Liouville Energy on a Topological Two Sphere

XiuXiong Chen · Meijun Zhu

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Abstract In this paper we shall give an analytic proof of the fact that the Liouville energy on a topological two sphere is bounded from below. Our proof does not rely on the uniformization theorem and the Onofri inequality, thus it is essentially needed in the alternative proof of the uniformization theorem via the Calabi flow. Such an analytic approach also sheds light on how to obtain the boundedness for E_1 energy in the study of general Kähler manifolds.

Keywords Uniformization theorem · Liouville energy · Moser–Trudinger–Onofri inequality · Blowup analysis

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1 Introduction

1.1 Description of the Problem

Let (M, g) be a smooth Riemann surface. For any conformal new metric $g_1 = e^u g$, the corresponding Liouville energy is defined by

$$L_g(g_1) = \int_M \ln \frac{g_1}{g} \cdot (R_{g_1} dV_{g_1} + R_g dV_g), \quad (1)$$

where R_g and R_{g_1} are twice the Gaussian curvatures K_g and K_{g_1} with respect to metrics g and g_1 . Since

$$R_{g_1} = e^{-u}(-\Delta_g u + R_g),$$

X.X. Chen (✉)

Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706-1388, USA
e-mail: xiuxiong.chen@gmail.com

M. Zhu

Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA

the Liouville energy of metric g_1 can also be represented by

$$L_g(g_1) = \int_M (|\nabla_g u|^2 + 2R_g u) dV_g.$$

It was showed by Chow [14] and Chen [7] that Ricci flow and Calabi flow can be viewed as the gradient flow of the Liouville energy. Therefore if M is a topological two sphere, it is important to know in the study of Ricci and Calabi flow that the Liouville energy is bounded from below for any metric with fixed volume. In fact such boundedness can be used to show the global existence of Calabi flow via integral estimates (see, for example, Chen [7]), thus one can give an alternative proof of the uniformization theorem (for the hardest case: positive curvature case) via Calabi flow.¹ However, present argument in the literature seems to be a tautology: the proof of the fact that the Liouville energy on a two sphere is bounded from below relies on the uniformization theorem. In fact, the proof can be summarized as follows. Let (M, g) be a topological sphere. From the uniformization theorem we know that (M, g) is conformally equivalent to the standard sphere (S^2, g_0) , that is, there is a $\varphi(x) \in C^2(S^2)$ such that $g = e^\varphi g_0$. For a given metric $g_1 = e^u g$, since $g_1 = e^{u+\varphi} g_0$ and $R_{g_0} = 2$, we have

$$\begin{aligned} L_g(g_1) &= \int_M (|\nabla_g u|^2 + 2R_g u) dV_g \\ &= L_{g_0}(g_1) - L_{g_0}(g) \\ &= \int_{S^2} (|\nabla_{g_0}(u + \varphi)|^2 + 4(u + \varphi)) dV_{g_0} - L_{g_0}(g) \\ &\geq 16\pi \cdot \ln \frac{1}{4\pi} \int_{S^2} e^{(u+\varphi)} dV_{g_0} - L_{g_0}(g). \end{aligned}$$

The last inequality follows from the well-known Onofri inequality on S^2 . Thus the finiteness of the volume of the metric g_1 implies that the Liouville energy of such metric is bounded from below.

1.2 Historic Note

It is always natural to ask if one can use variational approach to attack the existence of constant scalar curvature metrics in certain conformal class in 2-sphere. The elliptic approach might start with Berger [4], which eventually leads to the famous prescribing curvature problem on S^2 (the Nirenberg problem). The parabolic approach was first studied by Hamilton [17] and Chow [14] via Ricci flow, and later details of the approach via the Calabi flow were illustrated by the first author. The first step of such an approach would be to find a direct analytic proof of the fact the Liouville energy is bounded from below first. This question was raised in [7] as a key step in deriving a

¹ In a recent note, Chen, Lu and Tian [11] clarify that the Ricci flow did yield another proof of the uniformization theorem for spheres even without using current integral estimates. Nevertheless our result not only answer a conjecture of Chen [7], but also has broad applications in the study of Kähler manifolds (see the historic note).

new proof for uniformization theorem in topological two sphere via the Calabi flow. More importantly, this question is closely related to a far reaching program in the study of general Kähler manifolds: Can one use variational approach to derive the existence of constant scalar curvature metrics in each Kähler class?

The search of constant scalar curvature metric in each Kähler class was suggested by Calabi [6] in 1982. This is one of the key problems in Kähler geometry and there is extensive research in this and related problems. The analytic obstruction to the existence of such metric was first observed in the important work of Mabuchi-Bando [3] in 1987, where they showed that the existence of Kähler Einstein metric implies the existence of lower bound of the K energy in each Kähler class.² This obstruction was extended by the work of the first author in 1998 to the case of constant scalar curvature metric in Kähler manifold with non-positive first Chern class. More recently, this problem is completely settled for all Kähler classes (cf. [10] for more references). In [9], Chen and Tian introduced a set of $n + 1$ energy functionals $E_0, E_1, \dots, E_n$ in any n -dimensional Kähler manifold where E_0 is the K energy functional and E_1 is the Liouville energy on S^2 . A natural question asked in [7] (Question 7.3) and [8] is whether the existence of a Kähler Einstein metric implies the lower bound of E_1 . Concentrated effort by various geometers and rapid progress are made in this problem. For instance, the first author observed that, along the Kähler Ricci flow on Kähler Einstein manifold, this functional is bounded below. This is answered affirmatively by Song and Weinkove [24] that lower bound of E_1 is obstruction to the existence of Kähler Einstein metrics. Shortly after, Pali [22] gives a new proof which considerably simplified the Song–Weinkove’s argument. See also [25] for other related results. All of these recent results highlight the importance of obtaining an a priori lower bound of this functional E_1 (without the knowledge of Kähler Einstein metric or constant scalar curvature metric). Unfortunately, even in the simplest case of CP^n ($n > 1$), we even do not know how to prove directly why E_1 energy is bounded from below without using Mabuchi–Bando theorem. On S^2 , the previous proof (for example, given by Osgood–Phillips–Sarnak [23]) relies on the existence of constant scalar curvature metric (as we showed before). The present paper gives an a priori lower bound of E_1 (or Liouville energy) on the topological S^2 . This certainly sheds light on how to analytically derive the low bound of the E_1 in higher dimension and it gives us hope that the large program can be successful.

1.3 The Main Results

The main purpose of this note is to give such an analytic proof of

Theorem 1.1 *If (M, g) is a topological two sphere with volume 4π and bounded curvature, then there is a constant C_{low} (which depends on metric g) such that for any conformal metric $g_1 = e^{2u}g$ with volume 4π ,*

$$L(g_1) \geq C_{\text{low}}.$$

²On S^2 , a Kähler class is equivalent a pointwise conformal class of Riemannian metric with fixed total area.

Remark 1.1 The previous proof based on the uniformization theorem and the Onofri inequality does yield the best constant for C_{low} . Our proof cannot yield the best lower bound.

Comparing with some existing results for higher dimensional Yamabe type problem (see, for example, [18] and [19]), our proof seems quite standard. We start with a local sharp inequality (caution: this local inequality does not rely on Moser's inequality), which allows us to obtain a rough sharp inequality (or, sometimes, we refer it as an ϵ -level inequality) in Proposition 2.1. We then study the behavior of the minimizing sequence as the parameter ϵ goes to 0. It will be shown that the perturbed Liouville energy will stay uniformly bounded as ϵ goes to 0, and this yields the result in Theorem 1.1. It is worth mentioning that in higher dimensional case, similar argument was carried out by Hebey and Vaugon [18] in their proof of a conjecture due to Aubin [1], and by Li and Zhu in the proof of sharp trace inequality [19].

2 Preliminary Results

We start with a local sharp inequality which is proved in [20]. It is interesting to point out that such a local inequality also yields a much simpler proof of the Onofri inequality (see also a more recent work [21]).

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, Ω^* be the ball in $\mathbb{R}^2$ which has the same area as Ω , and denote

$$D_{a,b}(\Omega) = \left\{ f(x) - b \in H_0^1(\Omega) : \int_{\Omega} e^{2f} dx = a \right\}.$$

We have the following sharp inequality.

Theorem 2.1 [20]

$$\inf_{w \in D_{a,b}(\Omega)} \int_{\Omega} |\nabla w|^2 dx \geq 4\pi \cdot \left(\ln \frac{ae^{-2b}}{\pi r^2} + \frac{\pi r^2}{ae^{-2b}} - 1 \right),$$

where r is the radius of Ω^* .

As simple consequences, we have

Corollary 2.1 Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. For any $u \in H_0^1(\Omega)$,

$$\int_{\Omega} e^u dx \leq \pi r^2 e \exp \left\{ \frac{1}{16\pi} \int_{\Omega} |\nabla u|^2 dx \right\},$$

where r is the radius of Ω^* .

And

Corollary 2.2 *Let (M, g) be a smooth Riemann surface. For any given $\epsilon > 0$, there are $r_0 > 0$ and $C(\epsilon) > 0$ such that if $\Omega \subset B(x_0, r_0)$ for some point $x_0 \in M$, then*

$$\int_{\Omega} e^w dV_g \leq C(\epsilon) \exp \left\{ \left(\frac{1}{16\pi} + \epsilon \right) \int_{\Omega} |\nabla_g w|^2 dV_g \right\} \quad \forall w \in H_0^1(\Omega).$$

Remark 2.1 Inequalities in Corollaries 2.1 and 2.2 were proved early by Cherrier [13] based on Moser's inequality.

From these two corollaries we can derive

Lemma 2.1 *Let (M, g) be a given Riemann surface. For any given $\epsilon > 0$, there are constants $C_1 = C_1(\epsilon)$, $C_2 = C_2(\epsilon)$ such that*

$$\int_M e^w dV_g \leq C_1 \exp \left\{ \left(\frac{1}{16\pi} + \epsilon \right) \int_M |\nabla_g w|^2 dV_g + C_2 \int_M w^2 \right\}, \quad \forall w \in H^1(M).$$

Proof Let r_0 be the constant given in Corollary 2.2. Let $\bigcup_{i=1}^N B(r_i/2, x_i)$ be a covering of M , where $r_i < r_0$. Let $\{\psi_i(x)\}_{i=1}^N$ be smooth cutoff functions subordinate to this covering, satisfying

$$\psi_i(x) = \begin{cases} 1, & x \in B(x_i, r_i/2), \\ 0, & x \notin B(x_i, r_i), \end{cases}$$

and $0 \leq \psi_i^2 \leq 1$. We have, using Corollary 2.2,

$$\begin{aligned} \int_M e^u dV_g &\leq \sum_{i=1}^N \int_{B(x_i, r_i/2)} e^u dV_g \\ &\leq \sum_{i=1}^N \int_{B(x_i, r_i)} e^{u \cdot \psi_i} dV_g \\ &\leq \sum_{i=1}^N C(\epsilon) \exp \left\{ \left(\frac{1}{16\pi} + \epsilon \right) \int_{B(x_i, r_i)} |\nabla_g (u \cdot \psi_i)|^2 dV_g \right\}. \end{aligned}$$

Note

$$\begin{aligned} \int_{B(x_i, r_i)} |\nabla_g (u \cdot \psi_i)|^2 dV_g &\leq \int_{B(x_i, r_i)} (|\nabla_g u|^2 \cdot |\psi_i|^2 + C_2 u^2) dV_g \\ &\leq \int_M (|\nabla_g u|^2 + C_2 u^2) dV_g. \end{aligned}$$

It follows that

$$\int_M e^u dV_g \leq N \cdot C(\epsilon) \exp \left\{ \left(\frac{1}{16\pi} + \epsilon \right) \int_M (|\nabla_g u|^2 + C_2 u^2) dV_g \right\}.$$

We are now ready to establish the following ϵ -level inequality according to Aubin [2]. Similar type inequalities were discussed for higher dimensional cases. See, for example, [18] or [19]. $\square$

Proposition 2.1 *Let (M, g) be a topological two sphere with bounded Gaussian curvature K_g . Then for any given $\epsilon > 0$, there is a constant $C_3 = C_3(\epsilon)$ such that for any $u \in H^1(M)$ with $\int_M K_g \cdot u \, dV_g = 0$,*

$$\int_M e^u \, dV_g \leq C_3(\epsilon) \exp \left\{ \left(\frac{1}{16\pi} + \epsilon \right) \int_M |\nabla_g u|^2 \, dV_g \right\}. \quad (2)$$

To prove this proposition, we need the following Poincaré–Sobolev type inequality.

Lemma 2.2 *Let (M, g) be a topological two sphere with bounded Gaussian curvature K_g . Then for any $p \geq 1$, there is a constant c_p such that for any $u \in H^1(M)$ with $\int_M K_g \cdot u \, dV_g = 0$,*

$$\left(\int_M |u|^p \right)^{2/p} \leq c_p \int_M |\nabla_g u|^2 \, dV_g. \quad (3)$$

Proof The proof is standard. We prove it by contradiction. Suppose this is not true, then there is a sequence of functions $\{u_n\}$ in $H^1(M)$ with $\int_M K_g \cdot u_n \, dV_g = 0$ such that

$$\left(\int_M |u_n|^p \right)^{2/p} \geq n \int_M |\nabla_g u_n|^2 \, dV_g.$$

Let

$$v_n = \frac{u_n}{\left(\int_M |u_n|^p \right)^{1/p}}.$$

We know that

$$\|v_n\|_{L^p} = 1, \quad \|v_n\|_{H^1} \leq C \quad \text{and} \quad \|\nabla_g v_n\|_{L^2} \rightarrow 0.$$

Thus, $v_n \rightharpoonup v_0$ in $H^1(M)$ and v_0 is a constant. By the compact embedding, we know that $v_n \rightarrow v_0$ in L^q for any $q < \infty$, thus $\int_M K_g v_0 \, dV_g = 0$ and $\|v_0\|_{L^p} = 1$. On the other hand, since $\int_M K_g \, dV_g = 4\pi$, we conclude that $v_0 = 0$. Contradiction. $\square$

Remark 2.2 The topological assumption is obviously needed since the above inequality does not hold for a flat torus.

Remark 2.3 The condition on K_g could be relaxed to $\|K_g\|_{L^q(M)} \leq C$ for some $q > 1$. However, it is not obvious whether the result is still true or not by only assuming that $K_g \in L^1(M)$.

We now return to the proof of Proposition 2.1.

Proof of Proposition 2.1 Due to Lemma 1 in [1, p. 586], we only need to prove the proposition for functions in $C^1(M)$ with only non-degenerate critical points. Let $u(x)$ be such a function with $\int_M K_g \cdot u \, dV_g = 0$. For small enough $\eta > 0$, there exists a_η such that

$$\text{Vol}\{x \in M : u(x) > a_\eta\} = \eta. \quad (4)$$

Using Lemma 2.1 we have for any $\epsilon_1 > 0$, there are constants $C_1(\epsilon_1)$, $C_2(\epsilon_1) > 0$ such that

$$\begin{aligned} \int_M e^u \, dV_g &\leq e^{a_\eta} \cdot \int_M e^{(u-a_\eta)} \, dV_g \leq e^{a_\eta} \cdot \int_M e^{(u-a_\eta)_+} \, dV_g \\ &\leq e^{a_\eta} \cdot C_1(\epsilon_1) \exp\left\{\left(\frac{1}{16\pi} + \epsilon_1\right) \int_M |\nabla_g(u-a_\eta)_+|^2 \, dV_g\right. \\ &\quad \left.+ C_2(\epsilon_1) \int_M |(u-a_\eta)_+|^2 \, dV_g\right\}, \end{aligned} \quad (5)$$

where $f_+(x) = \max\{f(x), 0\}$. Since

$$a_\eta \cdot \eta \leq \int_{\{x \in M : u(x) > a_\eta\}} u \, dV_g \leq \int_M |u| \, dV_g \leq c_1^{1/2} \left(\int_M |\nabla_g u|^2 \right)^{1/2},$$

we have

$$a_\eta \leq \eta \|\nabla_g u\|_{L^2}^2 + \frac{C}{\eta^3}.$$

Also, using Lemma 2.2, we have

$$\begin{aligned} \int_M |(u-a_\eta)_+|^2 \, dV_g &= \int_{\{x \in M : u(x) > a_\eta\}} |(u-a_\eta)_+|^2 \, dV_g \\ &\leq \left(\int_{\{x \in M : u(x) > a_\eta\}} |(u-a_\eta)_+|^4 \, dV_g \right)^{1/2} \cdot \eta^{1/2} \\ &\leq \left(\int_M |u|^4 \, dV_g \right)^{1/2} \cdot \eta^{1/2} \\ &\leq c_4 \|\nabla_g u\|_{L^2}^2 \cdot \eta^{1/2}. \end{aligned}$$

If we choose ϵ_1 and η sufficiently small so that

$$\eta + \epsilon_1 + C_2(\epsilon_1) \cdot c_4 \eta^{1/2} < \epsilon,$$

we obtain inequality (2). $\square$

3 Proof of Theorem 1.1

In this section, we always assume that (M, g) is a smooth topological sphere with bounded Gaussian curvature K_g . For convenience, we also assume that $\int_M dV_g = 4\pi$

and use the simplified notations as follows:

$$\nabla := \nabla_g, \quad \Delta := \Delta_g, \quad \int_M f := \int_M f dV_g.$$

For any $\epsilon > 0$, we define a perturbed (modified also) Liouville energy functional by

$$I_\epsilon(u) = \frac{1}{2} \int_M |\nabla u|^2 + \frac{8\pi - \epsilon}{4\pi} \int_M K_g u - (8\pi - \epsilon) \ln \int_M e^u, \quad (6)$$

and denote

$$E_\epsilon = \inf_{u \in H^1(M), \int_M K_g u = 0} I_\epsilon(u).$$

It is easy to see that $I_\epsilon(u + C) = I_\epsilon(u)$ for any constant C . Due to Proposition 2.1 we know that E_ϵ is a finite number. Moreover, we can show

Lemma 3.1 E_ϵ is achieved by a function $u_\epsilon \in H^1(M)$.

Proof Let $\{u_n\}$ be a minimizing sequence with $\int_M K_g u_n = 0$. Choose a small positive number ϵ_1 so that

$$\left(\frac{1}{2} - \epsilon_1\right) \cdot \frac{1}{8\pi - \epsilon} > \frac{1}{16\pi} + \epsilon_1.$$

Using Proposition 2.1 we have

$$\begin{aligned} E_\epsilon &\leftarrow I_\epsilon(u_n) = \frac{1}{2} \int_M |\nabla u_n|^2 - (8\pi - \epsilon) \ln \int_M e^{u_n} \\ &= \left(\frac{1}{2} - \epsilon_1\right) \int_M |\nabla u_n|^2 - (8\pi - \epsilon) \ln \int_M e^{u_n} + \epsilon_1 \int_M |\nabla u_n|^2 \\ &\geq C(\epsilon_1) + \epsilon_1 \int_M |\nabla u_n|^2, \end{aligned}$$

thus $\|\nabla u_n\|_{L^2} < C$. It follows from Lemma 2.2 that $\|u_n\|_{H^1} < C$. Further, it implies that $u_n \rightharpoonup u_\epsilon$ in $H^1(M)$ and $e^{ku_n} \rightarrow e^{ku_\epsilon}$ in $L^1(M)$ for any positive number k by Trudinger's inequality (see, e.g., [20, Theorem A]). It follows that u_ϵ is the minimizer.

Let

$$v_\epsilon = u_\epsilon - \ln \int_M e^{u_\epsilon},$$

then v_ϵ is also a minimizer, which satisfies the following Euler–Lagrange equation:

$$-\Delta v_\epsilon + \frac{8\pi - \epsilon}{4\pi} \cdot K_g = (8\pi - \epsilon)e^{v_\epsilon} \quad \text{in } M, \quad (7)$$

and $\int_M e^{2v_\epsilon} \leq C$ (since $\|v_\epsilon\|_{H^1} \leq C$). The standard elliptic estimates yield $v_\epsilon \in C^\alpha(M)$ for some $\alpha \in (0, 1)$.

To prove Theorem 1.1, we only need to show that E_ϵ is uniformly bounded from below as $\epsilon \rightarrow 0$. We follow closely the similar arguments given early in [15] and [16].

We argue by contradiction. Suppose that E_ϵ is not bounded from below, that is, up to a subsequence (due to the nature of the proof, for convenience we will not distinguish subsequence $\{\epsilon_i\}$ and the original sequence $\{\epsilon\}$ in this section),

$$\lim_{\epsilon \rightarrow 0} E_\epsilon = -\infty. \quad (8)$$

Let $\lambda_\epsilon = v_\epsilon(x_\epsilon) := \max_{x \in M} v_\epsilon(x)$. We first claim $\square$

Lemma 3.2

$$\lim_{\epsilon \rightarrow 0} \lambda_\epsilon = +\infty.$$

Proof Let $v_\epsilon^a = \frac{1}{4\pi} \int_M v_\epsilon$. If $\lambda_\epsilon < C$, then $e^{v_\epsilon} < C$, thus $\|\nabla(v_\epsilon - v_\epsilon^a)\|_{L^2} < C$ from Eq. (7). From the Poincaré inequality, we know that $\|v_\epsilon - v_\epsilon^a\|_{L^2} < C$. Since $\|\Delta(v_\epsilon - v_\epsilon^a)\|_{L^\infty} \leq C$, we know from the standard elliptic estimates that $v_\epsilon - v_\epsilon^a \in C^1(M)$. If $v_\epsilon^a \rightarrow -\infty$ is unbounded, we know from Eq. (7) that $v_\epsilon - v_\epsilon^a \rightarrow w$, where w satisfies $\Delta w = 2K_g$, this is impossible since $\int_M K_g \neq 0$; If v_ϵ^a is bounded, then $v_\epsilon = (v_\epsilon - v_\epsilon^a) + v_\epsilon^a$ is bounded, thus E_ϵ is bounded from below, this is a contradiction to (8).

Assume that, up to a subsequence, $x_\epsilon \rightarrow \bar{x} \in M$. In a neighborhood of $\bar{x}$, we choose a normal coordinate system, and define

$$\varphi_\epsilon(x) = v_\epsilon(\tau_\epsilon^{-1}x + x_\epsilon) - 2 \ln \tau_\epsilon,$$

where $\tau_\epsilon := e^{\lambda_\epsilon/2}$, and without confusion, we use $\tau_\epsilon^{-1}x + x_\epsilon$ to represent $\exp_{x_\epsilon}(\tau_\epsilon^{-1}x)$. For any given $R > 0$, if ϵ is sufficiently small, φ_ϵ satisfies

$$-\Delta \varphi_\epsilon + \frac{8\pi - \epsilon}{4\pi \tau_\epsilon^2} \cdot K_g = (8\pi - \epsilon)e^{\varphi_\epsilon} \quad \text{in } B_{2R}(0) \subset \mathbb{R}^2. \quad (9)$$

The behavior of v_ϵ in a tiny neighborhood of $\bar{x}$ can be described by the behavior of φ_ϵ in a large ball $B_{2R}(0) \subset \mathbb{R}^2$. $\square$

Lemma 3.3 *For a fixed $R > 0$, there is a constant $C(R)$ such that*

$$|\varphi_\epsilon(x)| < C(R) \quad \forall x \in B_R(0).$$

Proof Let $\varphi_\epsilon^{(1)}$ be the unique solution to

$$\begin{cases} -\Delta \varphi_\epsilon^{(1)} + \frac{8\pi - \epsilon}{4\pi \tau_\epsilon^2} \cdot K_g = (8\pi - \epsilon)e^{\varphi_\epsilon} & \text{in } B_{2R}(0), \\ \varphi_\epsilon^{(1)} = 0 & \text{on } \partial B_{2R}(0). \end{cases}$$

Since $e^{\varphi_\epsilon} \leq 1$, we know that $|\varphi_\epsilon^{(1)}| < C_1(R)$ for $x \in B_R(0)$. Let $\varphi_\epsilon^{(2)} = \varphi_\epsilon - \varphi_\epsilon^{(1)}$. Then $\varphi_\epsilon^{(2)}$ is a harmonic function bounded above by $C_1(R)$. Applying the Harnack

inequality to $2C_1(R) + \varphi_\epsilon^{(2)}$ we have

$$\frac{1}{C} \leq \frac{2C_1(R) + \varphi_\epsilon^{(2)}(x)}{2C_1(R) + \varphi_\epsilon^{(2)}(0)} < C \quad \forall x \in B_R(0).$$

This and the bound of $\varphi_\epsilon^{(1)}$ yield Lemma 3.3.

Since φ_ϵ is uniformly bounded in any fixed ball, we know, based on the standard elliptic estimates, that

$$\varphi_\epsilon \rightarrow \varphi_0 \quad \text{in } C^{1,\alpha}(B_R(0)), \quad (10)$$

for some $\alpha \in (0, 1)$, where φ_0 satisfies

$$-\Delta \varphi_0 = 8\pi e^{\varphi_0} \quad \text{in } R^2. \quad (11)$$

On the other hand, we note that

$$\varphi_0(x) \leq \varphi_0(0) = 1 \quad \text{and} \quad \int_{R^2} e^{\varphi_0} \leq \lim_{R \rightarrow +\infty} \left(\overline{\lim}_{\epsilon \rightarrow 0} \int_{B_R} e^{\varphi_\epsilon} \right) \leq 1. \quad (12)$$

From the classification of constant curvature metric on S^2 or the PDE's result of Chen and Li [12] we know that

$$\varphi_0(x) = 2 \ln \frac{1}{1 + \pi |x|^2}. \quad (13)$$

Away from the singular point $\bar{x}$, we have the following global estimate. □

Lemma 3.4 *For any compact domain $K \subset\subset M \setminus \{\bar{x}\}$, there is a constant $C(K)$ such that*

$$\left\| v_\epsilon - \frac{1}{4\pi} \int_M v_\epsilon \right\|_{L^\infty(K)} < C(K).$$

The proof of Lemma 3.4 is based on a lemma due to Brezis and Merle [5]. Such a lemma plays the same role as the sharp Sobolev inequalities play in higher dimensional cases. To keep our argument intact, we relegate the proof of this lemma to the end of this section.

Note that for any $1 \leq q < 2$,

$$\|\nabla v_\epsilon\|_{L^q(M)} < C_q. \quad (14)$$

This is because for any $\varphi \in W^{1,q/(q-1)}(M)$ with $\int_M \varphi = 0$ and $\|\varphi\|_{W^{1,q/(q-1)}} = 1$ (thus $\varphi \in L^\infty(M)$),

$$\left| \int_M \nabla v_\epsilon \nabla \varphi \right| = \left| \int_M \Delta v_\epsilon \varphi \right| \leq C.$$

We derive from the above and (8) that

$$4\pi v_\epsilon^a = \int_M K_g v_\epsilon - \int_M K_g (v_\epsilon - v_\epsilon^a)$$

$$\leq \int_M K_g v_\epsilon - C \rightarrow -\infty. \quad (15)$$

From Lemma 3.4 and (15), we conclude, via the standard elliptic estimates, that

$$v_\epsilon - v_\epsilon^a \rightarrow G(x, \bar{x}) \quad \text{in } C^{1,\alpha}(K) \quad (16)$$

for any compact domain K in $M \setminus \{\bar{x}\}$, where $G(x, \bar{x})$ is the Green function in M satisfying

$$\begin{cases} -\Delta G + 2K_g = 8\pi\delta_{\bar{x}} & \text{in } M, \\ \int_M G = 0. \end{cases} \quad (17)$$

Since K_g is bounded, in a normal coordinate system centered at $\bar{x}$,

$$G(x, \bar{x}) = -4\ln|x| + A(\bar{x}) + o(1), \quad (18)$$

where $A(\bar{x})$ is a constant depending on the location of $\bar{x}$, $o(1) \rightarrow 0$ as $|x| \rightarrow 0$.

To complete the proof of the main theorem, we also need a lower bound for v_ϵ away from the singular point $\bar{x}$.

Lemma 3.5 *For any fixed $R > 0$, let $r_\epsilon = \tau_\epsilon^{-1}R$. Then*

$$v_\epsilon(x) \geq G(x, x_\epsilon) - \lambda_\epsilon - 2\ln \frac{1 + \pi R^2}{R^2} - A(\bar{x}) + o_\epsilon(1) \quad \forall x \in M \setminus B(x_\epsilon, r_\epsilon), \quad (19)$$

where $o_\epsilon(1) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proof On $\partial B(x_\epsilon, r_\epsilon)$, we define $C_\epsilon^* := (v_\epsilon - G(x, x_\epsilon))|_{|x|=r_\epsilon}$. From (10) and (13) we know that

$$C_\epsilon^* = -\lambda_\epsilon - 2\ln \frac{1 + \pi R^2}{R^2} - A(\bar{x}) + o_\epsilon(1).$$

Let

$$\tilde{K}_g(x) = \begin{cases} K_g(x), & \text{if } K_g(x) \leq 0, \\ \phi(x)K_g(x), & \text{if } K_g(x) > 0, \end{cases}$$

where $0 \leq \phi \leq 1$ is a measurable function such that $\int_M \tilde{K}_g = 0$. Let $h(x)$ be one solution to $\Delta h = \tilde{K}_g$ in M (which is bounded). We consider $v_\epsilon(x) - G(x, x_\epsilon) - C_\epsilon^* + \frac{\epsilon}{4\pi}h(x)$ in $M \setminus B(x_\epsilon, r_\epsilon)$. Since $-\Delta(v_\epsilon(x) - G(x, x_\epsilon) - C_\epsilon^* + \frac{\epsilon}{4\pi}h(x)) \geq \frac{\epsilon(K_g - \tilde{K}_g)}{4\pi} \geq 0$, we have (19) via the maximum principle.

We are now ready to complete the proof of the main theorem.

We need to estimate $E_\epsilon = I_\epsilon(v_\epsilon)$. For any fixed small $\delta > 0$, we assume that ϵ is sufficiently small so that $\delta > r_\epsilon$. Then

$$\begin{aligned} \int_M |\nabla v_\epsilon|^2 &= \int_{M \setminus B(x_\epsilon, \delta)} |\nabla v_\epsilon|^2 + \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} |\nabla v_\epsilon|^2 + \int_{B(x_\epsilon, r_\epsilon)} |\nabla v_\epsilon|^2 \\ &:= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3. \end{aligned}$$

First of all, it is easy to see from (10) and (13) that

$$\begin{aligned} \mathbf{I}_3 &= \int_{B(x_\epsilon, r_\epsilon)} |\nabla \varphi_0|^2 dx + o_\epsilon(1) = \int_{B(R)} \left| \nabla \left(2 \ln \frac{1}{1 + \pi |x|^2} \right) \right|^2 dx + o_\epsilon(1) \\ &= 32\pi^3 \int_0^R \frac{r^2 r}{(1 + \pi r^2)^2} dr + o_\epsilon(1) \\ &= 16\pi (\ln(1 + \pi R^2) - 1) + o_\epsilon(1) + o_R(1), \end{aligned} \quad (20)$$

where $o_R(1) \rightarrow 0$ as $R \rightarrow \infty$.

Since $v_\epsilon(x) - v_\epsilon^a \rightarrow G(x, x_\epsilon)$ in $C_{\text{loc}}^2(M \setminus \{x_\epsilon\})$, we obtain

$$\begin{aligned} \mathbf{I}_1 &= \int_{M \setminus B(x_\epsilon, \delta)} |\nabla v_\epsilon|^2 = \int_{M \setminus B(x_\epsilon, \delta)} |\nabla G|^2 + o_\epsilon(1) \\ &= - \int_{M \setminus B(x_\epsilon, \delta)} 2K_g G - \int_{\partial B(x_\epsilon, \delta)} G \frac{\partial G}{\partial \mathbf{n}} + o_\epsilon(1). \end{aligned} \quad (21)$$

To estimate $\mathbf{I}_2$, we first use Eq. (7) to get

$$\begin{aligned} \mathbf{I}_2 &= - \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} \Delta v_\epsilon \cdot v_\epsilon + \int_{\partial(B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon))} v_\epsilon \frac{\partial v_\epsilon}{\partial \mathbf{n}} \\ &= \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} \left(-\frac{8\pi - \epsilon}{4\pi} K_g \cdot v_\epsilon + (8\pi - \epsilon) v_\epsilon e^{v_\epsilon} \right) \\ &\quad + \int_{\partial(B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon))} v_\epsilon \frac{\partial v_\epsilon}{\partial \mathbf{n}}. \end{aligned}$$

Applying Lemma 3.5, we have

$$\begin{aligned} \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} (8\pi - \epsilon) v_\epsilon e^{v_\epsilon} &\geq -\lambda_\epsilon \cdot (8\pi - \epsilon) \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} e^{v_\epsilon} \\ &\quad + \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} (8\pi - \epsilon) G e^{v_\epsilon} + o_\epsilon(1) + o_R(1), \end{aligned}$$

where $o_\epsilon(1) \rightarrow 0$ as $\epsilon \rightarrow 0$ and $o_R(1) \rightarrow 0$ as $R \rightarrow +\infty$. From Eqs. (7), (16) and (17) we obtain

$$\begin{aligned} \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} (8\pi - \epsilon) G e^{v_\epsilon} &= - \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} 2K_g v_\epsilon \\ &\quad + \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} \frac{8\pi - \epsilon}{4\pi} K_g \cdot G \\ &\quad + \int_{\partial(B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon))} \left(v_\epsilon \frac{\partial G}{\partial \mathbf{n}} - G \frac{\partial v_\epsilon}{\partial \mathbf{n}} \right) \\ &= - \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} 2K_g v_\epsilon + \int_{\partial B(x_\epsilon, \delta)} v_\epsilon^a \frac{\partial G}{\partial \mathbf{n}} \end{aligned}$$

$$\begin{aligned}
& - \int_{\partial B(x_\epsilon, r_\epsilon)} \left(v_\epsilon \frac{\partial G}{\partial \mathbf{n}} - G \frac{\partial v_\epsilon}{\partial \mathbf{n}} \right) \\
& + o_\epsilon(1) + o_\delta(1),
\end{aligned}$$

where $o_\delta(1) \rightarrow 0$ as $\delta \rightarrow 0$, and

$$\begin{aligned}
-\lambda_\epsilon \cdot (8\pi - \epsilon) \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} e^{v_\epsilon} &= -\lambda_\epsilon \cdot \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} \frac{8\pi - \epsilon}{4\pi} \cdot K_g \\
&+ \lambda_\epsilon \cdot \int_{\partial(B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon))} \frac{\partial v_\epsilon}{\partial \mathbf{n}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{I}_2 &\geq -\frac{16\pi - \epsilon}{4\pi} \cdot \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} K_g v_\epsilon + \int_{\partial(B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon))} v_\epsilon \frac{\partial v_\epsilon}{\partial \mathbf{n}} \\
&+ \int_{\partial B(x_\epsilon, \delta)} v_\epsilon^a \frac{\partial G}{\partial \mathbf{n}} - \int_{\partial B(x_\epsilon, r_\epsilon)} \left(v_\epsilon \frac{\partial G}{\partial \mathbf{n}} - G \frac{\partial v_\epsilon}{\partial \mathbf{n}} \right) \\
&- \lambda_\epsilon \cdot \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} \frac{8\pi - \epsilon}{4\pi} \cdot K_g + \lambda_\epsilon \cdot \int_{\partial(B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon))} \frac{\partial v_\epsilon}{\partial \mathbf{n}} \\
&+ o_\epsilon(1) + o_\delta(1) + o_R(1) \\
&= -\frac{16\pi - \epsilon}{4\pi} \cdot \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} K_g v_\epsilon - \lambda_\epsilon \cdot \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} \frac{8\pi - \epsilon}{4\pi} \cdot K_g \\
&+ o_\epsilon(1) + o_\delta(1) + o_R(1) \\
&- \int_{\partial B(x_\epsilon, r_\epsilon)} \frac{\partial v_\epsilon}{\partial \mathbf{n}} \cdot (v_\epsilon - (G - \lambda_\epsilon)) - \int_{\partial B(x_\epsilon, r_\epsilon)} v_\epsilon \frac{\partial G}{\partial \mathbf{n}} \\
&+ \int_{\partial B(x_\epsilon, \delta)} v_\epsilon \cdot \frac{\partial v_\epsilon}{\partial \mathbf{n}} + v_\epsilon^a \cdot \int_{\partial B(x_\epsilon, \delta)} \frac{\partial G}{\partial \mathbf{n}} + \lambda_\epsilon \cdot \int_{\partial B(x_\epsilon, \delta)} \frac{\partial v_\epsilon}{\partial \mathbf{n}}.
\end{aligned}$$

We now estimate the boundary term in the right hand side of the above inequality. From (10), (13) and Lemma 3.5, we know that

$$- \int_{\partial B(x_\epsilon, r_\epsilon)} \frac{\partial v_\epsilon}{\partial \mathbf{n}} \cdot (v_\epsilon - (G - \lambda_\epsilon)) \geq \frac{8\pi^2 R^2}{1 + \pi R^2} \cdot \left(-A(\bar{x}) - 2 \ln \frac{1 + \pi R^2}{R^2} \right) + o_\epsilon(1),$$

and

$$\begin{aligned}
& - \int_{\partial B(x_\epsilon, r_\epsilon)} v_\epsilon \frac{\partial G}{\partial \mathbf{n}} \\
&= -\lambda_\epsilon \int_{\partial B(x_\epsilon, r_\epsilon)} \frac{\partial G}{\partial \mathbf{n}} - 16\pi \ln(1 + \pi R^2) + o_\epsilon(1) + o_R(1) \\
&= \lambda_\epsilon \int_{M \setminus B(x_\epsilon, r_\epsilon)} \Delta G - 16\pi \ln(1 + \pi R^2) + o_\epsilon(1) + o_R(1)
\end{aligned}$$

$$= 2\lambda_\epsilon \int_{M \setminus B(x_\epsilon, r_\epsilon)} K_g - 16\pi \ln(1 + \pi R^2) + o_\epsilon(1) + o_R(1).$$

From Eq. (7) we have

$$\begin{aligned} \lambda_\epsilon \cdot \int_{\partial B(x_\epsilon, \delta)} \frac{\partial v_\epsilon}{\partial \mathbf{n}} &= -\lambda_\epsilon \cdot \int_{M \setminus B(x_\epsilon, \delta)} \Delta v_\epsilon \\ &= -\frac{8\pi - \epsilon}{4\pi} \lambda_\epsilon \int_{M \setminus B(x_\epsilon, \delta)} K_g + \lambda_\epsilon \int_{M \setminus B(x_\epsilon, \delta)} (8\pi - \epsilon) e^{v_\epsilon} \\ &\geq -\frac{8\pi - \epsilon}{4\pi} \lambda_\epsilon \int_{M \setminus B(x_\epsilon, \delta)} K_g. \end{aligned}$$

Similarly, we have

$$v_\epsilon^a \cdot \int_{\partial B(x_\epsilon, \delta)} \frac{\partial G}{\partial \mathbf{n}} = -2v_\epsilon^a \int_{M \setminus B(x_\epsilon, \delta)} K_g$$

and

$$\begin{aligned} v_\epsilon^a \cdot \int_{\partial B(x_\epsilon, \delta)} \frac{\partial v_\epsilon}{\partial \mathbf{n}} &= -\frac{8\pi - \epsilon}{4\pi} v_\epsilon^a \int_{M \setminus B(x_\epsilon, \delta)} K_g + v_\epsilon^a \int_{M \setminus B(x_\epsilon, \delta)} (8\pi - \epsilon) e^{v_\epsilon} \\ &= -\frac{8\pi - \epsilon}{4\pi} v_\epsilon^a \int_{M \setminus B(x_\epsilon, \delta)} K_g + v_\epsilon^a e^{v_\epsilon^a} \int_{M \setminus B(x_\epsilon, \delta)} (8\pi - \epsilon) e^{v_\epsilon - v_\epsilon^a} \\ &= -\frac{8\pi - \epsilon}{4\pi} v_\epsilon^a \int_{M \setminus B(x_\epsilon, \delta)} K_g + o_\epsilon(1). \end{aligned}$$

For the interior terms, we note that

$$\begin{aligned} \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} K_g v_\epsilon &= v_\epsilon^a \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} K_g + \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} K_g (v_\epsilon - v_\epsilon^a) \\ &= v_\epsilon^a \int_{B(x_\epsilon, \delta) \setminus B(x_\epsilon, r_\epsilon)} K_g + o_\delta(1), \end{aligned}$$

where we use the Poincaré inequality and (14).

We conclude

$$\begin{aligned} \mathbf{I}_2 &\geq -(16\pi - \epsilon) v_\epsilon^a + \frac{16\pi - \epsilon}{4\pi} \int_{B(x_\epsilon, r_\epsilon)} K_g \cdot v_\epsilon^a \\ &\quad - 8\pi A(\bar{x}) - 16\pi \ln \pi - 16\pi \ln(1 + \pi R^2) \\ &\quad + \epsilon \lambda_\epsilon \cdot \left(1 - \frac{1}{4\pi} \int_{B(x_\epsilon, r_\epsilon)} K_g \right) \\ &\quad + \int_{\partial B(x_\epsilon, \delta)} G \frac{\partial G}{\partial \mathbf{n}} \\ &\quad + o_\epsilon(1) + o_\delta(1) + o_R(1). \end{aligned} \tag{22}$$

We have, from (20)–(22),

$$\begin{aligned} I_\epsilon(v_\epsilon) &\geq -4\pi A(\bar{x}) - 8\pi \ln \pi - 8\pi \\ &\quad + \epsilon \lambda_\epsilon \cdot \left(\frac{1}{2} - \frac{1}{8\pi} \int_{B(x_\epsilon, r_\epsilon)} K_g \right) \\ &\quad - \frac{1}{2} \epsilon v_\epsilon^a + \frac{16\pi - \epsilon}{8\pi} \int_{B(x_\epsilon, r_\epsilon)} K_g \cdot v_\epsilon^a \\ &\quad - \int_{M \setminus B(x_\epsilon, \delta)} K_g G + \frac{8\pi - \epsilon}{4\pi} \int_M K_g \cdot (v_\epsilon - v_\epsilon^a) \\ &\quad + o_\epsilon(1) + o_\delta(1) + o_R(1). \end{aligned}$$

From (15), we know that $v_\epsilon^a \rightarrow -\infty$. Also, since $\lambda_\epsilon \cdot \text{vol}(B(x_\epsilon, r_\epsilon)) = o_\epsilon(1)$, we know from Lemma 3.5 that $v_\epsilon^a \cdot \text{vol}(B(x_\epsilon, r_\epsilon)) = o_\epsilon(1)$. We finally have

$$\lim_{\epsilon \rightarrow 0} E_\epsilon \geq -4\pi A(\bar{x}) - 8\pi \ln \pi - 8\pi + C$$

for some constant C . Thus Theorem 1.1 is proved. $\square$

We are left to prove Lemma 3.4. We need the following lemma due to Brezis and Merle [5].

Lemma 3.6 *Let $\Omega \subset M$ be a bounded domain and u be a solution to the following equation:*

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $f(x) \in L^1(\Omega)$, then for any $0 < \delta < 4\pi$, there is a constant $C(\delta)$ such that

$$\int_\Omega \exp \left\{ \frac{(4\pi - \delta)|u(x)|}{\|f\|_{L^1(\Omega)}} \right\} \leq C(\delta).$$

Lemma 3.6 was originally proved for a bounded domain in R^2 , but it is easy to see that the same result holds on any domain in a compact Riemann surface. We now return to the

Proof of Lemma 3.4 Let $K \subset\subset M \setminus \{\bar{x}\}$. We choose other two compact sets K_1 and K_2 in $M \setminus \{\bar{x}\}$ such that $K \subset\subset K_1 \subset\subset K_2 \subset\subset M \setminus \{\bar{x}\}$. It follows from (10) that

$$\lim_{\epsilon \rightarrow 0} \int_{K_2} e^{v_\epsilon} = 0. \quad (23)$$

Let v_ϵ^1 be the unique solution to

$$\begin{cases} -\Delta v_\epsilon^1 = (8\pi - \epsilon)e^{v_\epsilon} & \text{on } K_2, \\ v_\epsilon^1 = 0 & \text{on } \partial K_2, \end{cases} \quad (24)$$

and $v_\epsilon^2 = v_\epsilon - v_\epsilon^1 - v_\epsilon^a$. From (23) and Lemma 3.6 we know that $e^{|v_\epsilon^1|} \in L^p(K_2)$ for some $1 < p < 2$; thus $\|v_\epsilon^1\|_{L^p(K_2)} \leq C$. On the other hand, we note v_ϵ^2 satisfying $\Delta v_\epsilon^2 = \frac{8\pi-\epsilon}{4\pi} K_g$ in K_2 , we have, from the interior L^p estimates, for $K_1 \subset\subset K_2$,

$$\begin{aligned} \|v_\epsilon^2\|_{L^\infty(K_1)} &\leq C \|v_\epsilon^2\|_{L^p(K_2)} \\ &\leq C (\|v_\epsilon - v_\epsilon^a\|_{L^p(K_2)} + \|v_\epsilon^1\|_{L^p(K_2)}) \\ &\leq C (\|\nabla v_\epsilon\|_{L^p(K_2)} + \|v_\epsilon^1\|_{L^p(K_2)}) \\ &\leq C, \end{aligned}$$

where we also use (14). In turn, we know that $\int_{K_1} e^{pv_\epsilon^1} \cdot e^{pv_\epsilon^2} \leq C$. Combining with (15) we have

$$\begin{aligned} \int_{K_1} e^{pv_\epsilon} &\leq \int_{K_1} e^{pv_\epsilon^a} \cdot e^{pv_\epsilon^1} \cdot e^{pv_\epsilon^2} \\ &\leq C, \end{aligned}$$

we then know from (24) that $\|v_\epsilon^1\|_{L^\infty(K)} \leq C$, thus $\|v_\epsilon - v_\epsilon^a\|_{L^\infty(K)} \leq C$. $\square$

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