Solutions of Nabla Fractional Difference Equations Using N-Transforms

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Received: 29 July 2013 / Revised: 7 November 2013 / Accepted: 10 March 2014 /

Published online: 25 April 2014

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Abstract In the present paper, we present some important properties of N-transform, which is the Laplace transform for the nabla derivative on the time scale of integers (Bohner and Peterson in Dynamic equations on time scales, Birkhauser, Boston, 2001; Advances in dynamic equations on time scales, Birkhauser, Boston, 2002). We obtain the N-transform of nabla fractional sums and differences and then apply this transform to solve some nabla fractional difference equations with initial value problems. Finally, using N-transforms, we prove that discrete Mittag-Leffler function is the eigen function of Caputo type nabla fractional difference operator ∇^α .

Keywords Fractional difference · Caputo type · Exponential order · N-transform · Discrete Mittag-Leffler function

Mathematical Subject Classification (2010) 39A10 · 39A99

1 An Introduction to Nabla Discrete Fractional Calculus

Fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering [1]. The analogous theory for nabla discrete fractional calculus was initiated and properties of the theory of fractional

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sums and differences were established [2,3]. Recently, a series of papers continuing this research has appeared [4–20].

In this section, we introduce some basic definitions and results concerning nabla discrete fractional calculus. Throughout the article, for notations and terminology we refer [21]. The extended binomial coefficient $\binom{a}{n}$, ($a \in \mathbb{R}$, $n \in \mathbb{Z}$) is defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & n > 0 \\ 1 & n = 0 \\ 0 & n < 0. \end{cases} \quad (1.1)$$

Definition 1.1 For any complex numbers α and β , $(\alpha)_\beta$ be defined as follows.

$$(\alpha)_\beta = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} & \text{when } \alpha \text{ and } \alpha + \beta \text{ are neither zero nor negative integers,} \\ 1 & \text{when } \alpha = \beta = 0, \\ 0 & \text{when } \alpha = 0, \beta \text{ is neither zero nor negative integer,} \\ \text{undefined otherwise.} \end{cases}$$

Lemma 1.1 For any $a, b \in \mathbb{C}$, the quotient expansion of two gamma functions at infinity is given by

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} = n^{a-b} \left[1 + O\left(\frac{1}{n}\right) \right], (|\arg(n+a)| < \pi, |n| \rightarrow \infty).$$

Let $u(n) : \mathbb{N}_0^+ \rightarrow \mathbb{R}$ and $m-1 < \alpha \leq m$ where $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{N}_1^+$.

Definition 1.2 [19] The fractional sum operator of order α is defined as

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) = \sum_{j=1}^n \binom{n-j+\alpha-1}{n-j} u(j). \quad (1.2)$$

Definition 1.3 [19] The Caputo type fractional difference operator of order α is defined as

$$\nabla^\alpha u(n) = \nabla^{\alpha-m} [\nabla^m u(n)] = \sum_{j=0}^{n-1} \binom{j-\alpha+m-1}{j} \nabla^m u(n-j). \quad (1.3)$$

Now, we simplify the above definition for our convenience as follows.

Corollary 1 The equivalent form of (1.3) is

$$\nabla^\alpha u(n) = \sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}. \quad (1.4)$$

Proof We prove the statement (1.4) using mathematical induction on m .

For $m = 1$,

$$\begin{aligned}
 \nabla^\alpha u(n) &= \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u(n-j) \\
 &= u(n) - \binom{n-\alpha-1}{n-1} u(0) - \alpha \sum_{j=1}^{n-1} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} u(n-j) \\
 &\quad \text{(using [4])} \\
 &= \sum_{j=0}^{n-1} \binom{j-\alpha-1}{j} u(n-j) - \binom{n-\alpha-1}{n-1} u(0) \\
 &= \sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j) - \binom{n-\alpha-1}{n-1} u(0). \tag{1.5}
 \end{aligned}$$

The statement is true for $m = 1$.

For $m = 2$,

$$\begin{aligned}
 \nabla^\alpha u(n) &= \sum_{j=0}^{n-1} \binom{j-\alpha+1}{j} \nabla^2 u(n-j) \\
 &= \sum_{j=0}^{n-1} \binom{j-\alpha+1}{j} \nabla [\nabla u(n-j)] \\
 &= \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u(n-j) - \binom{n-\alpha}{n-1} [\nabla u(n-j)]_{j=n} \text{ (using (1.5))} \\
 &= \sum_{j=0}^{n-1} \binom{j-\alpha-1}{j} u(n-j) - \binom{n-\alpha-1}{n-1} u(0) - \binom{n-\alpha}{n-1} [\nabla u(n-j)]_{j=n} \\
 &\quad \text{(using (1.5))} \\
 &= \sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}.
 \end{aligned}$$

The statement is also true for $m = 2$. We assume that the statement is true for $m - 1$. Then

$$\begin{aligned}
 \nabla^\alpha u(n) &= \sum_{j=0}^{n-1} \binom{j-\alpha+m-2}{j} \nabla^{m-1} u(n-j) \\
 &= \sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-2} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}. \tag{1.6}
 \end{aligned}$$

Now we prove the statement is true for m . Consider

$$\begin{aligned}
 \nabla^\alpha u(n) &= \sum_{j=0}^{n-1} \binom{j-\alpha+m-1}{j} \nabla^m u(n-j) \\
 &= \sum_{j=0}^{n-1} \binom{j-\alpha+m-1}{j} \nabla[\nabla^{m-1} u(n-j)] \\
 &= \sum_{j=0}^{n-1} \binom{j-\alpha+m-2}{j} \nabla^{m-1} u(n-j) - \binom{n-\alpha+m-2}{n-1} \\
 &\quad [\nabla^{m-1} u(n-j)]_{j=n} \text{ (using (1.5))} \\
 &= \sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0} \\
 &\quad \text{(using (1.6)).}
 \end{aligned}$$

Hence by the principle of mathematical induction on m , the statement (1.4) is true for $m \in \mathbb{N}_1^+$. $\square$

2 N-Transforms and Properties

Differential equations and difference equations are playing a vital role in every aspect of applied mathematics. Investigation and analysis of differential and difference equations arising in applications led to many deep mathematical problems, and there are so many different techniques in order to solve them. Integral transforms is one such important technique, and Laplace transform is a particular case of integral transform.

Laplace transform is a powerful technique for analyzing linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. This transformation not only provides a fundamentally different way to understand the behavior of the system, but it also drastically reduces the complexity of the mathematical calculations required to analyze the system. Z-transform, like the Laplace transform, is an indispensable mathematical tool for the design, analysis and monitoring of systems. The z-transform is the discrete-time counter-part of the Laplace transform.

Later Bohner and Peterson [22,23] introduced the Laplace transform for an arbitrary time scale (arbitrary closed subset of reals). Two particular choices of time scales, namely the reals and the integers, yield the concepts of the classical Laplace transform and the classical Z-transform. According to the definition given by Bohner and Peterson [22,23], Laplace transform is the classical Laplace transform on $\mathbb{R}$. R- and N-transforms are discrete analogs of the Laplace transform on the time scale of integers for the alpha derivative. In particular, R-transform is the discrete transform defined on the time scale of integers for the delta (Δ) derivative and N-transform is the discrete

transform defined on the time scale of integers for the nabla (∇) derivative. R- and N-transforms are not intended to be the more commonly employed Z-transform.

In this section, we derive some important properties of N-transform. Let $u(n) : \mathbb{N}_1^+ \rightarrow \mathbb{R}$. Atici and Eloe [7] defined the discrete transform (N-transform) by

$$N[u(n)] = \sum_{j=1}^{\infty} u(j)(1-z)^{j-1} = U(z) \quad (2.1)$$

for each $z \in \mathbb{C}$ for which the series converges.

Definition 2.1 A function $u(n)$ is said to be of exponential order r , $r > 0$ if there exists a constant $A > 0$ such that $|u(n)| \leq Ar^{-n}$ for sufficiently large n .

The following lemma discusses the convergence of N-transform.

Lemma 2.1 Suppose $u(n)$ is of exponential order r , $r > 0$. Then $N[u(n)]$ exists for each z lies inside the open ball $B_1(r)$.

Proof Suppose $u(n)$ is of exponential order r , $r > 0$. Then there exists some constant $A > 0$ and a natural number m such that $|u(n)| \leq Ar^{-n}$ for each $n > m$. Now, consider

$$\begin{aligned} |N[u(n)]| &\leq \sum_{j=1}^{\infty} |u(j)||1-z|^{j-1} \\ &= \sum_{j=0}^{\infty} |u(j+1)||1-z|^j \\ &= \sum_{j=0}^m |u(j+1)||1-z|^j + \sum_{j=m+1}^{\infty} |u(j+1)||1-z|^j \\ &\leq \sum_{j=0}^m |u(j+1)|(1-z)^j + \sum_{j=m+1}^{\infty} \frac{A}{r^{j+1}} |1-z|^j \\ &= \sum_{j=0}^m |u(j+1)|(1-z)^j + \frac{A}{r} \sum_{j=m+1}^{\infty} \left| \frac{1-z}{r} \right|^j \\ &= \sum_{j=0}^m |u(j+1)|(1-z)^j + \frac{A}{r} \frac{\left| \frac{1-z}{r} \right|^{m+1}}{1 - \left| \frac{1-z}{r} \right|} \\ &= \sum_{j=0}^m |u(j+1)|(1-z)^j + \frac{A}{r^{m+1}} \left[\frac{|1-z|^{m+1}}{r - |1-z|} \right] \\ &< \infty. \end{aligned}$$

Thus, if $u(n) : \mathbb{N}_1^+ \rightarrow \mathbb{R}$ is of exponential order r , $r > 0$, then $N[u(n)]$ exists for each $z \in B_1(r)$. $\square$

Remark 1 Clearly, for any $a \in \mathbb{R}$ and $p \in \mathbb{R} \setminus \{1\}$, the generalized nabla exponential function $\hat{e}_p(n, a) = \frac{1}{(1-p)^{n-a}}$ is of exponential order $|1-p|$. Thus, $N[\hat{e}_p(n, a)]$ exists for each $z \in B_1(|1-p|)$ and is given by $\frac{1}{z-p}$. Consequently, we have $N[1] = \frac{1}{z}$.

Now we derive some important properties of N-transforms. Let $u(n), v(n) : \mathbb{N}_1^+ \rightarrow \mathbb{R}$ are of exponential orders $r, s > 0$, respectively and a and b are any scalars.

Theorem 2.2 (Linearity) *For each $z \in B_1(\min\{r, s\})$,*

$$N[au(n) + bv(n)] = aN[u(n)] + bN[v(n)]. \quad (2.2)$$

Proof Let $p = \min\{r, s\}$. Since $u(n), v(n)$ are of exponential orders $r, s > 0$, respectively, we have $|u(n)| \leq r^{-n}$ and $|v(n)| \leq s^{-n}$ for sufficiently large $n \in \mathbb{N}_1^+$. Consequently,

$$\begin{aligned} |au(n) + bv(n)| &\leq |a||u(n)| + |b||v(n)| \\ &\leq |a|r^{-n} + |b|s^{-n} \\ &\leq [|a| + |b|]p^{-n} \end{aligned}$$

for sufficiently large $n \in \mathbb{N}_1^+$, implying $au(n) + bv(n)$ is of exponential order p , $p > 0$. Thus, from Lemma 2.1, $N[au(n) + bv(n)]$ exists for each $z \in B_1(\min\{r, s\})$. Now consider,

$$\begin{aligned} N[au(n) + bv(n)] &= \sum_{j=1}^{\infty} [au(j) + bv(j)](1-z)^{j-1} \\ &= a \sum_{j=1}^{\infty} u(j)(1-z)^{j-1} + b \sum_{j=1}^{\infty} v(j)(1-z)^{j-1} \\ &= aN[u(n)] + bN[v(n)]. \end{aligned}$$

□

The following lemma relates the shifted N-transform to the original.

Lemma 2.3 (Shifting Theorem) *Let $k \in \mathbb{N}_0^+$. Then for each $z \in B_1(r)$,*

$$N[u(n-k)] = (1-z)^k N[u(n)] \quad (2.3)$$

and

$$N[u(n+k)] = (1-z)^{-k} \left[N[u(n)] - u(1) - (1-z)^1 u(2) - \dots - (1-z)^{k-1} u(k) \right]. \quad (2.4)$$

Proof Consider

$$N[u(n-k)] = \sum_{j=1}^{\infty} u(j-k) (1-z)^{j-1}.$$

Take $j - k = i$ then i varies from 1 to ∞ . Then

$$\begin{aligned} N[u(n - k)] &= \sum_{i=1}^{\infty} u(i) (1 - z)^{i+k-1} = (1 - z)^k \sum_{i=1}^{\infty} u(i) (1 - z)^{i-1} \\ &= (1 - z)^k N[u(n)]. \end{aligned}$$

Now consider

$$N[u(n + k)] = \sum_{j=1}^{\infty} u(j + k) (1 - z)^{j-1}.$$

Take $j + k = i$ then i varies from $k + 1$ to ∞ . Then

$$\begin{aligned} N[u(n + k)] &= \sum_{i=k+1}^{\infty} u(i) (1 - z)^{i-k-1} \\ &= (1 - z)^{-k} \sum_{i=k+1}^{\infty} u(i) (1 - z)^{i-1} \\ &= (1 - z)^{-k} \left[\sum_{i=1}^{\infty} u(i) (1 - z)^{i-1} - u(1) - (1 - z)^1 u(2) \right. \\ &\quad \left. - \dots - (1 - z)^{k-1} u(k) \right] \\ &= (1 - z)^{-k} \left[N[u(n)] - u(1) - (1 - z)^1 u(2) - \dots - (1 - z)^{k-1} u(k) \right]. \end{aligned}$$

□

Definition 2.2 The convolution of $u(n)$ and $v(n)$ is defined as

$$u(n) * v(n) = \sum_{m=1}^n u(m) v(n - m + 1). \quad (2.5)$$

Lemma 2.4 (Convolution Theorem) *Let $u(n)$ and $v(n)$ are of exponential order r , $r > 0$. Then*

$$N[u(n) * v(n)] = N[u(n)] N[v(n)]. \quad (2.6)$$

Proof In order to prove (2.6), first we show that $u(n) * v(n)$ is of some exponential order. Since $u(n)$, $v(n)$ are of exponential order $r > 0$, there exist a natural number m such that $|u(n)| \leq r^{-n}$ and $|v(n)| \leq r^{-n}$ for each $n > m$. Consider

$$\begin{aligned}
|u(n) * v(n)| &\leq \sum_{k=1}^n |u(k)||v(n-k+1)| \\
&= \sum_{k=1}^m |u(k)||v(n-k+1)| + \sum_{k=m+1}^n |u(k)||v(n-k+1)| \\
&\leq \sum_{k=1}^m |u(k)||v(n-k+1)| + \sum_{k=m+1}^n r^{-k} r^{-(n-k+1)} \\
&= \sum_{k=1}^m |u(k)||v(n-k+1)| + r^{-(n+1)}(n-m) \\
&\leq \sum_{k=1}^m |u(k)||v(n-k+1)| + r^{-(n+1)}(n-1). \tag{2.7}
\end{aligned}$$

If $r > 1$, then $(n-1) < r^{n-1}$ for sufficiently large $n \in \mathbb{N}_1^+$ and hence

$$|u(n) * v(n)| < \sum_{k=1}^m |u(k)||v(n-k+1)| + r^{-2} \leq A$$

for some constant $A > 0$, implying $u(n) * v(n)$ is of exponential order one. If $0 < r \leq 1$, then for $1 \leq \frac{1}{r} < s$, $(n-1) < s^{(n-1)}$ for sufficiently large $n \in \mathbb{N}_1^+$ and hence

$$|u(n) * v(n)| < \sum_{k=1}^m |u(k)||v(n-k+1)| + s^{2n} < s^{3n} = \left(\frac{1}{s^3}\right)^{-n} < p^{-n}$$

where $0 < p < \frac{1}{s^3} < 1$. Thus, we have $u(n) * v(n)$ is of exponential order p , $0 < p < 1$. Hence $N[u(n) * v(n)]$ exists for each $z \in \bigcup_{0 < r \leq 1} B_1(r) = B_1(1)$. Now, consider

$$\begin{aligned}
N[u(n) * v(n)] &= \sum_{j=1}^{\infty} [u(j) * v(j)] (1-z)^{j-1} \\
&= \left[\sum_{m=1}^{\infty} u(m)(1-z)^{m-1} \right] \left[\sum_{j=1}^{\infty} v(j-m+1)(1-z)^{j-m} \right].
\end{aligned}$$

Take $j-m+1 = i$ then i varies from 1 to ∞ . Then

$$N[u(n) * v(n)] = \left[\sum_{m=1}^{\infty} u(m)(1-z)^{m-1} \right] \left[\sum_{i=1}^{\infty} v(i)(1-z)^{i-1} \right] = N[u(n)]N[v(n)].$$

□

Lemma 2.5 Let $\alpha \in \mathbb{R} \setminus \{\dots, -2, -1\}$ and $n \in \mathbb{N}_1^+$. Then for each $z \in B_1(1)$,

$$N \left[\binom{n+\alpha-2}{n-1} \right] = \frac{1}{z^\alpha}. \quad (2.8)$$

Proof From Lemma 2.1 we know that if $\binom{n+\alpha-2}{n-1}$ is of some exponential order r , $r > 0$, then $N \left[\binom{n+\alpha-2}{n-1} \right]$ exists for each $z \in B_1(r)$. Now we will show that $\binom{n+\alpha-2}{n-1}$ is of some exponential order.

First, we assume that $\alpha > 0$. Then there exists $m \in \mathbb{N}_1^+$ such that $\alpha \leq m$. For sufficiently large $n \in \mathbb{N}_1^+$ and $0 < r \leq \frac{1}{m} \leq 1$, we get

$$\begin{aligned} \binom{n+\alpha-2}{n-1} &= \frac{\Gamma(n+\alpha-1)}{\Gamma(n)\Gamma(\alpha)} = \frac{(-1)^{n-1}}{\Gamma(n)} \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha-\overline{n-1})} \\ &= \frac{(-1)^{n-1}}{\Gamma(n)} (-\alpha)(-\alpha-1) \cdots (-\alpha-\overline{n-2}) \\ &< \frac{(\alpha)^{n-1}}{\Gamma(n)} \\ &< m^n \\ &\leq r^{-n} \end{aligned}$$

which implies, $\binom{n+\alpha-2}{n-1}$ is of exponential order r , $0 < r \leq 1$. Now, suppose that $\alpha < 0$. Using Lemma 1.1, for sufficiently large $n \in \mathbb{N}_1^+$,

$$\binom{n+\alpha-2}{n-1} = \frac{\Gamma(n+\alpha-1)}{\Gamma(n)\Gamma(\alpha)} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \leq 1$$

implying $\binom{n+\alpha-2}{n-1}$ is of exponential order $r = 1$. Thus, we conclude that $\binom{n+\alpha-2}{n-1}$ is of exponential order r , $0 < r \leq 1$. Hence $N \left[\binom{n+\alpha-2}{n-1} \right]$ exists for each $z \in \bigcup_{0 < r \leq 1} B_1(r) = B_1(1)$. Now, consider

$$N \left[\binom{n+\alpha-2}{n-1} \right] = \sum_{j=1}^{\infty} \binom{j+\alpha-2}{j-1} (1-z)^{j-1} = [1 - (1-z)]^{-\alpha} = \frac{1}{z^\alpha}.$$

□

Lemma 2.6 Suppose $u(n)$ is of exponential order r , $0 < r \leq 1$ and let $\alpha \in \mathbb{R}^+$. Then, for each $z \in B_1(1)$,

$$N[\nabla^{-\alpha}u(n)] = z^{-\alpha}N[u(n)]. \quad (2.9)$$

Proof From Lemma 2.1 we know that if $\nabla^{-\alpha}u(n)$ is of some exponential order s , $s > 0$, then $N[\nabla^{-\alpha}u(n)]$ exists for all $z \in B_1(s)$. Now we will show that $N[\nabla^{-\alpha}u(n)]$ is of some exponential order. Since $u(n)$ and $\binom{n+\alpha-2}{n-1}$ are of exponential order r , $0 < r \leq 1$, from Lemma 2.5,

$$\nabla^{-\alpha}u(n) = \sum_{j=1}^n \binom{n-j+\alpha-1}{n-j} u(j) = u(n) * \binom{n+\alpha-2}{n-1}$$

is also of exponential order r , $0 < r \leq 1$. Now, consider

$$N[\nabla^{-\alpha}u(n)] = N\left[u(n) * \binom{n+\alpha-2}{n-1}\right] = z^{-\alpha}N[u(n)].$$

□

Lemma 2.7 Suppose $u(n)$ is of exponential order r , $0 < r \leq 1$ and let $\alpha \in \mathbb{R}^+$, $m \in \mathbb{Z}^+$ such that $m-1 < \alpha < m$. Then for each $z \in B_1(1)$,

$$N[\nabla^{\alpha}u(n)] = z^{\alpha} \left[N[u(n)] - \sum_{k=0}^{m-1} \frac{1}{z^{k+1}} [\nabla^k u(j)]_{j=0} \right]. \quad (2.10)$$

Proof From Definition 1.3, the Caputo type fractional difference operator of order α is given by

$$\nabla^{\alpha}u(n) = \nabla^{-(m-\alpha)}[\nabla^m u(n)]. \quad (2.11)$$

Since $u(n)$ is of exponential order r , $0 < r \leq 1$, $\nabla^m u(n)$ is also of exponential order r , $0 < r \leq 1$. Then, using Lemma 2.6, we conclude that $\nabla^{\alpha}u(n)$ is of exponential order r , $0 < r \leq 1$. Thus $N[\nabla^{\alpha}u(n)]$ exists for each $z \in B_1(1)$. Consider

$$\begin{aligned} N[\nabla^{\alpha}u(n)] &= N \left[\sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0} \right] \\ &= N_1 + N_2, \end{aligned}$$

where

$$N_1 = N \left[\sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j) \right] = z^{\alpha} N[u(n)]$$

and

$$N_2 = N \left[\sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0} \right] = \sum_{k=0}^{m-1} N \left[\binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0} \right].$$

Now we consider

$$N \left[\binom{n+k-\alpha-1}{n-1} \right] = \frac{1}{z^{k-\alpha+1}}.$$

Thus

$$N [\nabla^\alpha u(n)] = z^\alpha \left[N[u(n)] - \sum_{k=0}^{m-1} \frac{1}{z^{k+1}} [\nabla^k u(j)]_{j=0} \right].$$

□

Definition 2.3 Let $u(n) : \mathbb{N}_1^+ \rightarrow \mathbb{R}$ is of exponential order r , $r > 0$. Then $U(z) = \sum_{j=1}^{\infty} u(j)(1-z)^{j-1}$ exists for each z lies inside the open ball $B_1(r)$ and the inverse N-transform of $U(z)$ is defined as

$$N^{-1}(U(z)) = u(n). \quad (2.12)$$

The following are some important properties of inverse N-transforms.

Theorem 2.8 Let $u(n), v(n) : \mathbb{N}_1^+ \rightarrow \mathbb{R}$ are of exponential orders $r, s > 0$, respectively and a and b are any scalars such that $N[u(n)] = U(z)$ and $N[v(n)] = V(z)$. Then,

1. $N^{-1}[aU(z) + bV(z)] = au(n) + bv(n)$, for each $z \in B_1(\min\{r, s\})$;
2. $N^{-1}[U(z)V(z)] = u(n) * v(n)$, if $r = s$;
3. Let $\alpha \in \mathbb{R} \setminus \{\dots, -2, -1\}$. Then for each $z \in B_1(1)$, $N^{-1} \left[\frac{1}{z^\alpha} \right] = \binom{n+\alpha-2}{n-1}$.

3 Solutions of Fractional Difference Equations Using N-Transforms

In this section, we will illustrate the possible use of the N-transform by applying it to solve some fractional order initial value problems.

In 2003, Nagai [19] defined the discrete Mittag-Leffler function

$$F_\alpha(a, n) = \sum_{j=0}^{\infty} \left[a^j \binom{n+j\alpha-1}{n-j} \right] \quad (3.1)$$

which is a generalization of nabla exponential function on the time scale of integers [22, 23]. He also proved that $F_\alpha(a, n)$ is an eigen function of Caputo type fractional

difference operator defined in (1.3), that is,

$$\nabla^\alpha F_\alpha(a, n) = a F_\alpha(a, n). \quad (3.2)$$

Now we prove the same using N-transforms.

Example 1 Let $u(n)$ is of exponential order r , $0 < r \leq 1$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Then the solution of

$$\nabla^\alpha u(n) = au(n), \quad (3.3)$$

$$u(0) = a_0, \quad (3.4)$$

is $F_\alpha(a, n)$, where a and a_0 are constants.

Solution Taking N-transforms on both sides of (3.3), we have

$$\begin{aligned} z^\alpha \left[N[u(n)] - \frac{u(0)}{z} \right] &= a N[u(n)] \\ \text{or } N[u(n)] &= a_0 \left[\frac{z^{\alpha-1}}{z^\alpha - a} \right] \\ \text{or } N[u(n)] &= a_0 \left[\frac{1}{z} + az^{-\alpha-1} + a^2 z^{-2\alpha-1} + \dots \right]. \end{aligned}$$

Applying inverse N-transforms on both sides, we get

$$\begin{aligned} u(n) &= a_0 \left[1 + a \binom{n+\alpha-1}{n-1} + a^2 \binom{n+2\alpha-1}{n-2} + \dots \right] \\ &= a_0 \sum_{j=0}^{\infty} \left[a^j \binom{n+j\alpha-1}{n-j} \right] \\ &= a_0 F_\alpha(a, n). \end{aligned}$$

Remark 2 It is clear from the above example that

$$N[F_\alpha(a, n)] = \frac{z^{\alpha-1}}{z^\alpha - a}. \quad (3.5)$$

Example 2 Let $u(n)$ is of exponential order r , $0 < r \leq 1$ and let $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}^+$ such that $m-1 < \alpha < m$. Find the solution of

$$\nabla^\alpha u(n) = au(n), \quad (3.6)$$

$$\left[\nabla^k u(j)_{j=0} \right] = a_k, \quad k = 0, 1, 2, \dots, (m-1), \quad (3.7)$$

where $a, a_0, a_1, \dots, a_{m-1}$ are constants.

Solution Taking N-transforms on both sides of (3.6), we have

$$z^\alpha \left[N[u(n)] - \sum_{k=0}^{m-1} \frac{1}{z^{k+1}} [\nabla^k u(j)]_{j=0} \right] = aN[u(n)] \text{ or } N[u(n)] = \sum_{k=0}^{m-1} a_k \left[\frac{z^{\alpha-k-1}}{z^\alpha - a} \right].$$

Applying inverse N-transforms on both sides and then convolution theorem, we get

$$\begin{aligned} u(n) &= \sum_{k=0}^{m-1} a_k N^{-1} \left[\frac{z^{\alpha-k-1}}{z^\alpha - a} \right] = \sum_{k=0}^{m-1} a_k N^{-1} \left[\frac{z^{\alpha-1}}{z^\alpha - a} \times \frac{1}{z^k} \right] \\ &= \sum_{k=0}^{m-1} a_k \left[F_\alpha(a, n) * \binom{n+k-2}{n-1} \right] \\ &= \sum_{k=0}^{m-1} a_k \sum_{j=1}^n \left[F_\alpha(a, j) \times \binom{n-j+k-1}{n-j} \right] \\ &= \sum_{k=0}^{m-1} \sum_{j=1}^n \left[a_k F_\alpha(a, j) \binom{n-j+k-1}{n-j} \right]. \end{aligned}$$

Example 3 Let $u(n)$ and $v(n)$ are of exponential order r , $0 < r \leq 1$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Find the solution of

$$\nabla^\alpha u(n) = av(n), \quad (3.8)$$

$$u(0) = a_0, \quad (3.9)$$

where a and a_0 are constants.

Solution Taking N-transforms on both sides of (3.8), we have

$$\begin{aligned} z^\alpha \left[N[u(n)] - \frac{u(0)}{z} \right] &= aN[v(n)] \\ \text{or } N[u(n)] &= a_0 \left[\frac{1}{z} \right] + a \left[N[v(n)] \times \frac{1}{z^\alpha} \right]. \end{aligned}$$

Applying inverse N-transforms on both sides and then convolution theorem, we get

$$\begin{aligned} u(n) &= a_0 + aN^{-1} \left[N[v(n)] \times \frac{1}{z^\alpha} \right] \\ &= a_0 + a \left[v(n) * \binom{n+\alpha-2}{n-1} \right] \\ &= a_0 + \sum_{j=1}^n \left[av(j) \binom{n-j+\alpha-1}{n-j} \right]. \end{aligned}$$

Example 4 Let $u(n)$ and $v(n)$ are of exponential order r , $0 < r \leq 1$ and let $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}^+$ such that $m - 1 < \alpha < m$. Find the solution of

$$\nabla^\alpha u(n) = av(n), \quad (3.10)$$

$$[\nabla^k u(j)]_{j=0} = a_k, \quad k = 0, 1, 2, \dots, (m-1), \quad (3.11)$$

where $a, a_0, a_1, \dots, a_{m-1}$ are constants.

Solution Taking N-transforms on both sides of (3.10), we have

$$\begin{aligned} z^\alpha \left[N[u(n)] - \sum_{k=0}^{m-1} \frac{1}{z^{k+1}} [\nabla^k u(j)]_{j=0} \right] &= aN[v(n)] \\ \text{or } N[u(n)] &= \sum_{k=0}^{m-1} a_k \left[\frac{1}{z^{k+1}} \right] + a \left[N[v(n)] \times \frac{1}{z^\alpha} \right]. \end{aligned}$$

Applying inverse N-transforms on both sides and then convolution theorem, we get

$$\begin{aligned} u(n) &= \sum_{k=0}^{m-1} a_k N^{-1} \left[\frac{1}{z^{k+1}} \right] + a N^{-1} \left[N[v(n)] \times \frac{1}{z^\alpha} \right] \\ &= \sum_{k=0}^{m-1} a_k \binom{n+k-1}{n-1} + a \left[v(n) * \binom{n+\alpha-2}{n-1} \right] \\ &= \sum_{k=0}^{m-1} a_k \binom{n+k-1}{n-1} + \sum_{j=1}^n a \left[v(j) \binom{n-j+\alpha-1}{n-j} \right]. \end{aligned}$$

Example 5 Let $u(n)$ and $v(n)$ are of exponential order r , $0 < r \leq 1$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Find the solution of

$$\nabla^\alpha u(n) = au(n) + bv(n), \quad (3.12)$$

$$u(0) = a_0, \quad (3.13)$$

where a, b and a_0 are constants.

Solution Taking N-transforms on both sides of (3.12), we have

$$\begin{aligned} z^\alpha \left[N[u(n)] - \frac{u(0)}{z} \right] &= aN[u(n)] + bN[v(n)] \\ \text{or } N[u(n)] &= a_0 \left[\frac{z^{\alpha-1}}{z^\alpha - a} \right] + b \left[N[v(n)] \times \frac{1}{z^\alpha} \right]. \end{aligned}$$

Applying inverse N-transforms on both sides and applying convolution theorem, we get

$$u(n) = a_0 F_\alpha(a, n) + b \sum_{j=1}^n \left[v(j) \binom{n-j+\alpha-1}{n-j} \right].$$

Example 6 Let $u(n)$ and $v(n)$ are of exponential order r , $0 < r \leq 1$ and let $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}^+$ such that $m-1 < \alpha < m$. Find the solution of

$$\nabla^\alpha u(n) = au(n) + bv(n), \quad (3.14)$$

$$\left[\nabla^k u(j)_{j=0} \right] = a_k, \quad k = 0, 1, 2, \dots, (m-1), \quad (3.15)$$

where $a, b, a_0, a_1, \dots, a_{m-1}$ are constants.

Solution Taking N-transforms on both sides of (3.14), we have

$$\begin{aligned} z^\alpha \left[N[u(n)] - \sum_{k=0}^{m-1} \frac{1}{z^{k+1}} [\nabla^k u(j)]_{j=0} \right] &= aN[u(n)] + bN[v(n)] \\ \text{or } N[u(n)] &= \sum_{k=0}^{m-1} a_k \left[\frac{z^{\alpha-k-1}}{z^\alpha - a} \right] + b \left[N[v(n)] \times \frac{1}{z^\alpha} \right]. \end{aligned}$$

Applying inverse N-transforms on both sides and then convolution theorem, we get

$$\begin{aligned} u(n) &= \sum_{k=0}^{m-1} a_k N^{-1} \left[\frac{z^{\alpha-k-1}}{z^\alpha - a} \right] + b N^{-1} \left[N[v(n)] \times \frac{1}{z^\alpha} \right] \\ &= \sum_{k=0}^{m-1} \sum_{j=1}^n \left[a_k F_\alpha(a, j) \binom{n-j+k-1}{n-j} \right] + \sum_{j=1}^n b \left[v(j) \binom{n-j+\alpha-1}{n-j} \right]. \end{aligned}$$

Acknowledgments The authors are grateful to the referees for their suggestions and comments which considerably helped to improve the content of the paper.

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