cos \alpha + \omega(\bar{\nabla}_{e_1} X, X) \nabla_{e_2} \cos \alpha}{\cos^3 \alpha} d\mu_{\Sigma}, \end{aligned} \quad (3.12)$$

in the last step, we used integration by parts.

By the Bianchi equality, we have

$$\begin{aligned} & R(X, e_1, J e_2, X) + R(X, e_2, X, J e_1) + R(e_1, e_2, J X, X) \\ &= R(X, e_1, J X, e_2) + R(X, e_2, e_1, J X) + R(X, J X, e_2, e_1) = 0. \end{aligned} \quad (3.13)$$

Therefore, by (3.10)–(3.13), we obtain the following formula.

Corollary 3.2 *If X is a normal variation with compact support, then*

$$\begin{aligned} & \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} L(\phi_t) =: II(X) \\ &= 4 \int_{\Sigma} \frac{(X \cdot H)^2}{\cos \alpha} d\mu_{\Sigma} - 2 \int_{\Sigma} \frac{R(X, e_i, X, e_i)}{\cos \alpha} d\mu_{\Sigma} \\ & \quad + 2 \int_{\Sigma} \frac{|\bar{\nabla}_{e_i}^{\perp} X|^2}{\cos \alpha} d\mu_{\Sigma} - 2 \int_{\Sigma} \frac{(X \cdot A(e_i, e_j))^2}{\cos \alpha} d\mu_{\Sigma} \\ & \quad + 4 \int_{\Sigma} \frac{X \cdot H (\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X))}{\cos^2 \alpha} d\mu_{\Sigma} \end{aligned}$$

$$\begin{aligned}
& + 2 \int_{\Sigma} \frac{[\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X)]^2}{\cos^3 \alpha} d\mu_{\Sigma} \\
& - 2 \int_{\Sigma} \frac{\omega(X, \bar{\nabla}_{e_2} X) \nabla_{e_1} \cos \alpha + \omega(\bar{\nabla}_{e_1} X, X) \nabla_{e_2} \cos \alpha}{\cos^3 \alpha} d\mu_{\Sigma}.
\end{aligned} \quad (3.14)$$

Set $X = x_3 e_3 + x_4 e_4$, then

$$\begin{aligned}
\omega(X, \bar{\nabla}_{e_2} X) &= \langle JX, \bar{\nabla}_{e_2} X \rangle \\
&= \langle x_3 J e_3 + x_4 J e_4, x_{32} e_3 + x_{42} e_4 - x_3 h_{2i}^3 e_i - x_4 h_{2i}^4 e_i \rangle \\
&= x_3 x_{42} \langle J e_3, e_4 \rangle + x_4 x_{32} \langle J e_4, e_3 \rangle \\
&\quad + x_3 (x_3 h_{12}^3 + x_4 h_{12}^4) \langle e_3, J e_1 \rangle + x_3 (x_3 h_{22}^3 + x_4 h_{22}^4) \langle e_3, J e_2 \rangle \\
&\quad + x_4 (x_3 h_{12}^3 + x_4 h_{12}^4) \langle e_4, J e_1 \rangle + x_4 (x_3 h_{22}^3 + x_4 h_{22}^4) \langle e_4, J e_2 \rangle.
\end{aligned}$$

Notice that

$$\langle J e_1, e_2 \rangle = \langle J e_3, e_4 \rangle = \cos \alpha, \quad (3.15)$$

$$\langle J e_1, e_3 \rangle = -\langle J e_2, e_4 \rangle, \quad (3.16)$$

$$\langle J e_1, e_4 \rangle = \langle J e_2, e_3 \rangle. \quad (3.17)$$

Thus

$$\begin{aligned}
\omega(X, \bar{\nabla}_{e_2} X) &= (x_3 x_{42} - x_4 x_{32}) \cos \alpha \\
&\quad + (x_3^2 h_{12}^3 + x_3 x_4 h_{12}^4 - x_3 x_4 h_{22}^3 - x_4^2 h_{22}^4) \langle e_3, J e_1 \rangle \\
&\quad + (x_3^2 h_{22}^3 + x_3 x_4 h_{22}^4 + x_3 x_4 h_{12}^3 + x_4^2 h_{12}^4) \langle e_3, J e_2 \rangle
\end{aligned} \quad (3.18)$$

and

$$\begin{aligned}
\omega(X, \bar{\nabla}_{e_1} X) &= (x_3 x_{41} - x_4 x_{31}) \cos \alpha \\
&\quad + (x_3^2 h_{11}^3 + x_3 x_4 h_{11}^4 - x_3 x_4 h_{12}^3 - x_4^2 h_{12}^4) \langle e_3, J e_1 \rangle \\
&\quad + (x_3^2 h_{12}^3 + x_3 x_4 h_{12}^4 + x_3 x_4 h_{11}^3 + x_4^2 h_{11}^4) \langle e_3, J e_2 \rangle.
\end{aligned} \quad (3.19)$$

Similarly,

$$\begin{aligned}
\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X) &= -\cos \alpha (x_3 H^3 + x_4 H^4) \\
&\quad + (x_{32} + x_{41}) \langle J e_1, e_3 \rangle + (x_{42} - x_{31}) \langle J e_2, e_3 \rangle.
\end{aligned} \quad (3.20)$$

By (3.20), we get that

$$\begin{aligned}
& 4 \int_{\Sigma} \frac{X \cdot H (\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X))}{\cos^2 \alpha} + 2 \int_{\Sigma} \frac{[\omega(\bar{\nabla}_{e_1} X, e_2) + \omega(e_1, \bar{\nabla}_{e_2} X)]^2}{\cos^3 \alpha} \\
&= -2 \int_{\Sigma} \frac{(x_3 H^3 + x_4 H^4)^2}{\cos \alpha} d\mu_{\Sigma} + 2 \int_{\Sigma} \frac{(x_{31} + x_{41})^2 \langle J e_1, e_3 \rangle^2}{\cos^3 \alpha} d\mu_{\Sigma} \\
&\quad + 2 \int_{\Sigma} \frac{(x_{42} - x_{31})^2 \langle J e_2, e_3 \rangle^2}{\cos^3 \alpha} d\mu_{\Sigma} \\
&\quad + 4 \int_{\Sigma} \frac{(x_{32} + x_{41})(x_{42} - x_{31}) \langle J e_1, e_3 \rangle \langle J e_2, e_3 \rangle}{\cos^3 \alpha} d\mu_{\Sigma}.
\end{aligned} \quad (3.21)$$

Now $II(X)$ can be written as

$$\begin{aligned}
 II(X) = & -2 \int_{\Sigma} \frac{x_3^2 R_{3i3i} + x_4^2 R_{4i4i} + 2x_3 x_4 R_{3i4i}}{\cos \alpha} d\mu_{\Sigma} + 2 \int_{\Sigma} \frac{x_{3i}^2 + x_{4i}^2}{\cos \alpha} \\
 & -4 \int_{\Sigma} \frac{(x_3 h_{12}^3 + x_4 h_{12}^4)^2 - (x_3 h_{11}^3 + x_4 h_{11}^4)(x_3 h_{22}^3 + x_4 h_{22}^4)}{\cos \alpha} \\
 & + 2 \int_{\Sigma} \frac{(x_{32} + x_{41})^2 \langle J e_1, e_3 \rangle^2 + (x_{42} - x_{31})^2 \langle J e_2, e_3 \rangle^2}{\cos^3 \alpha} \\
 & + 4 \int_{\Sigma} \frac{(x_{32} + x_{41})(x_{42} - x_{31}) \langle J e_1, e_3 \rangle \langle J e_2, e_3 \rangle}{\cos^3 \alpha} d\mu_{\Sigma} \\
 & + III(X), \tag{3.22}
 \end{aligned}$$

where

$$III(X) = -2 \int_{\Sigma} \frac{\omega(X, \bar{\nabla}_{e_2} X) \nabla_{e_1} \cos \alpha + \omega(\bar{\nabla}_{e_1} X, X) \nabla_{e_2} \cos \alpha}{\cos^3 \alpha} d\mu_{\Sigma}.$$

In addition, if we set $Y = -J_{\nu} X = x_4 e_3 - x_3 e_4$, by (3.22), we get that

$$\begin{aligned}
 II(X) + II(Y) = & -2 \int_{\Sigma} \frac{|X|^2 (R_{3i3i} + R_{4i4i})}{\cos \alpha} d\mu_{\Sigma} + 4 \int_{\Sigma} \frac{x_{3i}^2 + x_{4i}^2}{\cos \alpha} d\mu_{\Sigma} \\
 & -4 \int_{\Sigma} \frac{|X|^2 ((h_{12}^3)^2 + (h_{12}^4)^2 - h_{11}^3 h_{22}^3 - h_{11}^4 h_{22}^4)}{\cos \alpha} d\mu_{\Sigma} \\
 & + 2 \int_{\Sigma} \frac{((x_{32} + x_{41})^2 + (x_{42} - x_{31})^2) (\langle J e_1, e_3 \rangle^2 + \langle J e_2, e_3 \rangle^2)}{\cos^3 \alpha} d\mu_{\Sigma} \\
 & + III(X) + III(Y). \tag{3.23}
 \end{aligned}$$

Using (3.18) and (3.19), it is easy to check that

$$\begin{aligned}
 III(X) + III(Y) = & 4 \int_{\Sigma} (x_3 x_{42} - x_4 x_{32}) \nabla_{e_1} \frac{1}{\cos \alpha} d\mu_{\Sigma} \\
 & -4 \int_{\Sigma} (x_3 x_{41} - x_4 x_{31}) \nabla_{e_2} \frac{1}{\cos \alpha} d\mu_{\Sigma} \\
 & -2 \int_{\Sigma} \frac{|X|^2 (h_{12}^3 - h_{12}^4) \nabla_{e_1} \cos \alpha \langle e_3, J e_1 \rangle}{\cos^3 \alpha} d\mu_{\Sigma} \\
 & -2 \int_{\Sigma} \frac{|X|^2 (h_{22}^3 + h_{12}^4) \nabla_{e_1} \cos \alpha \langle e_4, J e_1 \rangle}{\cos^3 \alpha} d\mu_{\Sigma} \\
 & + 2 \int_{\Sigma} \frac{|X|^2 (h_{11}^3 - h_{12}^4) \nabla_{e_2} \cos \alpha \langle e_3, J e_1 \rangle}{\cos^3 \alpha} d\mu_{\Sigma} \\
 & + 2 \int_{\Sigma} \frac{|X|^2 (h_{12}^3 + h_{11}^4) \nabla_{e_2} \cos \alpha \langle e_4, J e_1 \rangle}{\cos^3 \alpha} d\mu_{\Sigma}. \tag{3.24}
 \end{aligned}$$

Integrating the first two terms in (3.24) by parts, we get that

$$\begin{aligned} & 4 \int_{\Sigma} (x_3 x_{42} - x_4 x_{32}) \nabla_{e_1} \frac{1}{\cos \alpha} d\mu_{\Sigma} - 4 \int_{\Sigma} (x_3 x_{41} - x_4 x_{31}) \nabla_{e_2} \frac{1}{\cos \alpha} d\mu_{\Sigma} \\ &= 4 \int_{\Sigma} \frac{2(x_{41} x_{32} - x_{31} x_{42}) + (x_4 x_{321} - x_3 x_{421}) + (x_3 x_{412} - x_4 x_{312})}{\cos \alpha} d\mu_{\Sigma} \\ &= 8 \int_{\Sigma} \frac{x_{41} x_{32} - x_{31} x_{42}}{\cos \alpha} d\mu_{\Sigma} + 4 \int_{\Sigma} \frac{|X|^2 K_{3412}}{\cos \alpha} d\mu_{\Sigma}, \end{aligned}$$

where we have used (2.10) and (2.11).

We compute the last two terms at each point, so we may choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ around the point such that ω, J take the forms (cf. [1, 2]),

$$\omega = \cos \alpha \theta_1 \wedge \theta_2 + \cos \alpha \theta_3 \wedge \theta_4 + \sin \alpha \theta_1 \wedge \theta_3 - \sin \alpha \theta_2 \wedge \theta_4, \quad (3.25)$$

where $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ is the dual frame of $\{e_1, e_2, e_3, e_4\}$, and

$$J = \begin{pmatrix} 0 & \cos \alpha & \sin \alpha & 0 \\ -\cos \alpha & 0 & 0 & -\sin \alpha \\ -\sin \alpha & 0 & 0 & \cos \alpha \\ 0 & \sin \alpha & -\cos \alpha & 0 \end{pmatrix}. \quad (3.26)$$

By the equation of a symplectic critical surface [3], we have

$$H^4 = \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_1} \alpha, \quad \text{and} \quad H^3 = \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_2} \alpha.$$

Thus

$$\begin{aligned} & -2(h_{12}^3 - h_{22}^4) \nabla_{e_1} \cos \alpha + 2(h_{11}^3 - h_{12}^4) \nabla_{e_2} \cos \alpha \\ &= 2 \sin \alpha (h_{12}^3 + h_{11}^4 - H^4) \nabla_{e_1} \alpha + 2 \sin \alpha (h_{12}^4 + h_{22}^3 - H^3) \nabla_{e_2} \alpha \\ &= -2 \sin \alpha \left(\nabla_{e_1} \alpha + \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_1} \alpha \right) \nabla_{e_1} \alpha \\ &\quad - 2 \sin \alpha \left(\nabla_{e_2} \alpha + \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_2} \alpha \right) \nabla_{e_2} \alpha \\ &= -2 \frac{\sin \alpha}{\cos^2 \alpha} |\nabla \alpha|^2, \end{aligned}$$

and

$$\begin{aligned} & 2(h_{22}^3 + h_{12}^4) \nabla_{e_1} \cos \alpha - 2(h_{12}^3 + h_{11}^4) \nabla_{e_2} \cos \alpha \\ &= 2 \sin \alpha \nabla_{e_2} \alpha \nabla_{e_1} \alpha - 2 \sin \alpha \nabla_{e_2} \alpha \nabla_{e_1} \alpha = 0. \end{aligned}$$

By the Ricci equation

$$K_{3412} = R_{3412} + h_{1k}^3 h_{2k}^4 - h_{2k}^3 h_{1k}^4,$$

we obtain that

$$\begin{aligned} III(X) + III(Y) &= 8 \int_{\Sigma} \frac{x_{41}x_{32} - x_{31}x_{42}}{\cos \alpha} d\mu_{\Sigma} + 4 \int_{\Sigma} \frac{|X|^2 R_{3412}}{\cos \alpha} d\mu_{\Sigma} \\ &\quad - 2 \int_{\Sigma} \frac{|X|^2 \sin^2 \alpha}{\cos^5 \alpha} |\nabla \alpha|^2 d\mu_{\Sigma} \\ &\quad + 4 \int_{\Sigma} \frac{h_{1k}^3 h_{2k}^4 - h_{2k}^3 h_{1k}^4}{\cos \alpha} d\mu_{\Sigma}. \end{aligned} \quad (3.27)$$

Putting all these identities into (3.23), we obtain that

$$\begin{aligned} II(X) + II(Y) &= \\ &\quad -2 \int_{\Sigma} \frac{|X|^2 (R_{3i3i} + R_{4i4i} - 2R_{1234})}{\cos \alpha} d\mu_{\Sigma} \\ &\quad + 4 \int_{\Sigma} \frac{(x_{41} + x_{32})^2 + (x_{42} - x_{31})^2}{\cos \alpha} d\mu_{\Sigma} \\ &\quad + 2 \int_{\Sigma} \frac{[(x_{41} + x_{32})^2 + (x_{42} - x_{31})^2] \sin^2 \alpha}{\cos^3 \alpha} d\mu_{\Sigma} \\ &\quad - 2 \int_{\Sigma} \frac{|X|^2 \sin^2 \alpha}{\cos^5 \alpha} |\nabla \alpha|^2 d\mu_{\Sigma} \\ &\quad - 4 \int_{\Sigma} \frac{|X|^2 ((h_{12}^3)^2 + (h_{12}^4)^2 - h_{11}^3 h_{22}^3 - h_{11}^4 h_{22}^4 - h_{1k}^3 h_{2k}^4 + h_{2k}^3 h_{1k}^4)}{\cos \alpha} d\mu_{\Sigma}. \end{aligned}$$

By Lemma 3.2 in [3], we see that

$$R_{3i3i} + R_{4i4i} - 2R_{1234} = \frac{R}{2} \sin^2 \alpha.$$

By (2.14), we have

$$(x_{41} + x_{32})^2 + (x_{42} - x_{31})^2 = \frac{1}{2} |\bar{\partial} X|^2.$$

Using the fact that Σ is a symplectic critical surface again, we can simplify the last term in the above expression of $II(X) + II(Y)$,

$$\begin{aligned}
 & (h_{12}^3)^2 + (h_{12}^4)^2 - h_{11}^3 h_{22}^3 - h_{11}^4 h_{22}^4 - h_{1k}^3 h_{2k}^4 + h_{2k}^3 h_{1k}^4 \\
 &= (h_{11}^4 + h_{12}^3)(h_{12}^3 - h_{22}^4) + (h_{12}^4 + h_{22}^3)(h_{12}^4 - h_{11}^3) \\
 &= -\nabla_{e_1} \alpha (-\nabla_{e_1} \alpha - H^4) - \nabla_{e_2} \alpha (-\nabla_{e_2} \alpha - H^3) \\
 &= -\nabla_{e_1} \alpha \left(-\nabla_{e_1} \alpha - \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_1} \alpha \right) - \nabla_{e_2} \alpha \left(-\nabla_{e_2} \alpha - \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_2} \alpha \right) \\
 &= \frac{1}{\cos^2 \alpha} |\nabla \alpha|^2.
 \end{aligned}$$

Therefore, it concludes that

$$\begin{aligned}
 II(X) + II(Y) &= - \int_{\Sigma} \frac{R|X|^2 \sin^2 \alpha}{\cos \alpha} d\mu_{\Sigma} + 2 \int_{\Sigma} \frac{|\bar{\partial} X|^2}{\cos \alpha} d\mu_{\Sigma} + \int_{\Sigma} \frac{\sin^2 \alpha}{\cos^3 \alpha} |\bar{\partial} X|^2 d\mu_{\Sigma} \\
 &\quad - 2 \int_{\Sigma} \frac{\sin^2 \alpha}{\cos^5 \alpha} |X|^2 |\nabla \alpha|^2 d\mu_{\Sigma} - 4 \int_{\Sigma} \frac{|X|^2 |\nabla \alpha|^2}{\cos^3 \alpha} d\mu_{\Sigma} \\
 &= - \int_{\Sigma} \frac{R|X|^2 \sin^2 \alpha}{\cos \alpha} d\mu_{\Sigma} + \int_{\Sigma} \frac{|\bar{\partial} X|^2 (1 + \cos^2 \alpha)}{\cos^3 \alpha} d\mu_{\Sigma} \\
 &\quad - 2 \int_{\Sigma} \frac{(1 + \cos^2 \alpha)}{\cos^5 \alpha} |X|^2 |\nabla \alpha|^2 d\mu_{\Sigma}.
 \end{aligned}$$

Theorem 3.3 *If we choose $X = x_3 e_3 + x_4 e_4$, $Y = -J_{\nu} X = x_4 e_3 - x_3 e_4$, which are of compact supports, then the second variation of the functional L of a symplectic critical surface is*

$$\begin{aligned}
 II(X) + II(Y) &= - \int_{\Sigma} \frac{R|X|^2 \sin^2 \alpha}{\cos \alpha} d\mu_{\Sigma} + \int_{\Sigma} \frac{|\bar{\partial} X|^2 (1 + \cos^2 \alpha)}{\cos^3 \alpha} d\mu_{\Sigma} \\
 &\quad - 2 \int_{\Sigma} \frac{(1 + \cos^2 \alpha)}{\cos^5 \alpha} |X|^2 |\nabla \alpha|^2 d\mu_{\Sigma}.
 \end{aligned}$$

4 Stable Symplectic Critical Surfaces

In this section, as an application of the second variation formulae of the functional L , we study stable symplectic critical surfaces.

Proposition 4.1 *Let M be a complete Kähler surface with positive scalar curvature R . If Σ is a stable symplectic critical surface in M whose normal bundle admits a section X with*

$$\frac{|\bar{\partial} X|^2}{|X|^2} \leq \frac{|\nabla \alpha|^2}{\cos^2 \alpha}$$

and

$$\int_{\Sigma} \frac{|X|^2}{\cos^3 \alpha} d\mu_{\Sigma} < \infty,$$

then Σ is a holomorphic curve.

Proof Let $\psi \in C_0^{\infty}(M)$ be a cut-off function, with $\psi \equiv 1$ in $B_r(x_0)$, $\psi \equiv 0$ outside $B_{2r}(x_0)$ and $|\nabla \psi| \leq C/r$, where $x_0 \in \Sigma$. If Σ is a stable symplectic critical surface, we have

$$II(\psi X) + II(\psi Y) \geq 0,$$

where $Y = -J_{\nu}X$. By Theorem 3.3, we obtain

$$\int_{B_r(x_0)} \frac{R|X|^2 \sin^2 \alpha}{\cos \alpha} d\mu_{\Sigma} \leq \frac{C}{r^2}.$$

Since $R > 0$, by the last inequality, letting $r \rightarrow \infty$, we obtain $\sin \alpha \equiv 0$, that is Σ is a holomorphic curve. $\square$

The following corollary is obvious.

Corollary 4.2 *Let M be a complete Kähler surface with positive scalar curvature R . If Σ is a stable symplectic critical surface in M with $\cos \alpha \geq \delta > 0$, whose normal bundle admits a holomorphic section $X \in L^2(\Sigma)$, then Σ is a holomorphic curve.*

Corollary 4.3 *Let M be a complete Kähler surface with positive scalar curvature R . If Σ is a compact stable symplectic critical surface in M with $\chi(\nu) \geq g$ where $\chi(\nu)$ is the Euler characteristic of the normal bundle ν of Σ in M and g is the genus of Σ , then Σ is a holomorphic curve.*

Proof Let K be the canonical line bundle over Σ , let $H^0(\Sigma, \mathcal{O}(\nu))$ denote the space of holomorphic sections of ν and let $H^0(\Sigma, \mathcal{O}(\nu \otimes K))$ denote the space of holomorphic sections of $\nu \otimes K$, then by Riemann–Roch theorem, we have

$$\dim H^0(\Sigma, \mathcal{O}(\nu)) = \chi(\nu) - g + 1 + \dim H^0(\Sigma, \mathcal{O}(\nu \otimes K)) \geq 1.$$

We therefore have a holomorphic section on ν , so the corollary follows from Proposition 4.1. $\square$

5 Symplectic Critical Surfaces in $\mathbb{C}^2$

In this section, we construct some symplectic critical surfaces in $\mathbb{C}^2$ and prove a Liouville theorem for symplectic critical surfaces in $\mathbb{C}^2$. We start with a Liouville theorem.

Theorem 5.1 *If Σ is a complete symplectic critical surface in $\mathbb{C}^2$ with area quadratic growth, and $\cos \alpha \geq \delta > 0$, then it is a holomorphic curve.*

Proof By the equation of the Kähler angle, we have ([3])

$$\Delta \cos \alpha = \frac{3 \sin^2 \alpha - 2}{\cos \alpha} |\nabla \alpha|^2. \quad (5.1)$$

Thus,

$$-\Delta \log \cos \alpha = 2 |\nabla \alpha|^2. \quad (5.2)$$

We choose a cut-off function $\phi \in C_0^\infty(B_{2R}(0))$ such that $\phi \equiv 1$ in $B_R(0)$ and $|\nabla \phi| \leq C/R$, where 0 is a point in Σ , and $B_r(0)$ is the metric ball centered at 0 with radius r . Multiplying $\phi^2 \sin^2 \alpha$ on both sides of (5.2), integrating over Σ , one gets

$$\begin{aligned} 2 \int_{\Sigma} \phi^2 |\nabla \cos \alpha|^2 d\mu &= \int_{\Sigma} \frac{\sin^2 \alpha}{\cos \alpha} \phi \nabla \phi \cdot \nabla \cos \alpha d\mu \\ &\leq C \left(\int_{\Sigma} \phi^2 |\nabla \cos \alpha|^2 d\mu \right)^{1/2} \left(\int_{B_{2R}} |\nabla \phi|^2 d\mu \right)^{1/2}, \end{aligned}$$

which yields that

$$\int_{\Sigma} \phi^2 |\nabla \cos \alpha|^2 d\mu \leq C,$$

where we have used the assumption of the area quadratic growth. Letting $R \rightarrow \infty$, one gets that

$$\int_{\Sigma} |\nabla \cos \alpha|^2 d\mu \leq C. \quad (5.3)$$

We can rewrite the equation as

$$\Delta \cos \alpha = \frac{3 |\nabla \cos \alpha|^2}{\cos \alpha} - \frac{2 |\nabla \cos \alpha|^2}{\sin^2 \alpha \cos \alpha}. \quad (5.4)$$

Multiplying $\phi^2 (1 - \cos^2 \alpha)^p \cos \alpha$ on both sides of (5.4), where p is a positive constant which will be decided later, integrating over Σ , we obtain

$$\begin{aligned} &\int_{\Sigma} \phi^2 (1 - \cos^2 \alpha)^p |\nabla \cos \alpha|^2 d\mu - 2p \int_{\Sigma} \phi^2 \cos^2 \alpha (1 - \cos^2 \alpha)^{p-1} |\nabla \cos \alpha|^2 d\mu \\ &= 2 \int_{\Sigma} \phi^2 (1 - \cos^2 \alpha)^{p-1} |\nabla \cos \alpha|^2 d\mu - 3 \int_{\Sigma} \phi^2 (1 - \cos^2 \alpha)^p |\nabla \cos \alpha|^2 \phi^2 d\mu \\ &\quad - 2 \int_{\Sigma} \cos \alpha \phi (1 - \cos^2 \alpha)^p \nabla \cos \alpha \cdot \nabla \phi d\mu, \end{aligned}$$

and thus we have

$$\begin{aligned} & \int_{\Sigma} \phi^2 [(2p+4)\cos^2 \alpha - 2] (1 - \cos^2 \alpha)^{p-1} |\nabla \cos \alpha|^2 d\mu \\ &= 2 \int_{\Sigma} \cos \alpha \phi (1 - \cos^2 \alpha)^p \nabla \cos \alpha \cdot \nabla \phi d\mu \\ &\leq C \int_{B_{2R} \setminus B_R} |\nabla \cos \alpha|^2 d\mu. \end{aligned}$$

Applying (5.3) and letting $R \rightarrow \infty$, we know that the term on the right-hand side of the last inequality tends to zero. We choose p such that $(2p+4)\cos^2 \alpha - 2 \geq (2p+4)\delta^2 - 2 \geq 1$, then we get that

$$\int_{\Sigma} (1 - \cos^2 \alpha)^{p-1} |\nabla \cos \alpha|^2 d\mu = 0,$$

and it follows that $\nabla \cos \alpha \equiv 0$ on Σ or $\cos \alpha = 1$. This proves the theorem. $\square$

Then, we construct some symplectic critical surfaces in $\mathbf{C}^2$. We first consider the surface Σ which is defined by

$$F(u, v) = (v \cos u, v \sin u, -\ln v, 0), \quad u \in [0, 2\pi], \quad v > 0.$$

It is in fact a rotational symmetric surface $z = -\frac{1}{2} \log(x^2 + y^2)$ in $\mathbf{R}^3$, we consider it as a surface in $\mathbf{C}^2$.

It is clear that

$$\begin{aligned} F_v &= (\cos u, \sin u, -\frac{1}{v}, 0), \\ F_u &= (-v \sin u, v \cos u, 0, 0). \end{aligned}$$

We choose the normal basis

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{1+v^2}} (\cos u, \sin u, v, 0), \\ v_2 &= (0, 0, 0, 1). \end{aligned}$$

Thus the induced metric on Σ is

$$(g_{ij}) = \begin{pmatrix} \frac{1+v^2}{v^2} & 0 \\ 0 & v^2 \end{pmatrix}, \quad d\mu = \sqrt{1+v^2} du dv.$$

It is evident that

$$\begin{aligned} F_{vv} &= (0, 0, \frac{1}{v^2}, 0), \\ F_{vu} &= (-\sin u, \cos u, 0, 0), \\ F_{uu} &= (-v \cos u, -v \sin u, 0, 0). \end{aligned}$$

Using the standard complex structure J of $\mathbb{C}^2$,

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (5.5)$$

the Kähler angle of Σ is

$$\cos \alpha = \frac{\langle JF_v, F_u \rangle}{\sqrt{(\det(g))}} = \frac{v}{\sqrt{1+v^2}}.$$

It is easy to compute that

$$\begin{aligned} H^1 &= g^{11} \langle F_{vv}, v_1 \rangle + g^{22} \langle F_{uu}, v_1 \rangle \\ &= \frac{v}{(1+v^2)^{\frac{3}{2}}} - \frac{1}{v(1+v^2)^{\frac{1}{2}}} \\ &= -\frac{1}{v(1+v^2)^{\frac{3}{2}}}, \\ H^2 &= g^{11} \langle F_{vv}, v_2 \rangle + g^{22} \langle F_{uu}, v_2 \rangle \\ &= 0. \end{aligned}$$

Since

$$\begin{aligned} \nabla \cos \alpha &= g^{11} \frac{\partial \cos \alpha}{\partial v} F_v \\ &= \frac{v^2}{1+v^2} \left(\frac{1}{\sqrt{1+v^2}} - \frac{v^2}{(1+v^2)^{\frac{3}{2}}} \right) F_v \\ &= \frac{v^2}{(1+v^2)^{\frac{5}{2}}} F_v, \end{aligned}$$

we obtain that

$$\begin{aligned} (J \nabla \cos \alpha)^\top &= \frac{v^2}{(1+v^2)^{\frac{5}{2}}} \frac{\langle JF_v, F_u \rangle F_u}{|F_u|^2} \\ &= \frac{v^2}{(1+v^2)^{\frac{5}{2}}} \frac{\cos \alpha \sqrt{\det(g)}}{v^2} F_u \end{aligned}$$

$$\begin{aligned}
&= \frac{\cos \alpha}{(1+v^2)^2} F_u, \\
(J(J\nabla \cos \alpha)^\top)^\perp &= \frac{\cos \alpha}{(1+v^2)^2} [\langle JF_u, v_1 \rangle v_1 + \langle JF_u, v_2 \rangle v_2] \\
&= -\cos \alpha \frac{v}{(1+v^2)^{\frac{5}{2}}} v_1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\cos^3 \alpha \vec{H} &= -\cos^3 \alpha \frac{1}{v(1+v^2)^{\frac{3}{2}}} v_1 \\
&= -\cos \alpha \frac{v}{(1+v^2)^{\frac{5}{2}}} v_1 \\
&= (J(J\nabla \cos \alpha)^\top)^\perp.
\end{aligned}$$

So Σ is a symplectic critical surface which is not minimal.

The Gauss curvature of Σ is

$$K = \frac{-1}{(1+v^2)^2}.$$

It is not difficult to see that Σ is a complete surface. See the Fig. 1.

We then consider the surface Σ

$$F(u, v) = (v \sin u, v \cos u, f(v), u), \quad u \in (-\infty, +\infty), \quad v > 0.$$

It is clear that

$$\begin{aligned}
F_v &= (\sin u, \cos u, f_v, 0), \\
F_u &= (v \cos u, -v \sin u, 0, 1)
\end{aligned}$$

Fig. 1 Symplectic critical graph

and

$$d\mu = \sqrt{(1+v^2)(1+(f')^2)} du dv.$$

We choose the normal vectors

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{1+v^2}}(-\cos u, \sin u, 0, v), \\ v_2 &= \frac{1}{\sqrt{1+(f')^2}}(f' \sin u, f' \cos u, -1, 0). \end{aligned}$$

Using the complex structure of (5.5), we have

$$\cos \alpha = \frac{f' - v}{\sqrt{(1+v^2)(1+(f')^2)}}.$$

Straight computation yields that

$$\begin{aligned} H^1 &= 0, \\ H^2 &= -\frac{f''}{(1+(f')^2)^{\frac{3}{2}}} - \frac{vf'}{(1+v^2)(1+(f')^2)^{\frac{1}{2}}} \end{aligned}$$

and

$$(J(J\nabla \cos \alpha)^\top)^\perp = \frac{\cos \alpha(1+vf')}{(1+(f')^2)(1+v^2)^{\frac{1}{2}}} \frac{\partial \cos \alpha}{\partial v}.$$

One checks that F is a symplectic critical surface in $\mathbb{C}^2$ if and only if $f(v)$ satisfies the equation

$$(1+v^2)^2 f'' = 1 + (2v-v^3)f' + 3v^2(f')^2 - v(f')^3.$$

It is clear that $f_1(v) = \frac{1}{2}v^2$, $f_2(v) = -\ln v$ and $f_3(v) = v^2 + \ln v$ ($v > 0$) are solutions. The surface Σ with $f = f_1$ is Lagrangian which is not minimal. The surface Σ with $f = f_2$ is a holomorphic curve. If we choose $f = f_3$, then Σ is a symplectic critical surface which is not holomorphic.

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