***R*-Matrix Realization of Two-Parameter Quantum Group $U_{r,s}(\mathfrak{gl}_n)$**

Naihuan Jing · Ming Liu

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Abstract We provide a Faddeev–Reshetikhin–Takhtajan’s RTT approach to the quantum group $\text{Fun}(\text{GL}_{r,s}(n))$ and the quantum enveloping algebra $U_{r,s}(\mathfrak{gl}_n)$ corresponding to the two-parameter *R*-matrix. We prove that the quantum determinant $\det_{r,s} T$ is a quasi-central element in $\text{Fun}(\text{GL}_{r,s}(n))$ generalizing earlier results of Dipper–Donkin and Du–Parshall–Wang. The explicit formulation provides an interpretation of the deforming parameters, and the quantized algebra $U_{r,s}(R)$ is identified to $U_{r,s}(\mathfrak{gl}_n)$ as the dual algebra. We then construct $n - 1$ quasi-central elements in $U_{r,s}(R)$ which are analogs of higher Casimir elements in $U_q(\mathfrak{gl}_n)$.

Keywords Quantum groups · Determinants · Casimir elements · Yang–Baxter equations

Mathematics Subject Classification 17B37 · 20G42 · 16T25

1 Introduction

Quantum groups were discovered as certain noncommutative and noncocommutative Hopf algebras by Drinfeld [10] and Jimbo [15]. The standard definition of a quantum group $U_q(\mathfrak{g})$ is given as a *q*-deformation of universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$ generated by the Chevalley generators under the Serre relations based on the data coming from the corresponding Cartan matrix.

N. Jing

Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA

N. Jing (✉) · M. Liu

School of Mathematical Sciences, South China University of Technology, Guangzhou 510640, China
e-mail: jing@math.ncsu.edu

Faddeev et al. [12] gave another realization of the quantum groups using the solutions R of the Yang–Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (1.1)$$

They also studied the quantum function algebra $\text{Fun}(\text{GL}_q(n))$ for Lie group GL_n in [12]. In particular, the quantum determinant $\text{qdet}(T)$ for the quantum function algebra $\text{Fun}(\text{GL}_q(n))$ was introduced and proved to be a special central element. Furthermore, the authors studied the algebra $U(R)$ as a dual algebra of the quantum function algebra and proved that $U(R)$ is isomorphic to $U_q(\mathfrak{g})$. This approach can systematically provide a complete set of generators for the center of the quantum enveloping algebra $U_q(\mathfrak{sl}_n)$:

$$c_k = \sum_{\sigma, \sigma' \in S_n} (-q)^{l(\sigma) + l(\sigma')} l_{\sigma(1), \sigma'(1)}^+ \cdots l_{\sigma(k), \sigma'(k)}^+ l_{\sigma(k+1), \sigma'(k+1)}^- \cdots l_{\sigma(n), \sigma'(n)}^-, \quad (1.2)$$

where $l_{ij}^\pm$ are the q -analogs of the Weyl root vectors corresponding to the roots $\pm(\epsilon_i - \epsilon_j)$.

2-parameter general linear and special linear quantum groups were introduced by Takeuchi [20] in 1990. A special case of the quantum coordinate algebra was the subject of study in relationship with the Schur q -algebra [8, 11] where a quantum determinant was shown to be quasi-central. The two-parameter quantum enveloping algebras have later gained full attention after Benkart and Witherspoon's works [1, 2] on the (r, s) -deformed quantum algebras associated with $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$, where the quantum R -matrix and the Drinfeld doubles were obtained (see also [4]). They further showed that the representation theory of the 2-parameter quantum enveloping algebras can be similarly developed as the one-parameter case. In this context they established a two-parameter quantum Schur–Weyl duality (see also [16]). In [17] an interpretation of the two parameters was demonstrated through a combinatorial realization of the affine analog [14] of the quantum special linear algebra.

The quantum inverse scattering method has been very successful in studying various quantum groups [6] such as the Yangian algebras [18, 19] and finite W-algebras [5]. For quantum matrix algebras the method has been particularly useful [12, 13]. In this paper we will generalize the Faddeev–Reshetikhin–Takhtajan (FRT) approach [12] to the two-parameter case by studying the two-parameter quantum linear groups and the quantum general/special linear algebras as dual Hopf algebras. The anti-symmetric tensors [16] associated to the R -matrix [3] help one to define 2-parameter quantum determinant $\det_{r,s} T$ and the quantum Casimir elements. If t_{ij} are the generators of the 2-parameter quantum monoid $\text{Mat}_{r,s}(n)$, using the FRT method we obtain that the row determinant and the column determinant are given, respectively, by

$$\begin{aligned} \det_{r,s} T &= \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1), 1} \cdots t_{\sigma(n), n} \\ &= \sum_{\sigma \in S_n} (-r)^{-l(\sigma)} t_{1, \sigma(1)} \cdots t_{n, \sigma(n)}. \end{aligned}$$

This means that the two parameters r, s exactly correspond to column permutations and row permutations respectively.

By a similar method to FRT approach, we are able to define naturally and systematically the Weyl root vectors or Gauss generators for $U_{r,s}(\mathfrak{sl}_n)$ and their commutation relations can be compactly written as matrix equations in terms of the 2-parameter R -matrices. Moreover we can obtain the explicit formulae for the higher Casimir elements and show that they are quasi-central. In studying representation theory of 2-parameter quantum algebras, it seems to be more natural to use the quasi-commutative Casimir elements instead of using the central elements given the similarity of the two representation theories.

The paper is organized as follows. In Sect. 2, we study the 2-parameter quantum function algebra $\text{Fun}(\text{GL}_{r,s}(n))$ and its 2-parameter quantum determinant. Different from the 1-parameter case, we prove that the 2-parameter quantum determinant $\det_{r,s}(T)$ is not a central element but a quasi-central element. In Sect. 3, we study the algebra $U(R)$ as the dual of $\text{Fun}(\text{GL}_{r,s}(n))$, and analog to the 1-parameter case, we give $n - 1$ quasi-central elements c_k in $U(R)$ which generalize the classical higher Casimir elements in the center of the universal enveloping algebra. In Sect. 4, we study the commutation relations between the Cartan–Weyl generators of $U(R)$ by using the Gauss decomposition of the matrix $L^\pm$ inside $U(R)$. In Sect. 5, we show that $U_{r,s}(\mathfrak{gl}_n)$ in Drinfeld–Jimbo realization is isomorphic to $U(R)$ in FRT realization.

2 2-Parameter Quantum Algebra $\text{Fun}(\text{GL}_{r,s}(n))$

In this section, we study the 2-parameter quantum coordinate algebra $\text{Fun}(\text{GL}_{r,s}(n))$ associated to the general linear group GL_n using the FRT method.

Let $V = \mathbb{C}^n$, the n -dimensional complex space with the basis of column vectors $e^i = (0, \dots, 1, \dots, 0)^T$. Let e_{ij} be the unit matrices acting on V so that $e_{ij}e^k = \delta_{jk}e^i$. The 2-parameter quantum R -matrix $\in \text{End}(V \otimes V)$ is given as

$$\begin{aligned} R &= \sum_{ijkl} R_{kl}^{ij} e_{ik} \otimes e_{jl} \\ &= s \sum_{i=1}^n e_{ii} \otimes e_{ii} + rs \sum_{i>j} e_{ii} \otimes e_{jj} + \sum_{i<j} e_{ii} \otimes e_{jj} + (s-r) \sum_{i>j} e_{ij} \otimes e_{ji}, \end{aligned} \quad (2.1)$$

which satisfies the well-known Yang–Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{12} = R \otimes 1$, and R_{ij} acts on the i th and j th copies of V inside $V^{\otimes 3}$.

Remark 2.1 When $s = q = r^{-1}$, the R -matrix (2.1) is the 1-parameter R -matrix considered in [12]. We remark that the R -matrix can not be re-scaled to a one-parameter R -matrix.

We set

$$\widehat{R} = PR = s \sum_{i=1}^n e_{ii} \otimes e_{ii} + rs \sum_{i < j} e_{ij} \otimes e_{ji} + \sum_{i > j} e_{ij} \otimes e_{ji} + (s-r) \sum_{i < j} e_{ii} \otimes e_{jj}, \quad (2.2)$$

where P is the permutation operator $P(u \otimes v) = v \otimes u$. Then the Yang–Baxter equation (1.1) is equivalent to the braid relation

$$\widehat{R}_{12} \widehat{R}_{23} \widehat{R}_{12} = \widehat{R}_{23} \widehat{R}_{12} \widehat{R}_{23}. \quad (2.3)$$

Furthermore $\widehat{R}$ satisfies the following Hecke relation:

$$(\widehat{R} - s)(\widehat{R} + r) = 0.$$

For $r + s \neq 0$, $\widehat{R}$ has the following spectral decomposition

$$\widehat{R} = sP^+ - rP^-,$$

where $P^\pm$ are the idempotents such that $P^+P^- = P^-P^+ = 0$, $P^+ + P^- = 1$ and they are given by

$$P^+ = \frac{\widehat{R} + r}{r + s},$$

$$P^- = \frac{-\widehat{R} + s}{r + s}.$$

Following [12] we define the quantum algebra $A(R)$ of the matrix monoid for the 2-parameter R -matrix (2.1).

Definition 2.2 The algebra $A(R)$ is an associative algebra generated by t_{ij} , $1 \leq i, j \leq n$ subject to the quadratic relations defined by

$$RT_1T_2 = T_2T_1R, \quad (2.4)$$

where $T_1 = \sum_{ij} t_{ij} \otimes e_{ij} \otimes 1$, $T_2 = \sum_{ij} t_{ij} \otimes 1 \otimes e_{ij} \in A(R) \otimes \text{End}(V^{\otimes 2})$.

Thus an n -dimensional (defining) representation W of $A(R)$ is a linear map $T \rightarrow A \in \text{End}(W)$ such that $RA_1A_2 = A_2A_1R$ on $\text{End}(W \otimes V^{\otimes 2})$.

Proposition 2.3 [12] *The algebra $A(R)$ is a bialgebra with comultiplication*

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \quad i, j = 1, \dots, n,$$

and co-unit ϵ

$$\epsilon(t_{ij}) = \delta_{ij}, \quad i, j = 1, \dots, n.$$

Suppose there are elements t'_{ij} , $1 \leq i, j \leq n$ such that

$$\sum_k t'_{ik} t_{kj} = \sum_k t_{ik} t'_{kj} = \delta_{ij} \cdot 1.$$

Then the algebra $\text{Fun}(\text{GL}_{r,s}(n))$ is defined to be the associative algebra generated by t_{ij}, t'_{ij} subject to aforementioned relations involving the generators. Clearly one has $T'T = TT' = 1$ in $\text{Fun}(\text{GL}_{r,s}(n))$. Moreover $\text{Fun}(\text{GL}_{r,s}(n))$ is a Hopf algebra with the antipode given by $S(t_{ij}) = t'_{ij}$. We remark that the elements t'_{ij} will be shown to exist once we prove that the quantum determinant is a regular element in the ring $\text{Fun}(\text{GL}_{r,s}(n))$ as the elements t'_{ij} can be solved by the quantum Cramer rule.

Example 2.4 The 2-parameter quantum group $\text{Fun}(\text{GL}_{r,s}(2))$ is generated by t_{ij} , $\det_{r,s}^{\pm 1}$ subject to the relations

$$\begin{aligned} t_{11}t_{12} &= r^{-1}t_{12}t_{11}, & t_{11}t_{21} &= st_{21}t_{11} \\ t_{21}t_{22} &= r^{-1}t_{21}t_{22}, & t_{12}t_{22} &= st_{22}t_{12} \\ t_{12}t_{21} &= rst_{21}t_{12}, \\ t_{11}t_{22} - t_{22}t_{11} &= (s-r)t_{21}t_{12}, \\ \det_{r,s}\det_{r,s}^{-1} &= \det_{r,s}^{-1}\det_{r,s} = 1, \\ (\det_{r,s})t_{ij} &= (rs)^{i-j}t_{ij}(\det_{r,s}), \end{aligned}$$

where

$$\begin{aligned} \det_{r,s} &= t_{11}t_{22} - st_{21}t_{12} = t_{22}t_{11} - rt_{21}t_{12} \\ &= t_{11}t_{22} - r^{-1}t_{12}t_{21}. \end{aligned}$$

We now turn to the general quantum determinant of $\text{Fun}(\text{GL}_{r,s}(n))$ using the quantum inverse scattering method via anti-symmetric tensors (cf. [16]).

Let V^* be the dual space of V spanned by the row vectors $e_i = e_i^T$. The endomorphism space $\text{End}(V)$ is identified with $V \otimes V^*$, which has the natural basis elements $e_{ij} = e^i e_j := e^i \otimes e_j$.

Before introducing the quantum determinant, we introduce the (r, s) -deformed antisymmetric tensors $|\varepsilon\rangle \doteq |\varepsilon_n\rangle \in V^{\otimes n}$, $\langle\varepsilon| \doteq \langle\varepsilon_n| \in (V^*)^{\otimes n}$ as follows:

$$\langle\varepsilon|\varepsilon\rangle = 1, \quad (2.5)$$

$$\langle\varepsilon|P_{k,k+1}^+ = 0, \quad k = 1, 2, \dots, n-1, \quad (2.6)$$

$$P_{k,k+1}^+|\varepsilon\rangle = 0, \quad k = 1, 2, \dots, n-1, \quad (2.7)$$

where $P_{k,k+1}^+ = 1^{\otimes(k-1)} \otimes P^+ \otimes 1^{\otimes(n-k-1)}$ with P^+ at the $(k, k+1)$ -position. It is straightforward to prove that the above equations have a unique solution up to normalization, and the (r, s) -deformed antisymmetric tensors can be chosen as follows:

$$\langle \varepsilon | = \frac{1}{[n]_{r,s}!} \sum_{\sigma \in S_n} (-s)^{l(\sigma)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}, \quad (2.8)$$

$$|\varepsilon\rangle = r^{\frac{n(n-1)}{2}} \sum_{\sigma \in S_n} (-r)^{-l(\sigma)} e^{\sigma(1)} \otimes \cdots \otimes e^{\sigma(n)}, \quad (2.9)$$

where $[k]_{r,s} = \frac{s^k - r^k}{s - r}$, $[k]_{r,s}! = [1]_{r,s}[2]_{r,s} \cdots [k]_{r,s}$, and e_i the i -th row vector, e^j the j -th column vector.

In view of the RTT relations (2.4) of $A(R)$ and definition of P^+ , we see that $\langle \varepsilon | T_1 \cdots T_n$ and $T_1 \cdots T_n |\varepsilon\rangle$ satisfy the following equations, respectively:

$$\begin{aligned} \langle \varepsilon | T_1 \cdots T_n P_{k,k+1}^+ &= 0, \quad k = 1, 2, \dots, n-1, \\ P_{k,k+1}^+ T_1 \cdots T_n |\varepsilon\rangle &= 0, \quad k = 1, 2, \dots, n-1. \end{aligned}$$

From the uniqueness of the solutions to Eqs. (2.6) and (2.7) it follows that

$$\langle \varepsilon | T_1 \cdots T_n = c_T \langle \varepsilon |, \quad (2.10)$$

$$T_1 \cdots T_n |\varepsilon\rangle = c^T |\varepsilon\rangle, \quad (2.11)$$

where $c_T, c^T \in \text{Fun}_{r,s}(\text{GL}_n)$. Subsequently it follows from (2.5) that

$$c_T = c^T = \langle \varepsilon | T_1 \cdots T_n |\varepsilon\rangle. \quad (2.12)$$

Definition 2.5 The quantum determinant of $\text{Fun}(\text{GL}_{r,s}(n))$ is defined to be the normalization factor

$$\det_{r,s} T = \langle \varepsilon | T_1 \cdots T_n |\varepsilon\rangle.$$

To find an explicit formula of $\det_{r,s} T$ we consider the rank one tensor $A_n = |\varepsilon\rangle\langle \varepsilon|$ acting on $(\mathbb{C}^n)^{\otimes n}$. It is easily seen that $A_n^2 = A_n$. The following result gives some compact form of various commutation relations in $\text{Fun}(\text{GL}_{r,s}(n))$.

Proposition 2.6 $A_n T_1 T_2 \cdots T_n = T_1 T_2 \cdots T_n A_n = A_n T_1 T_2 \cdots T_n A_n = \det_{r,s} T A_n$.

Proof By definition T commutes with $|\varepsilon\rangle$ or $\langle \varepsilon|$. It follows from $A_n^2 = A_n$ that

$$\begin{aligned} A_n T_1 \cdots T_n &= A_n T_1 \cdots T_n A_n \\ &= |\varepsilon\rangle\langle \varepsilon | T_1 \cdots T_n |\varepsilon\rangle\langle \varepsilon | \\ &= (\langle \varepsilon | T_1 \cdots T_n |\varepsilon\rangle) |\varepsilon\rangle\langle \varepsilon | \\ &= \det_{r,s} T A_n. \end{aligned}$$

The other identities are proved similarly. □

Remark 2.7 The quantum determinant is also expressed as a partial trace.

$$\begin{aligned} Tr_{1,\dots,n}(A_n T_1 \cdots T_n) &= Tr_{1,\dots,n}(|\varepsilon\rangle\langle\varepsilon| T_1 \cdots T_n) \\ &= Tr_{1,\dots,n}(\langle\varepsilon| T_1 \cdots T_n) |\varepsilon\rangle \\ &= \det_{r,s} T. \end{aligned}$$

In the following proposition, we give the explicit expression of the 2-parameter quantum determinant $\det_{r,s} T$ in $\text{Fun}(\text{GL}_{r,s}(n))$.

Proposition 2.8 *For any fixed $\eta \in S_n$ we have*

$$\begin{aligned} \det_{r,s} T &= (-s)^{-l(\eta)} \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),\eta(1)} \cdots t_{\sigma(n),\eta(n)} \\ &= (-r)^{l(\eta)} \sum_{\sigma \in S_n} (-r)^{-l(\sigma)} t_{\eta(1),\sigma(1)} \cdots t_{\eta(n),\sigma(n)}. \end{aligned} \quad (2.13)$$

In particular,

$$\begin{aligned} \det_{r,s} T &= \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),1} \cdots t_{\sigma(n),n} \\ &= \sum_{\sigma \in S_n} (-r)^{-l(\sigma)} t_{1,\sigma(1)} \cdots t_{n,\sigma(n)}. \end{aligned} \quad (2.14)$$

Proof The two formulas of (2.13) are proved similarly, so we only consider the first one. Recall that Proposition 2.6 says that $A_n T_1 \cdots T_n = \det_{r,s} T A_n$. For any $\eta \in S_n$ we apply the left-hand side to the column vector $e^{\eta(1)} \otimes \cdots \otimes e^{\eta(n)}$:

$$\begin{aligned} A_n T_1 \cdots T_n (e^{\eta(1)} \otimes \cdots \otimes e^{\eta(n)}) &= A_n \sum_{i_1, \dots, i_n} t_{i_1, \eta(1)} \cdots t_{i_n, \eta(n)} (e^{i_1} \otimes \cdots \otimes e^{i_n}) \\ &= \sum_{i_1, \dots, i_n} t_{i_1, \eta(1)} \cdots t_{i_n, \eta(n)} |\varepsilon\rangle\langle\varepsilon| (e^{i_1} \otimes \cdots \otimes e^{i_n}) \\ &= \frac{1}{[n]_{r,s}!} \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),\eta(1)} \cdots t_{\sigma(n),\eta(n)} |\varepsilon\rangle. \end{aligned} \quad (2.15)$$

On the other hand, we have

$$\det_{r,s} T A_n (e^{\eta(1)} \otimes \cdots \otimes e^{\eta(n)}) = \frac{1}{[n]_{r,s}!} (-s)^{l(\eta)} \det_{r,s} T |\varepsilon\rangle. \quad (2.16)$$

Paring with $\langle\varepsilon|$ to the right-hand sides of (2.15 and 2.16) we obtain that

$$\det_{r,s} T = (-s)^{-l(\eta)} \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),\eta(1)} \cdots t_{\sigma(n),\eta(n)}.$$

□

Remark 2.9 One can also show the following

$$\begin{aligned}\det_{r,s} T &= \sum_{\sigma \in S_n} (-r)^{l(\sigma)} t_{\sigma(n),n} \cdots t_{\sigma(1),1} \\ &= \sum_{\sigma \in S_n} (-s)^{-l(\sigma)} t_{n,\sigma(n)} \cdots t_{1,\sigma(1)}.\end{aligned}\quad (2.17)$$

When $s = q = r^{-1}$, the 2-parameter quantum determinant is reduced to the 1-parameter quantum determinant

$$\det_q T = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{1,\sigma(1)} \cdots t_{n,\sigma(n)}.$$

Different from the 1-parameter case, the 2-parameter quantum determinant is not a central element, but a quasi-central element. It is still a regular element in the ring $\text{Fun}(\text{GL}_{r,s}(n))$, so the antipode is well-defined.

Theorem 2.10 *The 2-parameter quantum determinant $\det_{r,s} T$ is quasi-central in $\text{Fun}(\text{GL}_{r,s}(n))$. In fact,*

$$(\det_{r,s} T)T = M^{-1} T M (\det_{r,s} T), \quad (2.18)$$

where

$$M = \begin{pmatrix} (rs)^{n-1} & & & \\ & (rs)^{n-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

which implies that $(\det_{r,s} T)t_{ij} = (rs)^{i-j} t_{ij}(\det_{r,s} T)$.

Before we prove the theorem, we need the following lemma.

Lemma 2.11 *Let $M_{n+1} = 1^{\otimes n} \otimes M \in \text{End}(V)^{\otimes(n+1)}$, then $sM_{n+1}(A_n \otimes 1) = (A_n \otimes 1)R_{1,n+1} \cdots R_{n,n+1}$ on $V^{\otimes(n+1)}$.*

Proof We order the indices (ij) lexicographically. As $R_{ij}^{ji} = s - r$ when $i > j$, the matrix R is an upper triangular block matrix with diagonal blocks given by

$$R_{ik}^{ik} = \begin{cases} s, & i = k; \\ rs, & i > k; \\ 1, & i < k. \end{cases} \quad (2.19)$$

In the same manner $R_{1,n+1}R_{2,n+1} \cdots R_{n,n+1}$ is also an upper triangular block matrix with diagonal blocks given by

$$\sum_{i_1, \dots, i_n, k} R_{i_1 k}^{i_1 k} \cdots R_{i_n k}^{i_n k} e_{i_1, i_1} \otimes \cdots \otimes e_{i_n, i_n} \otimes e_{k, k}. \quad (2.20)$$

We then get that for $\langle \varepsilon | = \langle \varepsilon_n | \otimes 1$

$$\langle \varepsilon | R_{1,n+1} \cdots R_{n,n+1} | \varepsilon \rangle = \sum_{k=1}^n \frac{r^{\frac{n(n-1)}{2}}}{[n]_{r,s}!} \sum_{\sigma \in S_n} (sr^{-1})^{l(\sigma)} R_{\sigma(1)k}^{\sigma(1)k} \cdots R_{\sigma(n)k}^{\sigma(n)k} (e_{k,k})_{n+1}. \quad (2.21)$$

Moreover, by Eq. (2.19) we have $R_{\sigma(1)k}^{\sigma(1)k} \cdots R_{\sigma(n)k}^{\sigma(n)k} = s(rs)^{n-k}$ for any $\sigma \in S_n$. Therefore

$$\langle \varepsilon | R_{1,n+1} \cdots R_{n,n+1} | \varepsilon \rangle = \sum_{k=1}^n s(rs)^{n-k} (e_{k,k})_{n+1} = sM_{n+1}. \quad (2.22)$$

Note that $A_n = |\varepsilon_n\rangle\langle\varepsilon_n|$, we then obtain that

$$(A_n \otimes 1)R_{1,n+1} \cdots R_{n,n+1}(A_n \otimes 1) = sM_{n+1}(A_n \otimes 1). \quad (2.23)$$

On the other hand the map

$$T_i \mapsto R_{i,n+1}$$

defines a representation of $\text{Fun}(\text{GL}_{r,s}(n))$, so $A_n T_1 \cdots T_n = T_1 \cdots T_n A_n$ implies that

$$\begin{aligned} (A_n \otimes 1)R_{1,n+1} \cdots R_{n,n+1}(A_n \otimes 1) &= (A_n \otimes 1)^2 R_{1,n+1} \cdots R_{n,n+1} \\ &= (A_n \otimes 1)R_{1,n+1} \cdots R_{n,n+1} = sM_{n+1}(A_n \otimes 1). \end{aligned}$$

□

Now we prove Theorem 2.10.

Proof For brevity we write A_n for $(A_n \otimes 1)$ when it is clear from the context. Using the RTT relations of $\text{Fun}(\text{GL}_{r,s}(n))$ to move T_{n+1} , we have that

$$A_n T_1 \cdots T_n T_{n+1} = A_n (R_{1,n+1} \cdots R_{n,n+1})^{-1} T_{n+1} T_1 \cdots T_n (R_{1,n+1} \cdots R_{n,n+1}). \quad (2.24)$$

It follows from Lemma 2.11 that

$$A_n T_1 \cdots T_n T_{n+1} = s^{-1} M_{n+1}^{-1} T_{n+1} A_n T_1 \cdots T_n (R_{1,n+1} \cdots R_{n,n+1}). \quad (2.25)$$

Note that $A_n T_1 \cdots T_n = A_n T_1 \cdots T_n A_n$ (Prop. 2.6). Applying Lemma 2.11 again we obtain that

$$\begin{aligned} A_n T_1 \cdots T_n T_{n+1} &= s^{-1} M_{n+1}^{-1} T_{n+1} A_n T_1 \cdots T_n A_n (R_{1,n+1} \cdots R_{n,n+1}) \\ &= M_{n+1}^{-1} T_{n+1} M_{n+1} A_n T_1 \cdots T_n. \end{aligned}$$

Taking partial trace $Tr_{1,\dots,n}$ (see Remark 2.7), we finally get that $(\det_{r,s} T)T = M^{-1} T M (\det_{r,s} T)$. □

3 FRT Realization of 2-Parameter Quantum Groups

In this section we study the algebra $U(R)$ as the dual Hopf algebra of $\text{Fun}(\text{GL}_{r,s}(n))$. Consider the R -matrix $\in \text{End}(V \otimes V)$

$$R^{(+)} = PRP = R_{21} = s \sum_{i=1}^n e_{ii} \otimes e_{ii} + rs \sum_{i < j} e_{ii} \otimes e_{jj} + \sum_{i > j} e_{ii} \otimes e_{jj} + (s-r) \sum_{i < j} e_{ij} \otimes e_{ji},$$

which is another solution of the Yang–Baxter equation (1.1).

Definition 3.1 $U(R)$ is an associative algebra with generators l_{ij}^+, l_{ji}^- , $1 \leq i \leq j \leq n$ subject to the quadratic relations given by

$$R^{(+)} L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R^{(+)}, \quad (3.1)$$

$$R^{(+)} L_1^+ L_2^- = L_2^- L_1^+ R^{(+)}, \quad (3.2)$$

where $L_1^{\pm} = \sum l_{ij}^{\pm} e_{ij} \otimes 1$, $L_2^{\pm} = \sum l_{ij}^{\pm} 1 \otimes e_{ij}$ and $L^{\pm} = (l_{ij}^{\pm})$ ($1 \leq i, j \leq n$) are invertible triangular matrices with $l_{ij}^+ = l_{ji}^- = 0$ for $1 \leq j < i \leq n$.

Proposition 3.2 *The algebra $U(R)$ is a Hopf algebra with comultiplication, antipode and counit given by*

$$\Delta(l_{ij}^{\pm}) = \sum_k l_{ik}^{\pm} \otimes l_{kj}^{\pm}, \quad (3.3)$$

$$S(L^{\pm}) = (L^{\pm})^{-1}, \quad (3.4)$$

$$\epsilon(l_{ij}^{\pm}) = \delta_{ij}. \quad (3.5)$$

As in the dual algebra case, let $\widehat{R}^{(+)} = PR^{(+)} = \widehat{R}_{21}$, then

$$\widehat{R}^{(+)} = s \sum_{i=1}^n e_{ii} \otimes e_{ii} + rs \sum_{i > j} e_{ij} \otimes e_{ji} + \sum_{i < j} e_{ij} \otimes e_{ji} + (s-r) \sum_{i > j} e_{ii} \otimes e_{jj}.$$

Moreover it satisfies the Hecke relation

$$(\widehat{R}^{(+)} - s)(\widehat{R}^{(+)} + r) = 0. \quad (3.6)$$

For this R -matrix $\hat{R}^{(+)}$, we introduce antisymmetric tensors $\langle \varepsilon^{(+)} |, |\varepsilon^{(+)} \rangle$ by the equations:

$$\langle \varepsilon^{(+)} | \varepsilon^{(+)} \rangle = 1, \quad (3.7)$$

$$\langle \varepsilon^{(+)} | (\hat{R}_{i,i+1}^{(+)} + r) = 0, \quad i = 1, 2, \dots, n-1, \quad (3.8)$$

$$(\hat{R}_{i,i+1}^{(+)} + r) |\varepsilon^{(+)} \rangle = 0, \quad i = 1, 2, \dots, n-1. \quad (3.9)$$

These equations determine the antisymmetric tensors up to a constant. It is easy to see that $\langle \varepsilon^{(+)} |, |\varepsilon^{(+)} \rangle$ can be chosen as follows:

$$\langle \varepsilon^{(+)} | = \frac{1}{[n]_{r,s}!} \sum_{\sigma \in S_n} (-s)^{-l(\sigma)} e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(n)}, \quad (3.10)$$

$$|\varepsilon^{(+)} \rangle = s^{\frac{n(n-1)}{2}} \sum_{\sigma \in S_n} (-r)^{l(\sigma)} e^{\sigma(1)} \otimes \dots \otimes e^{\sigma(n)}. \quad (3.11)$$

Define the rank one matrix $A_n^{(+)} = |\varepsilon^{(+)} \rangle \langle \varepsilon^{(+)} |$. Explicitly we have that

$$A_n^{(+)} = \frac{s^{\frac{n(n-1)}{2}}}{[n]_{r,s}!} \sum_{\sigma, \tau \in S_n} (-r)^{l(\tau)} (-s)^{-l(\sigma)} e_{\tau(1)\sigma(1)} \otimes \dots \otimes e_{\tau(n)\sigma(n)}. \quad (3.12)$$

The following result gives commutation relations among the Weyl generators of $U(R)$.

Proposition 3.3 *In $U(R)$ the following identities are satisfied for $k = 1, 2, \dots, n$:*

$$\begin{aligned} A_n^{(+)} L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- A_n^{(+)} &= A_n^{(+)} L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- \\ &= L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- A_n^{(+)}. \end{aligned}$$

Proof Using the RTT defining relations (3.1 and 3.2), we have that

$$\begin{aligned} \langle \varepsilon^{(+)} | L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- (\hat{R}_{i,i+1}^{(+)} + r) &= \langle \varepsilon^{(+)} | (\hat{R}_{i,i+1}^{(+)} + r) L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^-, \\ (\hat{R}_{i,i+1}^{(+)} + r) L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- |\varepsilon^{(+)} \rangle &= L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- (\hat{R}_{i,i+1}^{(+)} + r) |\varepsilon^{(+)} \rangle \end{aligned}$$

for $k = 1, 2, \dots, n-1$.

By the uniqueness of the solution to (3.8 and 3.9) we can assume that

$$\begin{aligned} \langle \varepsilon^{(+)} | L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- &= a_k \langle \varepsilon^{(+)} |, \\ L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- |\varepsilon^{(+)} \rangle &= b_k |\varepsilon^{(+)} \rangle, \end{aligned}$$

where $a_k, b_k \in U(R)$. By the normalization (3.7) it follows that

$$a_k = b_k = \langle \varepsilon^{(+)} | L_1^+ \dots L_k^+ L_{k+1}^- \dots L_n^- |\varepsilon^{(+)} \rangle.$$

Subsequently we get

$$A_n^{(+)} L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- = L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- A_n^{(+)} = a_k A_n^{(+)}.$$

□

The following lemma will be needed to compute a partial trace in Theorem 3.5.

Lemma 3.4 *In $\text{End}(V)^{\otimes(n+1)}$ one has that*

$$(A_n^{(+)} \otimes 1) R_{1,n+1}^{(+)} \cdots R_{n,n+1}^{(+)} = s M'_{n+1} (A_n^{(+)} \otimes 1), \quad (3.13)$$

$$R_{0,n}^{(+)} \cdots R_{0,1}^{(+)} (1 \otimes A_n^{(+)}) = s M_0 (1 \otimes A_n^{(+)}), \quad (3.14)$$

where the indices of V in $V^{\otimes(n+1)}$ in the first (resp. second) identity are $1, \dots, n+1$ (resp. $0, 1, \dots, n$) and R_{ij} and M_{n+1} (resp. M'_0) are defined accordingly. Here

$$M' = \begin{pmatrix} 1 & & & \\ & rs & & \\ & & \ddots & \\ & & & (rs)^{n-1} \end{pmatrix} \quad (3.15)$$

and

$$M = \begin{pmatrix} (rs)^{n-1} & & & \\ & (rs)^{n-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}. \quad (3.16)$$

Proof These two identities are proved similarly as Lemme 2.11. Note that $R^{(+)}$ is also a triangular block matrix with diagonal blocks given by

$$(R^{(+)})_{ik}^{ik} = \begin{cases} s, & i = k; \\ rs, & i < k; \\ 1, & i > k. \end{cases} \quad (3.17)$$

Therefore for any $\sigma \in S_n$ we have $(R^{(+)})_{i\sigma(1)}^{i\sigma(1)} \cdots (R^{(+)})_{i\sigma(n)}^{i\sigma(n)} = s(rs)^{n-i}$. Subsequently for $\langle \varepsilon^{(+)} | = 1 \otimes \langle \varepsilon_n^{(+)} | \in (V^*)^{\otimes(n+1)}$ and $|\varepsilon^{(+)}\rangle = 1 \otimes |\varepsilon_n^{(+)}\rangle \in V^{\otimes(n+1)}$ one has as in (2.20)

$$\begin{aligned} & \langle \varepsilon^{(+)} | R_{0,n}^{(+)} \cdots R_{0,1}^{(+)} | \varepsilon^{(+)} \rangle \\ &= \sum_{i=1}^n \frac{s^{\frac{n(n-1)}{2}}}{[n]_{r,s}!} \sum_{\sigma \in S_n} (s^{-1}r)^{l(\sigma)} (R^{(+)})_{i\sigma(1)}^{i\sigma(1)} \cdots (R^{(+)})_{i\sigma(n)}^{i\sigma(n)} (e_{ii})_0 \\ &= \sum_{i=1}^n s(rs)^{n-i} (e_{ii})_0 = s M_0. \end{aligned}$$

Applying $A_n^{(+)} = |\varepsilon_n^{(+)}\rangle\langle\varepsilon_n^{(+)}|$, we immediately get

$$(1 \otimes A_n^{(+)})R_{0,n}^{(+)} \cdots R_{0,1}^{(+)}(1 \otimes A_n^{(+)}) = sM_0(1 \otimes A_n^{(+)}). \quad (3.18)$$

Now the map $U(R) \otimes (\text{End} V)^{\otimes n} \rightarrow \text{End} V \otimes (\text{End} V)^{\otimes n}$ given by

$$L_i^{\pm} \mapsto R_{0,i}^{\pm}$$

is an algebra homomorphism, then the identity $A_n^{(+)}L_1^{+} \cdots L_n^{+} = L_1^{+} \cdots L_n^{+}A_n^{(+)}$ implies that

$$(1 \otimes A_n^{(+)})R_{0,n}^{(+)} \cdots R_{0,1}^{(+)}(1 \otimes A_n^{(+)}) = (1 \otimes A_n^{(+)})R_{0,n}^{(+)} \cdots R_{0,1}^{(+)} = sM_0(1 \otimes A_n^{(+)}).$$

□

Theorem 3.5 *The elements*

$$c_k = \sum_{\sigma, \sigma' \in S_n} (-s)^{-l(\sigma)} (-r)^{l(\sigma')} l_{\sigma(1), \sigma'(1)}^{+} \cdots l_{\sigma(k), \sigma'(k)}^{+} l_{\sigma(k+1), \sigma'(k+1)}^{-} \cdots l_{\sigma(n), \sigma'(n)}^{-} \quad (3.19)$$

are quasi-central elements of $U(R)$ for $k = 1, \dots, n-1$. Explicitly, we have

$$L^{+}c_k = c_k M^{-1} L^{+} M \quad (3.20)$$

and

$$c_k L^{-} = M'^{-1} L^{-} M' c_k. \quad (3.21)$$

Proof The quantum Casimir elements can be expressed as traces. In fact, using the choice (3.10 and 3.11) of $\langle \varepsilon^{(+)} |$ and $|\varepsilon^{(+)}\rangle$, we can get

$$\begin{aligned} c_k &= [n]_{r,s}! s^{\frac{n(1-n)}{2}} \langle \varepsilon^{(+)} | L_1^{+} \cdots L_k^{+} L_{k+1}^{-} \cdots L_n^{-} | \varepsilon^{(+)} \rangle \\ &= [n]_{r,s}! s^{\frac{n(1-n)}{2}} \text{tr}(A_n^{(+)} L_1^{+} \cdots L_k^{+} L_{k+1}^{-} \cdots L_n^{-}) \\ &= [n]_{r,s}! s^{\frac{n(1-n)}{2}} \text{tr}(L_1^{+} \cdots L_k^{+} L_{k+1}^{-} \cdots L_n^{-} A_n^{(+)}). \end{aligned}$$

We take an auxiliary copy L^{+} in the zero position and consider

$$L_0^{+} L_1^{+} \cdots L_k^{+} L_{k+1}^{-} \cdots L_n^{-} A_n^{(+)}.$$

Moving L_0^{+} to the extreme right by the RTT defining relations (3.1 and 3.2), we have

$$\begin{aligned} &L_0^{+} L_1^{+} \cdots L_k^{+} L_{k+1}^{-} \cdots L_n^{-} A_n^{(+)} \\ &= (R_{0n}^{(+)} \cdots R_{01}^{(+)})^{-1} L_1^{+} \cdots L_k^{+} L_{k+1}^{-} \cdots L_n^{-} L_0^{+} R_{0n}^{(+)} \cdots R_{01}^{(+)} A_n^{(+)} \\ &= s(R_{0n}^{(+)} \cdots R_{01}^{(+)})^{-1} L_1^{+} \cdots L_k^{+} L_{k+1}^{-} \cdots L_n^{-} A_n^{(+)} L_0^{+} M_0, \end{aligned}$$

where the last identity uses Lemma 3.4.

Now we move $A_n^{(+)}$ to the left by Proposition 3.3 and use Lemma 3.4 again:

$$\begin{aligned} & s(R_{0n}^{(+)} \cdots R_{01}^{(+)})^{-1} L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- A_n^{(+)} L_0^+ M_0 \\ &= s(R_{0n}^{(+)} \cdots R_{01}^{(+)})^{-1} A_n^{(+)} L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- L_0^+ M_0 \\ &= A_n^{(+)} M_0^{-1} L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- L_0^+ M_0 \\ &= A_n^{(+)} L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- M_0^{-1} L_0^+ M_0. \end{aligned}$$

Finally taking partial trace $Tr_{1\dots n}$ in $(\mathbb{C}^n)^{\otimes(n+1)}$ for the right-hand side, we obtain the result that $L^+ c_k = c_k M^{-1} L^+ M$. The identity (3.21) is proved similarly. $\square$

Remark 3.6 When $rs = 1$, c_k become the quantum Casimir elements in the center.

4 Gauss Decomposition of $L^\pm$

In this section, we use the Gauss decomposition of $L^\pm$ to study the commuting relations among the quantum Cartan–Weyl generators. Gauss decomposition was used by Ding–Frenkel [7] to show that the RTT version of the quantum algebras $U_q(\mathfrak{gl}_n)$ is isomorphic to the Drinfeld–Jimbo realization.

Proposition 4.1 *The matrices $L^\pm$ can be decomposed as follows.*

$$L^+ = \begin{pmatrix} K_1^+ & & & \\ & K_2^+ & & \\ & & \ddots & \\ & & & K_n^+ \end{pmatrix} \begin{pmatrix} 1 & E_{12} & \cdots & E_{1n} \\ & 1 & E_{23} & \vdots \\ & & \ddots & E_{n-1,n} \\ & & & 1 \end{pmatrix}, \quad (4.1)$$

$$L^- = \begin{pmatrix} 1 & & & \\ F_{21} & 1 & & \\ \vdots & & \ddots & \\ F_{n1} & \cdots & F_{n,n-1} & 1 \end{pmatrix} \begin{pmatrix} K_1^- & & & \\ & K_2^- & & \\ & & \ddots & \\ & & & K_n^- \end{pmatrix}, \quad (4.2)$$

where the elements $K_i^\pm$ ($1 \leq i \leq n$), E_{ij} , F_{ji} ($i < j$) are exclusively defined by $L_{ij}^\pm$ through the equations.

The following relations are obtained by the RTT defining relations (3.1 and 3.2) and the Gauss decomposition (4.1 and 4.2).

Proposition 4.2 *In $U(R)$ we have*

$$K_i^\pm K_j^\pm = K_j^\pm K_i^\pm, \quad (4.3)$$

$$K_i^\pm K_j^\mp = K_j^\mp K_i^\pm. \quad (4.4)$$

Next we compute the commutation relations between $K_i^\pm$ and $E_{j,j+1}$.

Proposition 4.3 *In $U(R)$ we have that*

$$K_i^+ E_{i,i+1} = r E_{i,i+1} K_i^+, \quad (4.5)$$

$$K_i^+ E_{i-1,i} = s E_{i-1,i} K_i^+, \quad (4.6)$$

$$K_i^+ E_{j,j+1} = E_{j,j+1} K_i^+, \quad \text{for } i \neq j, j+1; \quad (4.7)$$

$$K_i^- E_{i,i+1} = s E_{i,i+1} K_i^-, \quad (4.8)$$

$$K_i^- E_{i-1,i} = r E_{i-1,i} K_i^-, \quad (4.9)$$

$$K_i^- E_{j,j+1} = E_{j,j+1} K_i^-, \quad \text{for } i \neq j, j+1. \quad (4.10)$$

Proof Here we just prove Eqs. (4.5) and (4.7), as the other relations can be obtained similarly.

From the defining relation (3.1) of $U(R)$ it follows that

$$l_{ii}^+ l_{i,i+1}^+ = r l_{i,i+1}^+ l_{ii}^+.$$

Plugging in the Gauss-decomposition (4.1), we get Eq. (4.5).

For $i \neq j, j+1$ the defining relation (3.1) implies that

$$l_{ii}^+ l_{j,j+1}^+ = l_{j,j+1}^+ l_{ii}^+,$$

which is equivalent to

$$K_i^+ K_j^+ E_{j,j+1} = K_j^+ E_{j,j+1} K_i^+,$$

by using the Gauss decomposition of L^+ . Finally the invertibility of $K_j^\pm$ gives Eq. (4.7). $\square$

Similarly we can obtain the commuting relations between $K_i^\pm$ and $F_{j+1,j}$.

Proposition 4.4 *In $U(R)$ we have that*

$$K_i^+ F_{i+1,i} = r^{-1} F_{i+1,i} K_i^+, \quad (4.11)$$

$$K_i^+ F_{i,i-1} = s^{-1} F_{i,i-1} K_i^+, \quad (4.12)$$

$$K_i^+ F_{j+1,j} = E_{j+1,j} K_i^+, \quad \text{for } i < j; \quad (4.13)$$

$$K_i^- F_{i+1,i} = s^{-1} F_{i+1,i} K_i^-, \quad (4.14)$$

$$K_i^- F_{i,i-1} = r^{-1} F_{i,i-1} K_i^-, \quad (4.15)$$

$$K_i^- F_{j+1,j} = E_{j+1,j} K_i^-, \quad \text{for } i < j. \quad (4.16)$$

Now we compute the commuting relations between $E_{i,i+1}$ and $F_{j+1,j}$.

Proposition 4.5 *In $U(R)$ we have that*

$$[E_{i,i+1}, F_{j+1,j}] = \delta_{ij} (r^{-1} - s^{-1}) (K_{i+1}^- (K_i^-)^{-1} - K_{i+1}^+ (K_i^+)^{-1}). \quad (4.17)$$

Proof It follows from (3.2) that

$$rs l_{i,i+1}^+ l_{i+1,i}^- + (s-r) l_{i+1,i+1}^+ l_{i,i}^- = l_{i+1,i}^- l_{i,i+1}^+ + (s-r) l_{i+1,i+1}^- l_{i,i}^+. \quad (4.18)$$

By Gauss decomposition of $L^\pm$ both sides of (4.18) can be written as

$$rs K_i^+ E_{i,i+1} F_{i+1,i} K_i^- + (s-r) K_{i+1}^+ K_i^- = F_{i+1,i} K_i^- K_i^+ E_{i,i+1} + (s-r) K_{i+1}^- K_i^+.$$

Taking account of Eqs. (4.8) and (4.11), we see that the above is reduced to (4.17). $\square$

The analog of Serre relations is given below for E_{ij} .

Proposition 4.6 *In $U(R)$ we have that*

$$E_{i,i+1}^2 E_{i+1,i+2} - (r+s) E_{i,i+1} E_{i+1,i+2} E_{i,i+1} + rs E_{i+1,i+2} E_{i,i+1}^2 = 0, \quad (4.19)$$

$$E_{i,i+1} E_{i+1,i+2}^2 - (r+s) E_{i+1,i+2} E_{i,i+1} E_{i+1,i+2} + rs E_{i+1,i+2}^2 E_{i,i+1} = 0, \quad (4.20)$$

$$E_{i,i+1} E_{j,j+1} = E_{j,j+1} E_{i,i+1}, \quad \text{if } |i-j| > 1. \quad (4.21)$$

Proof We first prove the following commutation relation:

$$E_{i+1,i+2} E_{i,i+1} = s^{-1} E_{i,i+1} E_{i+1,i+2} + (r^{-1} - s^{-1}) E_{i,i+2}. \quad (4.22)$$

In fact, from the defining relation (3.1) it follows that

$$rs l_{i,i+1}^+ l_{i+1,i+2}^+ + (s-r) l_{i+1,i+1}^+ l_{i,i+2}^+ = rs l_{i+1,i+2}^+ l_{i,i+1}^+.$$

Plugging in the Gauss decomposition, this becomes

$$rs K_i^+ E_{i,i+1} K_{i+1}^+ E_{i+1,i+2} + (s-r) K_{i+1}^+ K_i^+ E_{i,i+2} = rs K_{i+1}^+ E_{i+1,i+2} K_i^+ E_{i,i+1}. \quad (4.23)$$

Then Eq. (4.22) is obtained by using Eqs. (4.6) and (4.7).

Multiplying $E_{i,i+1}$ from the left and the right of (4.22), we have that

$$\begin{aligned} E_{i,i+1}^2 E_{i+1,i+2} &= s E_{i,i+1} E_{i+1,i+2} E_{i,i+1} - (r^{-1}s - 1) E_{i,i+1} E_{i,i+2}, \\ rs E_{i+1,i+2} E_{i,i+1}^2 &= r E_{i,i+1} E_{i+1,i+2} E_{i,i+1} + (s-r) E_{i,i+2} E_{i,i+1}. \end{aligned}$$

On the other hand, the defining relation (3.1) also gives that

$$E_{i,i+1} E_{i,i+2} = r E_{i,i+2} E_{i,i+1}.$$

Then (4.19) immediately follows. Equation (4.20) is proved similarly. $\square$

In the same way we obtain the Serre relations for F_{ij} .

Proposition 4.7 *In $U(R)$ we have*

$$F_{i+1,i}^2 F_{i+2,i+1} - (r^{-1} + s^{-1}) F_{i+1,i} F_{i+2,i+1} F_{i+1,i} + r^{-1} s^{-1} F_{i+2,i+1} F_{i+1,i}^2 = 0, \quad (4.24)$$

$$F_{i+1,i} F_{i+2,i+1}^2 - (r^{-1} + s^{-1}) F_{i+2,i+1} F_{i+1,i} F_{i+2,i+1} + r^{-1} s^{-1} F_{i+2,i+1}^2 F_{i+1,i} = 0, \quad (4.25)$$

$$F_{i+1,i} F_{j+1,j} = F_{j+1,j} F_{i+1,i}, \quad \text{if } |i - j| > 1. \quad (4.26)$$

5 Isomorphism Between the Quantum Group $U_{r,s}(\mathfrak{gl}_n)$ and $U(R)$

The current version of 2-parameter quantum group was given by Benkart and Witherspoon [1] in terms of Chevalley generators and Serre relations in connection with the down-up algebras. In [2, 3] they further developed the representation theory of the 2-parameter quantum general and special linear algebras and constructed the corresponding R -matrix. We now identify their version with our FRT version given in earlier sections.

Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be the orthonormal basis of the Euclidean space $\mathbb{C}^n$ with inner product $\langle \cdot, \cdot \rangle$. Let $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} | j = 1, 2, \dots, n-1\}$ and $\Phi = \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$. Then Φ realizes the root system of type A_{n-1} with Π a base of simple roots.

Definition 5.1 The algebra $U_{r,s}(\mathfrak{gl}_n)$ is a unital associated algebra over $\mathbb{C}$ generated by $e_j, f_j, (1 \leq j < n)$, and $a_i^{\pm 1}, b_i^{\pm 1} (1 \leq i \leq n)$ subject to the following relations.

R1: $a_i^{\pm 1}, b_j^{\pm 1} (1 \leq i \leq n)$ commute with each other and $a_i a_i^{-1} = b_i b_i^{-1} = 1$;

R2: $a_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} e_j a_i$, and $a_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} f_j a_i$;

R3: $b_i e_j = s^{\langle \epsilon_i, \alpha_j \rangle} e_j b_i$, and $b_i f_j = s^{-\langle \epsilon_i, \alpha_j \rangle} f_j b_i$;

R4: $[e_i, f_j] = \frac{\delta_{ij}}{r-s} (a_i b_{i+1} - a_{i+1} b_i)$;

R5: $[e_i, e_j] = [f_i, f_j] = 0$ if $|i - j| > 1$;

R6: $e_i^2 e_{i+1} - (r + s) e_i e_{i+1} e_i + r s e_{i+1} e_i^2 = 0$,

$$e_{i+1}^2 e_i - (r + s) e_{i+1} e_i e_{i+1} + r s e_i e_{i+1}^2 = 0;$$

R7: $f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + r^{-1} s^{-1} f_{i+1} f_i^2 = 0$,

$$f_{i+1}^2 f_i - (r^{-1} + s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1} s^{-1} f_i f_{i+1}^2 = 0.$$

The algebra $U_{r,s}(\mathfrak{gl}_n)$ is a Hopf algebra such that $a_i^{\pm 1}, b_i^{\pm 1}$ are the group-like elements and the remaining Hopf algebra structure is given by

$$\delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i,$$

$$\delta(f_i) = f_i \otimes \omega'_i + 1 \otimes f_i,$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0,$$

$$S(e_i) = -\omega_i^{-1} e_i,$$

$$S(f_i) = -f_i \omega_i'^{-1}.$$

Remark 5.2 When $r = q = s^{-1}$, the Hopf algebra $U_{r,s}(\mathfrak{gl}_n)$ modulo the ideal generated by $b_i - a_i^{-1}$, $1 \leq i \leq n$ is isomorphic to $U_q(\mathfrak{gl}_n)$.

Analogous with the 1-parameter case, we have the following theorem.

Theorem 5.3 The mapping $\psi : U_{r,s}(\mathfrak{gl}_n) \rightarrow U(R)$ given by

$$e_i \mapsto \frac{r}{r-s} E_{i,i+1}, f_i \mapsto \frac{s}{s-r} F_{i+1,i}, a_1 \mapsto K_1^+, b_1 \mapsto K_1^-, \quad (5.1)$$

and

$$a_i \mapsto K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}, \quad (5.2)$$

$$b_i \mapsto K_i^- \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}, \quad (5.3)$$

is an isomorphism.

Proof By Propositions 4.2, 4.6 and 4.7, relations R1, R5, R6 and R7 hold. We only need to check relations R2, R3 and R4.

First let consider R2. It follows from Proposition 4.3 that

$$\begin{aligned} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} E_{i,i+1} &= K_i^+ E_{i,i+1} \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} \\ &= r E_{i,i+1} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}. \end{aligned} \quad (5.4)$$

$$\begin{aligned} &K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} E_{i-1,i} \\ &= K_i^+ (K_{i-1}^+)^{-1} (K_{i-1}^-)^{-1} E_{i-1,i} \prod_{l=2}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} \\ &= r^{-1} s^{-1} K_i^+ E_{i-1,i} (K_{i-1}^+)^{-1} (K_{i-1}^-)^{-1} \prod_{l=2}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} \\ &= r^{-1} E_{i-1,i} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}. \end{aligned} \quad (5.5)$$

When $j \neq i, i-1$, we have

$$K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} E_{j,j+1} = E_{j,j+1} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}. \quad (5.6)$$

Similarly we also have that

$$\begin{aligned} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} F_{i+1,i} &= r^{-1} F_{i+1,i} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}, \\ K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} F_{i,i-1} &= r F_{i,i-1} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}, \end{aligned}$$

and for $j \neq i, i-1$

$$K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} F_{j+1,i} = F_{j+1,i} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}.$$

Then relation R2 is satisfied. Relation R3 is proved similarly.

Next we consider relation R4. It follows from Proposition 4.5 that

$$\frac{r}{r-s} \frac{s}{s-r} [E_{i,i+1}, F_{i+1,i}] = \frac{1}{r-s} (K_{i+1}^- (K_i^-)^{-1} - K_{i+1}^+ (K_i^+)^{-1}).$$

It is easy to check that

$$\begin{aligned} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} K_{i+1}^- \prod_{l=1}^i (K_{i+1-l}^+ K_{i+1-l}^-)^{(-1)^l} &= K_{i+1}^- (K_i^-)^{-1}, \\ K_{i+1}^+ \prod_{l=1}^i (K_{i+1-l}^+ K_{i+1-l}^-)^{(-1)^l} K_i^- \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} &= K_{i+1}^+ (K_i^+)^{-1}. \end{aligned}$$

So R4 holds. Note that all the E_{ij} , F_{ji} , $i < j$, can be generated by $E_{i,i+1}$, $F_{i+1,i}$. Therefore ψ is a surjective homomorphism.

The injectivity can be proved verbatim as in [7] for the 1-parameter case. $\square$

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