Some Source Coding Theorems and 1:1 Coding Based on Generalized Inaccuracy Measure of Order α and Type β

Satish Kumar · Arun Choudhary · Arvind Kumar

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Abstract In this paper, we have established some noiseless coding theorems for a generalized parametric ‘useful’ inaccuracy measure of order α and type β and generalized mean codeword length. Further, lower bounds on exponentiated useful code length for the best 1:1 code have been obtained in terms of the useful inaccuracy of order α and type β and the generalized average useful codeword length.

Keywords Generalized inaccuracy measures · Mean codeword length · Holder’s inequality

Mathematics Subject Classification 94A15 · 94A17 · 94A24 · 26D15

1 Introduction

It is well known that information measures are important for practical applications of information processing. For measuring information, a general approach is provided in a statistical framework based on information entropy introduced by Shannon [30].

S. Kumar

Department of Mathematics, College of Natural Sciences, Arba-Minch University,
Arba-Minch, Ethiopia
e-mail: drsatish74@rediffmail.com

A. Choudhary (✉)

Department of Mathematics, Geeta Institute of Management & Technology, Kanipla,
Kurukshetra 136131, Haryana, India
e-mail: arunchoudhary07@gmail.com

A. Kumar

Department of Mathematics, BRCM College of Engineering & Technology, Bahal,
Bhiwani, Haryana, India
e-mail: math_arvi@yahoo.in

As a measure of information, the Shannon entropy satisfies some desirable axiomatic requirements and also it can be assigned operational significance in important practical problems, for instance, in coding and telecommunication. In coding theory, the problem of coding is that of associating the messages that have to be sent with the sequences of symbols in a one to one fashion. In coding theory, generally we come across the problem of efficient coding of messages to be sent over a noiseless channel. We do not consider the problem of error correction, but attempt to maximize the number of messages that can be sent through a channel in a given time. Therefore, we find the minimum value of a mean codeword length subject to a given constraint on codeword lengths. As the codeword lengths are integers, the minimum value lies between two bounds, so a noiseless coding theorem seeks to find these bounds for a given mean and a given constraint. For uniquely decipherable codes, Shannon [30] found the lower bounds for the arithmetic mean by using his entropy. A coding theorem analogous to Shannon's noiseless coding theorem has been established by Campbell [6], in terms of Renyi's entropy [29]:

$$H_{\alpha}(P) = \frac{1}{1-\alpha} \log_D \sum_{i=1}^N p_i^{\alpha}, \quad \alpha > 0 (\neq 1). \quad (1.1)$$

Kieffer [18] defined class rules and showed that $H_{\alpha}(P)$ is the best decision rule for deciding which of the two sources can be coded with least expected cost of sequences of length N when $N \rightarrow \infty$, where the cost of encoding a sequence is assumed to be a function of length only. Further, in Jelinek [13] it is shown that coding with respect to Campbell's mean length is useful in minimizing the problem of buffer overflow which occurs when the source symbol is produced at a fixed rate and the code words are stored temporarily in a finite buffer. Concerning Campbell's mean length the reader can consult Campbell [6]. Longo [23] obtained the lower bound for useful mean codeword length in terms of quantitative-qualitative measure of entropy, introduced by Belis and Guiasa [3]. Guiasa and Picard [9] proved a noiseless coding theorem and obtained the lower bounds for similar useful mean codeword length. Gurdial and Pessow [10] extended this by finding lower bounds for useful mean codeword length of order α in terms of useful measures of information of order α . It is important to note that for other standard means like the geometric mean, the harmonic mean and the power mean, the lower bounds can be found in principle, but except for the arithmetic mean, no closed expressions for the lower bounds can be obtained. Kraft's [20] inequality plays an important role in proving the noiseless coding theorem. It is uniquely determined by the condition for unique decipherability.

Chapeau-Blondeau et al. [7] have presented an extension to source coding theorem traditionally based upon Shannon's entropy and later generalized to Renyi's entropy. Chapeau-Blondeau et al. [8] have described a practical problem of source coding and investigated an important relation stressing that Renyi's entropy emerges at an order α differing from the traditional Shannon's entropy. Some new lower and upper bounds for compression rate of binary prefix codes optimized over memoryless sources have been provided by Baer [2]. Singh et al. [33] have provided the application of weighted measures of entropy to the field of coding theory.

Ramamoorthy [28] considered the problem of transmitting multiple compressible sources over a network at minimum cost with the objective to find the optimal rates at which the sources should be compressed. Tu et al. [37] have presented a new scheme based on variable length coding, capable of providing reliable resolutions for flow media data transmission in spatial communication. Some interesting work for the construction of information theoretic source network coding in the presence of eavesdroppers has been presented by Luo et al. [24]. Wu et al. [38] have constructed a space trellis and designed a low-complexity joint decoding algorithm with a variable length symbol-a posteriori probability algorithm in resource constrained deep space communication networks. The applications of coding theory to the field of marketing have been provided by O'Neill [26]. Some other related work concerned with the coding theory has been provided by Sharma and Raina [31] and Koski and Persson [19], etc.

Kumar and Choudhary [21] have been defined a new measure L_α^β , called average code word length of order α and type β and its relationship with a result of generalized Havrda–Charvat and Tsallis's entropy has been discussed. Using L_α^β , some coding theorem for discrete noiseless channel has been proved.

Parkash and Kakkar [27] have been defined two new mean codeword lengths $L(\alpha, \beta)$ and $L(\beta)$ and it is shown that these lengths satisfy desirable properties as a measure of typical codeword lengths. Consequently, two new noiseless coding theorems subject to Kraft's inequality have been proved.

In this paper, we study some coding theorems by considering a new function depending on the parameters α and β and a utility function. Our motivation for studying this function is that it generalizes some information measures already existing in the literature.

Throughout the paper $\mathbb{N}$ denotes the set of the natural numbers and for $N \in \mathbb{N}$ we set

$$\Delta_N = \left\{ (p_1, \dots, p_N) / p_i \geq 0, i = 1, \dots, N, \sum_{i=1}^N p_i = 1 \right\}.$$

In case there is no rise to misunderstanding we write $P \in \Delta_N$ instead of $(p_1, \dots, p_N) \in \Delta_N$.

Throughout this paper, $\sum$ will stand for $\sum_{i=1}^N$ unless otherwise stated and logarithms are taken to the base D ($D > 1$).

Consider the following model for a random experiment S ,

$$S_K = [E; P; U; K], \quad (1.2)$$

where $E = (E_1, E_2, \dots, E_N)$ is a finite system of events, $P = (p_1, p_2, \dots, p_N)$, $0 \leq p_i \leq 1$, $\sum_{i=1}^N p_i = 1$ the probability distribution and $U = (u_1, u_2, \dots, u_N)$, $u_i \geq 0$ is the utility distribution. The u_i are nonnegative real numbers.

Now let us suppose that experimenter asserts that the i th outcome E_i has the probability q_i whereas the true probability is p_i , with $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i = 1$. Thus we have two utility information schemes, (1.2) of a set of K events after an experiment and

$$S_K^* = [E; Q; U; K], \quad 0 \leq q_i \leq 1, \quad \sum_{i=1}^N q_i = 1, \quad u_i \geq 0 \quad (1.3)$$

of the same set of K events before the experiment.

The quantitative-qualitative measure of inaccuracy [35] associated with the statement of an experiment is given by

$$I(P, Q; U) = - \sum_{i=1}^N u_i p_i \log q_i. \quad (1.4)$$

By considering the weighted mean code word length [9]

$$L(U) = \frac{\sum_{i=1}^N u_i p_i n_i}{\sum_{i=1}^N u_i p_i} \quad (1.5)$$

where $n_1, n_2, \dots, n_N$ are the code lengths of $x_1, x_2, \dots, x_N$, respectively.

Taneja and Tuteja [35] derived the lower and upper bounds on $L(U)$ in terms of $I(P, Q; U)$.

In the derivation of the cost function (1.5) clearly the assumption that the cost varies linearly with the code length is taken. But, this is not always the case. However, there may be occasions when the cost does not vary linearly with code lengths. For example, this may be the case when the cost of encoding or decoding equipment was an important factor. In this case, cost is more nearly an exponential function of n_i . Such types of functions occur frequently in market equilibrium and growth models in economics. (Economic population growth is not always linear and it can be exponential also. For example, if u_i is taken as cost of encoding a code word, then $\sum_{i=1}^N u_i p_i n_i$ is average cost of encoding of a code which can be exponential also if n_i is exponential. In case u_i and p_i are uniformly distributed, then system attains equilibrium and vice versa. Concerning applications in economics the reader can consult Hooda and Sharma [12] and Theil [36].) Obviously, linear dependence is the limiting case of this measure (exponential measure). Thus, sometimes, it is more appropriate to minimize the quantity:

$$c = \sum_{i=1}^N p_i^\beta u_i^{t+1} D^{n_i t}, \quad (1.6)$$

where $-1 < t < \infty$, $\beta \geq 1$ are the parameters related to the cost.

In order to make the result of this paper more directly comparable with the usual noiseless coding theorem, instead of minimizing (1.6), we shall minimize

$$L_\beta^t(U) = \frac{1}{t} \log \left[\sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{t+1} D^{n_i t} \right], \quad -1 < t < \infty, \quad \beta \geq 1. \quad (1.7)$$

Being monotone function of c , (1.7) would be termed as order t and type β cost length. Clearly (1.7) is the generalization of the 'useful' average codeword length (1.5).

Remarks

- (i) For $\beta = 1$, $L_\beta^t(U)$ in (1.7) reduces to the ‘useful’ mean length $L^t(U)$ of the code given by Bhatia [4].
- (ii) when $\beta = 1$, $u_i = 1$, $\forall i = 1, 2, \dots, N$, $L_\beta^t(U)$ in (1.7) reduces to the mean length given by Campbell [6].
- (iii) When $\beta = 1$, $u_i = 1$, $\forall i = 1, 2, \dots, N$ and $\alpha \rightarrow 1$, $L_\beta^t(U)$ in (1.7) reduces to the optimal code length identical to Shannon [30].
- (iv) For $u_i = 1$, $\forall i = 1, 2, \dots, N$, $L_\beta^t(U)$ in (1.7) reduces to the mean length given by Khan and Ahmed [17].

We derive lower and upper bounds for the measures (1.7) in terms of generalized information measure as:

$$I_\alpha^\beta(P, Q; U) = \frac{1}{1 - \alpha} \log \left[\frac{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}}{\sum_{i=1}^N u_i p_i^\beta} \right], \quad \alpha > 0 (\neq 1), \quad \beta \geq 1, \quad (1.8)$$

under the condition

$$\sum_{i=1}^N p_i^\beta q_i^{-1} D^{-n_i} \leq 1. \quad (1.9)$$

Clearly the inequalities (1.9) is the generalization of Kraft’s inequality. A code satisfying (1.9) would be termed as a ‘useful’ personal probability code. $D (D \geq 2)$ is the size of the code alphabet.

Remarks

- (i) When $\beta = 1$, (1.8) reduces to a measure of ‘useful’ information measure of order α due to Bhatia [4].
- (ii) when $\beta = 1$, $u_i = 1$, $\forall i = 1, 2, \dots, N$, (1.8) reduces to the inaccuracy measure given by Nath [25], further it reduces to Renyi’s [29] entropy by taking $p_i = q_i$, $\forall i = 1, 2, \dots, N$.
- (iii) When $\beta = 1$, $u_i = 1$, $\forall i = 1, 2, \dots, N$ and $\alpha \rightarrow 1$, (1.8) reduces to the measure due to Kerridge [16].
- (iv) When $u_i = 1$, $\forall i = 1, 2, \dots, N$ and $p_i = q_i$, $\forall i = 1, 2, \dots, N$ the measure (1.8) becomes Aczel and Daroczy [1] and Kapur [14] entropy.

2 Coding Theorems

In this section, we prove the following theorems:

Theorem 2.1 *For every code whose lengths $n_1, n_2, \dots, n_N$ satisfy (1.9), the average code length satisfies*

$$L_\beta^t(U) \geq I_\alpha^\beta(P, Q; U), \quad (2.1)$$

where $\alpha = \frac{1}{1+t}$, the equality occurs if and only if

$$n_i = -\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{\beta(\alpha-1)}}. \quad (2.2)$$

Proof By Holder's inequality [32]

$$\left(\sum_{i=1}^N x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^N y_i^q \right)^{\frac{1}{q}} \leq \sum_{i=1}^N x_i y_i, \quad (2.3)$$

for all $x_i, y_i > 0$, $i = 1, 2, \dots, N$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p < 1$ ($\neq 0$), $q < 0$, or $q < 1$ ($\neq 0$), $p < 0$.

Making the substitutions

$$p = -t, \quad q = \frac{t}{t+1},$$

$$x_i = p_i^{-\frac{\beta}{t}} \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{-\left(\frac{t+1}{t}\right)} D^{-n_i}, \quad y_i = p_i^{\beta\left(\frac{1+t}{t}\right)} \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{\left(\frac{1+t}{t}\right)} q_i^{-1}.$$

in (2.3) and using (1.9), we get

$$\left[\sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{t+1} D^{n_i t} \right]^{\frac{1}{t}} \geq \left[\frac{\sum_{i=1}^N u_i p_i^\beta q_i^{\beta(\alpha-1)}}{\sum_{i=1}^N u_i p_i^\beta} \right]^{\frac{1+t}{t}}.$$

Taking logarithm of both the sides with base D , we obtain (2.1).

It is clear that the equality in (2.1) is true if $n_i = -\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{\beta(\alpha-1)}}$. The necessity of this condition for equality in (2.1) follows from the condition for equality in Holder's inequality: In the case of the Holder's inequality given above, equality holds if and only if for some c ,

$$x_i^p = c y_i^q. \quad (2.4)$$

Plugging this condition into our situation, with the x_i , y_i , and p , q as specified, and using the fact that the $\sum_{i=1}^N p_i = 1$, the necessity is proven. $\square$

Next, we obtain a result giving an upper bound to the generalized average 'useful' codeword length.

Theorem 2.2 *By properly choosing the lengths $n_1, n_2, \dots, n_N$ in the code of theorem 2.1, $L_\beta^t(U)$ can be made to satisfy the inequality*

$$L_\beta^t(U) < I_\alpha^\beta(P, Q; U) + 1. \quad (2.5)$$

Proof Let n_i be the positive integer satisfying the inequalities

$$-\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}} \leq n_i < -\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}} + 1. \quad (2.6)$$

Consider the interval

$$\delta_i = \left[-\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}}, -\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}} + 1 \right]$$

of length 1. In every δ_i , there is exactly one positive integer n_i such that

$$0 < -\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}} \leq n_i < -\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}} + 1. \quad (2.7)$$

It can be shown that the sequence $\{n_i\}$, $i = 1, 2, \dots, N$ thus defined, satisfies (1.9). $\square$

From the right inequality of (2.7), we have

$$\begin{aligned} n_i &< -\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}} + 1 \\ \Rightarrow D^{n_i \left(\frac{1-\alpha}{\alpha} \right)} &< D^{\left(\frac{1-\alpha}{\alpha} \right)} \left[\frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}} \right]^{\left(\frac{\alpha-1}{\alpha} \right)}. \end{aligned} \quad (2.8)$$

Multiplying both sides of (2.8) by $p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{\frac{1}{\alpha}}$, summing over $i = 1, 2, \dots, N$ and after that raising to the power $\left(\frac{\alpha}{1-\alpha} \right)$, we get

$$\left[\sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{n_i \left(\frac{1-\alpha}{\alpha} \right)} \right]^{\left(\frac{\alpha}{1-\alpha} \right)} < D \left[\frac{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}}{\sum_{i=1}^N u_i p_i^\beta} \right]^{\left(\frac{1}{1-\alpha} \right)}.$$

Taking logarithms on both sides and using the relation $\alpha = \frac{1}{1+t}$, we get (2.5).

Theorem 2.3 For arbitrary $N \in \mathbb{N}$, $\alpha > 0 (\neq 1)$, $\beta \geq 1$ and for every code word lengths n_i , $i = 1, \dots, N$ of theorem 2.1, $L_\beta^t(U)$ can be made to satisfy

$$L_\beta^t(U) \geq I_\alpha^\beta(P, Q; U) > I_\alpha^\beta(P, Q; U) + \frac{1}{t} \text{Log } D. \quad (2.9)$$

Proof Suppose

$$n_i = -\log \frac{u_i q_i^\alpha}{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}}, \quad \alpha > 0 (\neq 1), \quad \beta \geq 1.$$

Clearly $\bar{n}_i$ and $\bar{n}_i + 1$ satisfy ‘equality’ in Holder’s inequality (2.3). Moreover, $\bar{n}_i$ satisfies (1.9). Suppose n_i is the unique integer between $\bar{n}_i$ and $\bar{n}_i + 1$, then obviously, n_i satisfies (1.9).

Since $\alpha > 0$, $\beta \geq 1$, we have

$$\begin{aligned} \left[\sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{t+1} D^{n_i t} \right] &\leq \left[\sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{t+1} D^{\bar{n}_i t} \right] \\ &< D \left[\sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{t+1} D^{\bar{n}_i t} \right]. \end{aligned} \quad (2.10)$$

Since $\left[\sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{t+1} D^{\bar{n}_i t} \right] = \left[\frac{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}}{\sum_{i=1}^N u_i p_i^\beta} \right]^{1+t}$, (2.10) becomes

$$\begin{aligned} \left[\sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^\beta} \right)^{t+1} D^{n_i t} \right] &\leq \left[\frac{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}}{\sum_{i=1}^N u_i p_i^\beta} \right]^{1+t} \\ &< D \left[\frac{\sum_{i=1}^N u_i p_i^\beta q_i^{(\alpha-1)}}{\sum_{i=1}^N u_i p_i^\beta} \right]^{1+t}. \end{aligned}$$

Raising to the power $\frac{1}{t}$ and taking logarithms on both sides, we get (2.9). $\square$

3 Codeword Length for the Best 1:1 Code and its Lower Bound

The mean length of a noiseless uniquely decodable code for a discrete random variable X satisfies

$$H(X) \leq L_{UD} < H(X) + 1, \quad (3.1)$$

where

$$H(X) = -\sum_{i=1}^N p_i \log p_i \quad (3.2)$$

is the Shannon’s entropy [30] of the random variable X . Shannon’s restriction of coding of X to prefix codes is highly justified by the implicit assumption that the description will be concatenated and thus must be uniquely decodable and instantaneous codes, cf. [1, 4], the expected codeword length is the same for both the set of codes.

There are some communication situations in which a random variable X is being transmitted rather than a sequence of random variables. For this context Leung-Yang-Cheong and Cover [22] considered one to one codes i.e., codes which assign a distinct binary code to each outcome of the random variable X without regard to the condition that concatenations of the descriptions must be uniquely decipherable.

Bhatia [4] and Bhatia et al. [5] have extended the idea of the one to one code to the Kerridge inaccuracy [16], they have obtained the relations

$$L_{1:1} \geq H(P; Q) - \log_2 \sum_{i=1}^N \left(\frac{2}{i+2} \right), \quad (3.3)$$

where $L_{1:1}$ and $H(P; Q)$ denote the best 1:1 code length (cf. [16]) and Kerridge inaccuracy

$$H(P; Q) = - \sum_{i=1}^N p_i \log_2 q_i, \quad (3.4)$$

respectively, and

$$L_{1:1} \geq L_{UD} - \log_2 \log_2 K - 3, \quad (3.5)$$

where K is the total number of outcomes.

Further, they have also derived lower bounds to the exponential mean codeword length for the best one to one codes in terms of generalized inaccuracy of order α .

In the above section, we have proved in Theorems 2.1 and 2.2 that new codeword length $L_\beta^t(U)$ satisfies $I_\alpha^\beta(P, Q; U) \leq L_\beta^t(U) < I_\alpha^\beta(P, Q; U) + 1$ for all uniquely decipherable codes. We consider this codeword length for the best 1:1 code and find its lower bound. Let X be a random variable taking on a finite number of values $(x_1, x_2, \dots, x_N)$ which have to be encoded in terms of an alphabet of size D with probabilities $(p_1, p_2, \dots, p_N)$ and utilities $(u_1, u_2, \dots, u_N)$, respectively. Without any loss of generality, we may assume that $p_1 \geq p_2 \geq p_3 \geq \dots \geq p_N$. Here we take $D = 2$, that is, the set of available codewords is $(0, 1)$, the set of all finite binary sequences. Let n_i , $i = 1, 2, \dots, N$ denote the length of the codeword for x_i . For the best 1:1 code, it is clear that $n_1 \leq n_2 \leq n_3 \leq \dots \leq n_N$. Since there are 2^n 1:1 codewords of length n so $n_1 = 1$, $n_2 = 1$, $n_3 = 2, \dots$ and in general $n_i = \lceil \log_D \left(\frac{i}{2} + 1 \right) \rceil$ [15, 22], where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . Thus, the average 'useful' codeword length for the best 1:1 code is given by

$$L_{1:1}(U) = \frac{\sum_{i=1}^N u_i p_i \left\lceil \log_2 \left(\frac{i+2}{2} \right) \right\rceil}{\sum_{i=1}^N u_i p_i}. \quad (3.6)$$

When utilities are ignored, (3.6) reduces to $L_{1:1}$, cf. [22].

From (1.7), the exponentiated average ‘useful’ codeword length for binary codes can be given by

$$L_{\beta}^t(U) = \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^{\beta} \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^{\beta}} \right)^{t+1} 2^{n_i t} \right]. \quad (3.7)$$

When $\beta = 1$, (3.7) reduces to Bhatia’s [4] ‘useful’ codeword length for the best 1:1 code. Thus, the exponentiated average ‘useful’ codeword length for the best 1:1 code is given by

$$L_{\beta,1:1}^t(U) = \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^{\beta} \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^{\beta}} \right)^{t+1} 2^{t \left[\log_2 \left(\frac{i+2}{2} \right) \right]} \right]. \quad (3.8)$$

We will now prove the following theorem, which gives a lower bound on $L_{\beta,1:1}^t$.

Theorem 3.1 For $I_{\alpha}^{\beta}(P, Q; U)$, $L_{\beta}^t(U)$ and $L_{\beta,1:1}^t$ as given in (1.8), (3.7) and (3.8), respectively, the following estimates hold:

$$L_{\beta,1:1}^t(U) \geq I_{\alpha}^{\beta}(P, Q; U) - \log_2 \left[\sum_{i=1}^N \frac{p_i^{\beta}}{q_i} \left(\frac{2}{i+2} \right) \right] \quad (3.9)$$

and

$$L_{\beta,1:1}^t(U) \geq L_{\beta}^t(U) - \log_2 \left[\sum_{i=1}^N \frac{p_i^{\beta}}{q_i} \left(\frac{1}{i+2} \right) \right] - 2. \quad (3.10)$$

Proof From (3.8), we have

$$L_{\beta,1:1}^t(U) \geq \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^{\beta} \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^{\beta}} \right)^{t+1} \left(\frac{i+2}{2} \right)^t \right]. \quad (3.11)$$

Now

$$\begin{aligned} I_{\alpha}^{\beta}(P, Q; U) - L_{\beta,1:1}^t(U) &\leq \left(\frac{t+1}{t} \right) \log_2 \left[\frac{\sum_{i=1}^N u_i p_i^{\beta} q_i^{-\left(\frac{t+1}{t} \right)}}{\sum_{i=1}^N u_i p_i^{\beta}} \right] \\ &\quad - \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^{\beta} \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^{\beta}} \right)^{t+1} \left(\frac{i+2}{2} \right)^t \right] \\ &= \log_2 \left[\frac{\sum_{i=1}^N u_i p_i^{\beta} q_i^{-\left(\frac{t+1}{t} \right)}}{\sum_{i=1}^N u_i p_i^{\beta}} \right]^{\left(\frac{t+1}{t} \right)} \left[\sum_{i=1}^N p_i^{\beta} \left(\frac{u_i}{\sum_{i=1}^N u_i p_i^{\beta}} \right)^{t+1} \left(\frac{i+2}{2} \right)^t \right]^{-\frac{1}{t}}. \end{aligned} \quad (3.12)$$

Applying Holder's inequality to (3.12), we obtain

$$I_{\alpha}^{\beta}(P, Q; U) - L_{\beta, 1:1}^t(U) \leq \log_2 \left[\sum_{i=1}^N \frac{p_i^{\beta}}{q_i} \left(\frac{2}{i+2} \right) \right],$$

which gives (3.9).

Now from (2.5)

$$L_{\beta}^t(U) < I_{\alpha}^{\beta}(P, Q; U) + 1,$$

so

$$\begin{aligned} L_{\beta}^t(U) - L_{\beta, 1:1}^t(U) &< I_{\alpha}^{\beta}(P, Q; U) - L_{\beta, 1:1}^t(U) + 1 \\ &\leq 2 + \log_2 \left[\sum_{i=1}^N \frac{p_i^{\beta}}{q_i} \left(\frac{1}{i+2} \right) \right], \end{aligned}$$

which proves (3.10). $\square$

4 Particular Cases

- (1) When $\beta = 1$, (3.9) and (3.10) reduce to the results due to Bhatia [4].
- (2) When $\beta = 1$, $u_i = 1$, (3.9) and (3.10) reduce to the following bounds on exponential mean codeword length of 1 : 1 code:
i.e.,

$$L_{1:1}(t) \geq I_{\alpha}(P, Q) - \log_2 \left[\sum_{i=1}^N \frac{p_i}{q_i} \left(\frac{2}{i+2} \right) \right] \quad (4.1)$$

and

$$L_{1:1}(t) \geq L(t) - \log_2 \left[\sum_{i=1}^N \frac{p_i}{q_i} \left(\frac{1}{i+2} \right) \right] - 2, \quad (4.2)$$

where

$$L(t) = \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i 2^{n_{it}} \right], \quad (4.3)$$

$$L_{1:1}(t) = \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i 2^{t \lceil \log_2 \left(\frac{i+2}{2} \right) \rceil} \right] \quad (4.4)$$

and

$$I_{\alpha}(P, Q) = \frac{1}{1-\alpha} \log \sum_{i=1}^N p_i q_i^{(\alpha-1)}, \quad (4.5)$$

where (4.3) and (4.4) are studied by Taneja [34] and (4.5) is Nath [25] inaccuracy.

(3) When $u_i = 1$, (3.9) and (3.10) reduce to the results as:

$$L_{1:1}^{\beta}(t) \geq I_{\alpha}^{\beta}(P, Q) - \log_2 \left[\sum_{i=1}^N \frac{p_i^{\beta}}{q_i} \left(\frac{2}{i+2} \right) \right] \quad (4.6)$$

and

$$L_{1:1}^{\beta}(t) \geq L^{\beta}(t) - \log_2 \left[\sum_{i=1}^N \frac{p_i^{\beta}}{q_i} \left(\frac{1}{i+2} \right) \right] - 2, \quad (4.7)$$

where

$$L_{1:1}^{\beta}(t) = \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^{\beta} \left(\frac{1}{\sum_{i=1}^N p_i^{\beta}} \right)^{t+1} 2^{t \left[\log_2 \left(\frac{i+2}{2} \right) \right]} \right],$$

$$L_{\beta}(t) = \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^{\beta} \left(\frac{1}{\sum_{i=1}^N p_i^{\beta}} \right)^{t+1} 2^{n_i t} \right]$$

and

$$I_{\alpha}^{\beta}(P, Q) = \frac{1}{1-\alpha} \log \frac{\sum_{i=1}^N p_i^{\beta} q_i^{(\alpha-1)}}{\sum_{i=1}^N p_i^{\beta}}.$$

- (4) When $u_i = 1$ and $p_i = q_i$, (3.9) and (3.10) reduce to the results due to Hooda and Ram [11].
- (5) When $\beta = 1$, $u_i = 1$ and $p_i = q_i$, (3.9) and (3.10) reduce to the results due to Taneja [34].
- (6) When $\beta = 1$, $u_i = 1$ and $\alpha \rightarrow 1$, (3.9) and (3.10) reduce to the results due to Bhatia et al. [5].
- (7) When $\beta = 1$, $u_i = 1$, $p_i = q_i$ and $\alpha \rightarrow 1$, (3.9) and (3.10) reduce to the results due to Leung-Yan-Cheong and Cover [22].

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