Non-uniform Berry–Esseen Bounds for Weighted U-Statistics and Generalized L-Statistics

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Abstract Weighted U-statistics and generalized L-statistics are commonly used in statistical inference and their asymptotic properties have been well developed. In this paper sharp non-uniform Berry–Esseen bounds for weighted U-statistics and generalized L-statistic are established.

Keywords Weighted U-statistics · Generalized L-statistic · L-statistic · U-statistic · Non-uniform Berry–Esseen bound · Normal approximation

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1 Introduction and Main Results

Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of independent and identically distributed (i.i.d.) real-valued random variables and let $h(x_1, \dots, x_m)$ be a kernel of degree m , that is, h is a real-valued measurable symmetric function in m arguments. The main purpose of this paper is to establish non-uniform Berry–Esseen bounds for weighted U-statistics and generalized L-statistics.

1.1 Weighted U-Statistics

Let $a : N^m \rightarrow R$ be a symmetric function. A weighted U-statistic is given by

$$U_n = \sum_{1 \leq i_1 < \dots < i_m \leq n} a(i_1, \dots, i_m) h(X_{i_1}, \dots, X_{i_m}). \quad (1.1)$$

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Clearly, it reduces to the Hoeffding U-statistics when $a \equiv 1$. The weighted U-statistic was first introduced by Shapiro and Hubert [10] for $m = 2$ and its asymptotic normality and non-normality were discussed in O'Neil and Redner [8]. The results of O'Neil and Redner [8] were extended by Major [7] for general m . We refer to Rifi and Utzet [9] for the asymptotic behavior of the weighted U-statistics.

To present a non-uniform Berry–Esseen bound for the weighted U-statistic, assume

$$Eh(X_1, \dots, X_m) = 0, \quad Eh^2(X_1, \dots, X_m) < \infty. \quad (1.2)$$

Put

$$\sigma_0^2 = Eh^2(X_1, \dots, X_m) < \infty, \quad g_1(x) = E[h(X_1, \dots, X_m) | X_1 = x], \quad (1.3)$$

$$\sigma_1^2 = Eg_1^2(X_1),$$

$$A_l = \sum_{\substack{1 \leq i_1 < \dots < i_{m-1} \leq n, \\ i_j \neq l, j \in \{1, \dots, m-1\}}} a(i_1, \dots, i_{m-1}, l), \quad (1.4)$$

$$B_k = \left| \sum_{(i_1, \dots, i_m), (j_1, \dots, j_m) \in \mathcal{H}_k} a(i_1, \dots, i_m) a(j_1, \dots, j_m) \right|, \quad (1.5)$$

and

$$C_{k,l} = \left| \sum_{(i_1, \dots, i_{m-1}), (j_1, \dots, j_{m-1}) \in \mathcal{H}_{k,l}} a(i_1, \dots, i_{m-1}, l) a(j_1, \dots, j_{m-1}, l) \right|, \quad (1.6)$$

where $\mathcal{H}_k$ denotes the set of $1 \leq i_1 < \dots < i_m \leq n, 1 \leq j_1 < \dots < j_m \leq n$ with k common elements between the two sets $\{i_1, \dots, i_m\}$ and $\{j_1, \dots, j_m\}$, and $\mathcal{H}_{k,l}$ denotes the set of $1 \leq i_1 < \dots < i_{m-1} \leq n, 1 \leq j_1 < \dots < j_{m-1} \leq n, i_s \neq l, j_s \neq l$ for each $1 \leq s \leq m-1$ with k common elements between the two sets $\{i_1, \dots, i_{m-1}\}$ and $\{j_1, \dots, j_{m-1}\}$.

The following theorem gives a non-uniform Berry–Esseen bound for U_n .

Theorem 1.1 *If $Eh(X_1, \dots, X_m) = 0$ and $E|g_1(X_1)|^p < \infty$ for $2 < p \leq 3$, then there exists a universal constant C such that*

$$\begin{aligned} & \left| P\left(\frac{U_n}{(\sum_{j=1}^n A_j^2)^{1/2} \sigma_1} \leq z\right) - \Phi(z) \right| \\ & \leq \frac{9\sigma_0^2 \sum_{k=2}^m k B_k}{m(|z| + 1)^2 \sigma_1^2 \sum_{j=1}^n A_j^2} + \frac{C}{(|z| + 1)^p} \left(\frac{\sigma_0 (\sum_{k=2}^m k B_k)^{1/2}}{\sigma_1 (m \sum_{j=1}^n A_j^2)^{1/2}} \right. \\ & \quad \left. + \frac{\sigma_0 (\sum_{l=1}^n |A_l| (\sum_{k=2}^m k C_{k-1,l})^{1/2})}{\sigma_1 m^{1/2} \sum_{j=1}^n A_j^2} + \frac{E|g(X_1)|^p \sum_{j=1}^n |A_j|^p}{\sigma_1^p (\sum_{j=1}^n A_j^2)^{p/2}} \right). \quad (1.7) \end{aligned}$$

Remark 1.1 It is easy to see that when $a \equiv 1$,

$$\begin{aligned} A_l &= \frac{m}{n} \binom{n}{m}, \quad B_k = \binom{n}{m} \binom{m}{k} \binom{n-m}{m-k}, \\ C_{k,l} &= \binom{n-1}{m-1} \binom{m-1}{k} \binom{n-m}{m-1-k}. \end{aligned}$$

Result (1.7) recovers Theorem 3.1 of Chen and Shao [4].

Remark 1.2 For $m = 2$ letting $a(l, l) = 0$, we have

$$A_l = \sum_{j=1}^n a(j, l), \quad B_2 = \sum_{1 \leq i < j \leq n} a^2(i, j)$$

and

$$C_{1,l} = \sum_{j=1}^n a^2(j, l).$$

So the non-uniform Berry–Esseen bound (1.7) is reduced to

$$\frac{9\sigma_0^2 B_2}{(|z|+1)^2 \sigma_1^2 \sum_{j=1}^n A_j^2} + \frac{C}{(|z|+1)^p} \left(\frac{\sigma_0 B_2^{1/2}}{\sigma_1 (\sum_{j=1}^n A_j^2)^{1/2}} + \frac{E|g(X_1)|^p \sum_{j=1}^n |A_j|^p}{\sigma_1^p (\sum_{j=1}^n A_j^2)^{p/2}} \right).$$

In particular, if

$$\left(B_2 + \max_{1 \leq i \leq n} A_i^2 \right) / \sum_{j=1}^n A_j^2 \rightarrow 0,$$

then $U_n / (\sum_{j=1}^n A_j^2)^{1/2} \sigma_1$ is asymptotically normal.

1.2 Generalized L-Statistics

Let $R_{n,1} \leq \dots \leq R_{n,C_{n,m}}$ be the order values of $h(X_{i_1}, \dots, X_{i_m})$ taken over m -tuples $\{(i_1, \dots, i_m) : 1 \leq i_1 < \dots < i_m \leq n\}$, where $C_{n,m} = \binom{n}{m}$. A generalized L-statistic is defined by

$$T_n = \sum_{i=1}^{C_{n,m}} c_{n,i} R_{n,i}, \quad (1.8)$$

where

$$c_{n,i} = \int_{(i-1)/C_{n,m}}^{i/C_{n,m}} J(t) dt \quad (1.9)$$

and J is a weight function on $[0, 1]$. The generalized L-statistic was first introduced by Serfling [12]. It includes Hodges–Lehmann location estimator, U -statistics, trimmed U -statistics, Winsorized U -statistics as special cases. The asymptotic properties of the generalized L-statistics have been extensively studied. We refer to Serfling [12, 13] Borovshikh and Weber [1] for the asymptotic normality and for the general theory and applications, Cai [2] for the moderate and large deviations, and Helmers and Ruymgaart [6], Helmers, Janssen and Serfling [5] for the uniform Berry–Esseen bounds.

Let $H(y)$ be the distribution function of the random variable function $h(X_1, \dots, X_m)$ and denote the empirical distribution function by

$$H_n(y) = \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \dots < i_m \leq n} I(h(X_{i_1}, \dots, X_{i_m}) \leq y), \quad y \in \mathbb{R}. \quad (1.10)$$

Let T be an L-functional

$$T(G) = \int_0^1 J(t)G^{-1}(t) dt$$

with $G^{-1}(t) = \inf\{y : G(y) \geq t\}$ for any distribution function G . Clearly,

$$T_n = T(H_n)$$

and the generalized L-statistic T_n is actually estimating the parameter $T(H)$.

Let

$$A(x_1, \dots, x_m) = \int_{-\infty}^{+\infty} (I(h(x_1, \dots, x_m) \leq y) - H(y))J(H(y)) dy, \quad (1.11)$$

$$g_2(x) = E(A(X_1, \dots, X_m) | X_1 = x). \quad (1.12)$$

Put

$$\sigma_0^2 = Eh^2(X_1, \dots, X_m), \quad \sigma_2^2 = \text{Var}(g_2(X_1)). \quad (1.13)$$

Then we have the following non-uniform Berry–Esseen bound for T_n .

Theorem 1.2 Assume that $1 \leq m \leq n/5$ and there exists a constant c_0 such that

$$|J(t) - J(s)| \leq c_0|t - s| \quad \text{for } 0 \leq s, t \leq 1. \quad (1.14)$$

If $Eh(X_1, \dots, X_m) = 0$ and $E|g_2(X_1)|^p < \infty$ for $2 < p \leq 3$, then there exists a universal constant C such that

$$\begin{aligned} & \left| P\left(\frac{\sqrt{n}}{m\sigma_2}(T(H_n) - T(H)) \leq z\right) - \Phi(z) \right| \\ & \leq \frac{36m\sigma_0^2((|J(0)| + c_0)^2 + 6mc_0^2)}{n\sigma_2^2(|z| + 1)^2} \\ & \quad + \frac{C}{(|z| + 1)^p} \left(\frac{(|J(0)| + c_0)m\sigma_0}{n^{1/2}\sigma_2} + \frac{E|g_2(X_1)|^p}{n^{(p-2)/2}\sigma_2^p} \right). \end{aligned} \quad (1.15)$$

Theorem 1.2 recovers the uniform Berry–Esseen bound of Helmers, Janssen and Serfling [5] with a smaller order in terms of m .

2 Proofs

Our proof is based on the following non-uniform Berry–Esseen inequality of Chen and Shao [4]:

Proposition 2.1 Let $\xi_1, \dots, \xi_n$ be independent random variables satisfying $E\xi_i = 0$ for $i = 1, \dots, n$, and $\sum_{i=1}^n E(\xi_i^2) = 1$. Let $W = \sum_{i=1}^n \xi_i$ and Δ be a measurable function of $\{\xi_i, 1 \leq i \leq n\}$. If $E|\xi_i|^p < \infty$ for some $2 < p \leq 3$, then there exists an absolute constant C such that

$$\begin{aligned}
& |P(W + \Delta \leq z) - \Phi(z)| \\
& \leq \frac{9\|\Delta\|_2^2}{(1+|z|)^2} + \frac{C}{(|z|+1)^p} \left(\|\Delta\|_2 + \sum_{i=1}^n \|\xi_i\|_2 \|\Delta - \Delta_i\|_2 + \sum_{i=1}^n E|\xi_i|^p \right), \quad (2.1)
\end{aligned}$$

where Δ_i is any random variable such that ξ_i and $(W - \xi_i, \Delta_i)$ are independent, and $\|\xi\|_2$ denotes $(E\xi^2)^{1/2}$.

2.1 Proof of Theorem 1.1

To apply Proposition 2.1, the first step is to rewrite

$$\frac{1}{(\sum_{j=1}^n A_j^2)^{1/2} \sigma_1} U_n = W + \Delta,$$

where W is a sum of independent random variables and Δ is a negligible term. For $1 \leq k \leq m$, let

$$\begin{aligned}
h_k(x_1, \dots, x_k) &= E(h(X_1, \dots, X_m) | X_1 = x_1, \dots, X_k = x_k), \\
\bar{h}_k(x_1, \dots, x_k) &= h_k(x_1, \dots, x_k) - \sum_{i=1}^k g_1(x_i), \\
\bar{h}(x_1, \dots, x_m) &= \bar{h}_m(x_1, \dots, x_m) = h(x_1, \dots, x_m) - \sum_{j=1}^m g_1(x_j).
\end{aligned}$$

Then we can take

$$W = \sum_{i=1}^n \xi_i, \quad \xi_i = \frac{A_j}{(\sum_{j=1}^n A_j^2)^{1/2} \sigma_1} g_1(X_j),$$

and

$$\Delta = \frac{1}{(\sum_{j=1}^n A_j^2)^{1/2} \sigma_1} \sum_{1 \leq i_1 < \dots < i_m \leq n} a(i_1, \dots, i_m) \bar{h}(X_{i_1}, \dots, X_{i_m}).$$

Clearly, $\xi_1, \dots, \xi_n$ are independent random variable with $E\xi_i = 0$ and $\sum_{i=1}^n E(\xi_i^2) = 1$. Set

$$\Delta_l = \frac{1}{(\sum_{j=1}^n A_j^2)^{1/2} \sigma_1} \sum_{\substack{1 \leq i_1 < \dots < i_m \leq n, \\ i_j \neq l, j \in \{1, \dots, m\}}} a(i_1, \dots, i_m) \bar{h}(X_{i_1}, \dots, X_{i_m}).$$

Now by Proposition 2.1, Theorem 1.1 follows by the next two lemmas.

Lemma 2.1 *We have*

$$E\Delta^2 \leq \frac{\sigma_0^2 \sum_{k=2}^m k B_k}{m \sigma_1^2 \sum_{j=1}^n A_j^2}. \quad (2.2)$$

Lemma 2.2 *We have*

$$E(\Delta - \Delta_l)^2 \leq \frac{\sigma_0^2 \sum_{k=2}^m k C_{k-1,l}}{m \sigma_1^2 \sum_{j=1}^n A_j^2}. \quad (2.3)$$

Proof of Lemma 2.1 Observe that

$$\begin{aligned} E\Delta^2 &= \frac{1}{\sum_{j=1}^n A_j^2 \sigma_1^2} E \left(\sum_{1 \leq i_1 < \dots < i_m \leq n} a(i_1, \dots, i_m) \bar{h}(X_{i_1}, \dots, X_{i_m}) \right)^2 \\ &= \frac{1}{\sum_{j=1}^n A_j^2 \sigma_1^2} \sum_{k=2}^m \left(\sum_{((i_1, \dots, i_m), (j_1, \dots, j_m)) \in \mathcal{H}_k} a(i_1, \dots, i_m) a(j_1, \dots, j_m) \right) \\ &\quad \times E \bar{h}_k^2(X_1, \dots, X_k) \\ &= \frac{1}{\sum_{j=1}^n A_j^2 \sigma_1^2} \sum_{k=2}^m B_k E \bar{h}_k^2(X_1, \dots, X_k). \end{aligned}$$

By Eq. (10.80) in [3], we have

$$E \bar{h}_k^2(X_1, \dots, X_k) \leq \frac{k}{m} \sigma_0^2$$

and hence (2.2) holds. $\square$

Proof of Lemma 2.2 Similarly, we have

$$\begin{aligned} \sigma_1^2 \left(\sum_{j=1}^n A_j^2 \right) E(\Delta - \Delta_l)^2 &= E \left(\left\{ \sum_{1 \leq i_1 < \dots < i_m \leq n} - \sum_{\substack{1 \leq i_1 < \dots < i_m \leq n, \\ i_j \neq l, j \in \{1, \dots, m\}}} \right\} a(i_1, \dots, i_m) \bar{h}(X_{i_1}, \dots, X_{i_m}) \right)^2 \\ &= \sum_{k=1}^{m-1} \left(\sum_{((i_1, \dots, i_{m-1}), (j_1, \dots, j_{m-1})) \in \mathcal{H}_{k,l}} a(i_1, \dots, i_{m-1}, l) a(j_1, \dots, j_{m-1}, l) \right) \\ &\quad \times E \bar{h}_{k+1}^2(X_1, \dots, X_k, X_n) \\ &= \sum_{k=1}^{m-1} C_{k,l} E \bar{h}_{k+1}^2(X_1, \dots, X_k, X_n). \end{aligned}$$

By Eq. (10.80) in [3] again, we have

$$E \bar{h}_{k+1}^2(X_1, \dots, X_k, X_n) \leq \frac{k+1}{m} \sigma_0^2$$

and hence (2.3) holds. $\square$

2.2 Proof of Theorem 1.2

As in the proof of Theorem 1.1, the first step is to rewrite $\sqrt{n}(T(H_n) - T(H))/(m\sigma_2)$ as $W + \Delta$, where W is a sum of independent random variables and Δ is negligible. To this end, let $A(x_1, \dots, x_m)$ and $g_2(x)$ be defined as in (1.11) and (1.12) and put

$$\bar{A}_m(x_1, \dots, x_m) = A(x_1, \dots, x_m) - \sum_{j=1}^m g_2(x_j). \quad (2.4)$$

Let $\varphi(t) = \int_0^t J(s) ds$. Observing that by the integration by parts (see, e.g., [11, p. 265])

$$T(H_n) - T(H) = - \int_{-\infty}^{+\infty} (\varphi(H_n(y)) - \varphi(H(y))) dy,$$

we have

$$\frac{\sqrt{n}}{m\sigma_2} (T(H_n) - T(H)) = W + \Delta, \quad (2.5)$$

where

$$W = \sum_{i=1}^n \xi_i, \quad \xi_i = -g_2(X_i)/(\sqrt{n}\sigma_2),$$

and

$$\begin{aligned} \Delta = & - \binom{n}{m}^{-1} \frac{\sqrt{n}}{m\sigma_2} \sum_{1 \leq i_1 < \dots < i_m \leq n} \bar{A}_m(x_{i_1}, \dots, x_{i_m}) \\ & - \frac{\sqrt{n}}{m\sigma_2} \int_{-\infty}^{+\infty} [\varphi(H_n(y)) - \varphi(H(y)) - (H_n(y) - H(y))J(H(y))] dy. \end{aligned} \quad (2.6)$$

Clearly, $\xi_1, \dots, \xi_n$ are i.i.d. random variable with $E\xi_i = 0$ and $\sum_{i=1}^n E(\xi_i^2) = 1$. Set

$$\begin{aligned} \Delta_l = & - \binom{n}{m}^{-1} \frac{\sqrt{n}}{m\sigma_2} \sum_{\substack{1 \leq i_1 < \dots < i_m \leq n, \\ i_j \neq l, j \in \{1, \dots, m\}}} \bar{A}_m(x_{i_1}, \dots, x_{i_m}) \\ & - \frac{\sqrt{n}}{m\sigma_2} \int_{-\infty}^{+\infty} [\varphi(H_{n,l}(y)) - \varphi(H(y)) - (H_{n,l}(y) - H(y))J(H(y))] dy, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} H_{n,l}(y) = & \binom{n}{m}^{-1} \left(\sum_{\substack{1 \leq i_1 < \dots < i_m \leq n, \\ i_j \neq l, j \in \{1, \dots, m\}}} \left(I[h(X_{i_1}, \dots, X_{i_m}) \leq y] + \binom{n-1}{m-1} H(y) \right) \right). \end{aligned} \quad (2.9)$$

Now by Proposition 2.1, Theorem 1.2 follows by the next two lemmas.

Lemma 2.3 *We have*

$$E\Delta^2 \leq \frac{4(|J(0)| + c_0)^2 + 6mc_0^2)m\sigma_0^2}{n\sigma_2^2}. \quad (2.10)$$

Lemma 2.4 *We have*

$$E(\Delta - \Delta_I)^2 \leq \frac{16m^2(|J(0)| + 8c_0)^2\sigma_0^2}{n^2\sigma_2^2}. \quad (2.11)$$

The proofs of the two lemmas are presented in the next two subsections.

2.2.1 Proof of Lemma 2.3

Let $\sigma^2 = EA^2(X_1, \dots, X_m)$. We first show that

$$\sigma^2 \leq (|J(0)| + c_0)^2\sigma_0^2. \quad (2.12)$$

Let $\xi = h(X_1, \dots, X_m)$ and η be an independent copy of ξ . Then

$$\begin{aligned} \sigma^2 &= E \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (I(\xi \leq x) - H(x))J(H(x))(I(\xi \leq y) - H(y))J(H(y)) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (H(x \wedge y) - H(x)H(y))J(H(x))J(H(y)) dx dy \\ &\leq (|J(0)| + c_0)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (H(x \wedge y) - H(x)H(y)) dx dy \end{aligned}$$

by the fact that (1.14) implies $|J(t)| \leq (|J(0)| + c_0)$ for $0 \leq t \leq 1$. Also note that

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (H(x \wedge y) - H(x)H(y)) dx dy \\ &= 2 \int_{-\infty}^{\infty} \int_{x \leq y} H(x)(1 - H(y)) dx dy \\ &= 2E \int_{-\infty}^{\infty} \int_{x \leq y} I(\xi \leq x)(1 - I(\eta \leq y)) dx dy \\ &= 2E \int_{\xi \leq x < y < \eta} I(\xi \leq x)I(y < \eta) dx dy \\ &= E(\eta - \xi)^2 I(\xi < \eta) = (1/2)E(\xi - \eta)^2 \\ &\leq E\xi^2 = \sigma_0^2. \end{aligned} \quad (2.13)$$

This proves (2.12).

Let

$$\begin{aligned} h_1(x_1, \dots, x_{2m}) &= \int_{-\infty}^{\infty} \{I(h(x_1, \dots, x_m) \leq y) - H(y)\} \\ &\quad \times \{I(h(x_{m+1}, \dots, x_{2m}) \leq y) - H(y)\} dy \end{aligned} \quad (2.14)$$

and $\sigma_3^2 = Eh_1^2(X_1, \dots, X_{2m})$. Then

$$\sigma_3^2 \leq \sigma_0^2. \quad (2.15)$$

In fact, by the Hölder inequality

$$\begin{aligned} \sigma_3^2 &\leq E \left(\int_{-\infty}^{\infty} (I(h(X_1, \dots, X_m) \leq y) - H(y))^2 dy \right. \\ &\quad \times \left. \int_{-\infty}^{\infty} (I(h(X_{m+1}, \dots, X_{2m}) \leq y) - H(y))^2 dy \right) \\ &= \left(\int_{-\infty}^{\infty} H(y)(1 - H(y)) dy \right)^2 \\ &\leq (E|h(X_1, \dots, X_m)|)^2 \leq Eh^2(X_1, \dots, X_m) = \sigma_0^2, \end{aligned}$$

as desired.

Next we obtain a similar result for $Eh_1^2(X_1, \dots, X_m, Y_1, \dots, Y_m)$, where $(Y_1, \dots, Y_m)$ has the same distribution as that of $(X_1, \dots, X_m)$. Let $\xi = h(X_1, \dots, X_m)$ and $\eta = h(Y_1, \dots, Y_m)$. Then

$$\begin{aligned} Eh_1^2(X_1, \dots, X_m, Y_1, \dots, Y_m) &= E \left(\int_{-\infty}^{\infty} (I(\xi \leq x) - H(x))(I(\eta \leq x) - H(x)) dx \right)^2 \\ &\leq 0.25E \left(\int_{-\infty}^{\infty} (I(\xi \leq x) - H(x))^2 + (I(\eta \leq x) - H(x))^2 dx \right)^2 \\ &\leq E \left(\int_{-\infty}^{\infty} (I(\xi \leq x) - H(x))^2 dx \right)^2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(I(\xi \leq x) - H(x))^2 (I(\xi \leq y) - H(y))^2 dx dy. \end{aligned}$$

A direct calculation shows that for $x \leq y$

$$\begin{aligned} &E(I(\xi \leq x) - H(x))^2 (I(\xi \leq y) - H(y))^2 \\ &= H(x)(1 - 2H(x))(1 - 2H(y)) + H(x)H^2(y)(1 - 2H(x)) \\ &\quad + H^2(x)H(y)(1 - 2H(y)) + H^2(x)H^2(y) \\ &= H(x)(1 - 2H(x) - 2H(y) + 5H(x)H(y) - 3H(x)H^2(y) + H^2(y)) \\ &\leq H(x)(1 - H(y) + 3H(x)H(y) - 3H(x)H^2(y)) \\ &= H(x)(1 - H(y))(1 + 3H(x)H(y)) \\ &\leq 4H(x)(1 - H(y)). \end{aligned}$$

Thus by (2.13), we have

$$Eh_1^2(X_1, \dots, X_m, Y_1, \dots, Y_m) \leq 4E\xi^2 = 4\sigma_0^2. \quad (2.16)$$

To estimate $E\Delta^2$, write

$$\Delta = A + B,$$

where

$$A = -\binom{n}{m}^{-1} \frac{\sqrt{n}}{m\sigma_2} \sum_{1 \leq i_1 < \dots < i_m \leq n} \bar{A}_m(x_{i_1}, \dots, x_{i_m}),$$

$$B = -\frac{\sqrt{n}}{m\sigma_2} \int_{-\infty}^{+\infty} [\varphi(H_n(y)) - \varphi(H(y)) - (H_n(y) - H(y))J(H(y))] dy.$$

Clearly,

$$E\Delta^2 \leq 2E(A^2 + B^2). \quad (2.17)$$

Following the proof in [3, pp. 284–286], we have

$$\begin{aligned} & E \left(\sum_{1 \leq i_1 < \dots < i_m \leq n} \bar{A}_m(x_{i_1}, \dots, x_{i_m}) \right)^2 \\ &= \binom{n}{m} \sum_{j=2}^m \binom{m}{j} \binom{n-m}{m-j} E \bar{A}_j^2(X_1, \dots, X_n) \\ &\leq \frac{m(m-1)^2}{n(n-m+1)} \binom{n}{m}^2 \sigma^2 \end{aligned}$$

and hence

$$\begin{aligned} EA^2 &\leq \left(\binom{n}{m}^{-1} \frac{\sqrt{n}}{m\sigma_2} \right)^2 \frac{m(m-1)^2}{n(n-m+1)} \binom{n}{m}^2 \sigma^2 \\ &\leq \frac{(m-1)^2 \sigma^2}{m(n-m+1) \sigma_2^2} \\ &\leq \frac{2m(|J(0)| + c_0)^2 \sigma_0^2}{n \sigma_2^2}. \end{aligned} \quad (2.18)$$

To finish the proof of (2.10) by (2.17) and (2.18), it suffices to show that

$$EB^2 \leq \frac{12m^2 c_0^2 \sigma_0^2}{n \sigma_2^2}. \quad (2.19)$$

Recall that

$$EB^2 = \frac{n}{m^2 \sigma_2^2} E \left(\int_{-\infty}^{+\infty} [\varphi(H_n(y)) - \varphi(H(y)) - (H_n(y) - H(y))J(H(y))] dy \right)^2. \quad (2.20)$$

By the definition of $\varphi(t)$ and the Lipschitz condition (1.14)

$$|\varphi(t) - \varphi(s) - (t-s)J(s)| = \left| \int_0^{t-s} (J(u+s) - J(s)) du \right| \leq 0.5c_0(t-s)^2,$$

we have

$$\begin{aligned}
& E \left(\int_{-\infty}^{+\infty} [\varphi(H_n(y)) - \varphi(H(y)) - (H_n(y) - H(y))J(H(y))] dy \right)^2 \\
& \leq 0.25c_0^2 E \left(\int_{-\infty}^{+\infty} (H_n(y) - H(y))^2 dy \right)^2.
\end{aligned} \tag{2.21}$$

Write

$$\int_{-\infty}^{+\infty} (H_n(y) - H(y))^2 dy = Q_n + R_n,$$

where

$$\begin{aligned}
Q_n &= \binom{n}{m}^{-2} \sum_1^* h_1(X_{i_1}, \dots, X_{i_m}, X_{j_1}, \dots, X_{j_m}) \\
R_n &= \binom{n}{m}^{-2} \sum_1^{**} h_1(X_{i_1}, \dots, X_{i_m}, X_{j_1}, \dots, X_{j_m}),
\end{aligned}$$

here $\sum_1^*$ and $\sum_1^{**}$ denote the sum over all pairs of m -tuples $(i_1, \dots, i_m)$, $(j_1, \dots, j_m) \in C_{n,m}$ having all indices different and at least one ordered index equal, respectively, and h_1 is defined in (2.14). We further write

$$Q_n = \binom{n}{2m}^{-1} \sum_{C_{n,2m}} h_{1n}(X_{k_1}, \dots, X_{k_{2m}}),$$

where

$$h_{1n}(X_{k_1}, \dots, X_{k_{2m}}) = \binom{n}{2m} \binom{n}{m}^{-2} \sum^+ h_1(X_{i_1}, \dots, X_{i_m}, X_{j_1}, \dots, X_{j_m})$$

and $\sum^+$ denotes the sum over all $\binom{2m}{m}$ summands $h_1(X_{i_1}, \dots, X_{i_m}, X_{j_1}, \dots, X_{j_m})$ with $\{i_1, \dots, i_m, j_1, \dots, j_m\} = \{k_1, \dots, k_{2m}\}$. Observe that

$$\begin{aligned}
& E[E(h_{1n}(X_1, \dots, X_{2m})|X_1, \dots, X_k)]^2 \\
&= E \left(\left[E \left(\binom{n}{2m} \binom{n}{m}^{-2} \sum^+ h_1(X_{i_1}, \dots, X_{i_m}, X_{j_1}, \dots, X_{j_m}) | X_k \right) \right]^2 \right) \\
&= \binom{n}{2m}^2 \binom{n}{m}^{-4} \binom{2m}{m}^2 E[E(h_1(X_1, \dots, X_{2m})|X_1, \dots, X_k)]^2 \\
&\leq E[E(h_1(X_1, \dots, X_{2m})|X_1, \dots, X_k)]^2.
\end{aligned}$$

By Eq. (10.80) in [3], we have for $2 \leq k \leq 2m$,

$$E[E(h_1(X_1, \dots, X_{2m})|X_1, \dots, X_k)]^2 \leq \frac{k}{2m} \sigma_3^2$$

and by some simple calculation, we have

$$E[E(h_1(X_1, \dots, X_{2m})|X_1)] = 0,$$

hence

$$\begin{aligned}
EQ_n^2 &= \binom{n}{2m}^{-1} \sum_{k=1}^{2m} \binom{2m}{k} \binom{n-2m}{2m-k} E[E(h_{1n}(X_1, \dots, X_{2m})|X_1, \dots, X_k)]^2 \\
&\leq \binom{n}{2m}^{-1} \sum_{k=2}^{2m} \binom{2m}{k} \binom{n-2m}{2m-k} \frac{k}{2m} \sigma_3^2 \\
&\leq \frac{2m(2m-1)^2}{n(n-2m+1)} \sigma_3^2 \\
&\leq \frac{8m^3 \sigma_0^2}{n^2}.
\end{aligned} \tag{2.22}$$

Next we give an upper bound of ER_n^2 . Notice that the number of terms in $\sum_1^{**}$ equals

$$\begin{aligned}
\text{Card}(\sum_1^{**}) &= \binom{n}{m}^2 - \binom{n}{m} \binom{n-m}{m} \\
&= \binom{n}{m}^2 \left(1 - \left(1 - \frac{m}{n}\right) \left(1 - \frac{m}{n-1}\right) \cdots \left(1 - \frac{m}{n-m+1}\right)\right) \\
&\leq \binom{n}{m}^2 m \left(\frac{1}{n} + \cdots + \frac{1}{n-m+1}\right) \\
&\leq \binom{n}{m}^2 \frac{m^2}{(n-m+1)} \leq \frac{2m^2}{n} \binom{n}{m}^2.
\end{aligned}$$

Hence, by (2.16) and the Minkowski inequality

$$ER_n^2 \leq \frac{16m^4 \sigma_0^2}{n^2}. \tag{2.23}$$

Thus, by (2.22) and (2.23)

$$\begin{aligned}
EB^2 &\leq \frac{0.5nc_0^2}{m^2\sigma_2^2} E(Q_n^2 + R_n^2) \\
&\leq \frac{0.5nc_0^2}{m^2\sigma_2^2} \left(\frac{8m^3\sigma_0^2}{n^2} + \frac{16m^4\sigma_0^2}{n^2}\right) \\
&\leq \frac{12m^2c_0^2\sigma_0^2}{n\sigma_2^2}.
\end{aligned}$$

This proves (2.19). □

2.2.2 Proof of Lemma 2.4

Let $\Delta_l = C + D$, where

$$\begin{aligned}
C &= -\binom{n}{m}^{-1} \frac{\sqrt{n}}{m\sigma_2} \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq n, \\ i_j \neq l, j \in \{1, \dots, m\}}} \bar{A}_m(x_{i_1}, \dots, x_{i_m}), \\
D &= -\frac{\sqrt{n}}{m\sigma_2} \int_{-\infty}^{+\infty} [\varphi(H_{n,l}(y)) - \varphi(H(y)) - (H_{n,l}(y) - H(y))J(H(y))] dy.
\end{aligned}$$

Clearly, we have

$$E(\Delta - \Delta_l)^2 \leq 2E(A - C)^2 + 2E(B - D)^2.$$

Note that

$$E(A - C)^2 = \left(-\binom{n}{m}^{-1} \frac{\sqrt{n}}{m\sigma_2} \right)^2 E \left(\left(\sum_{1 \leq i_1 < \dots < i_m \leq n} - \sum_{\substack{1 \leq i_1 < \dots < i_m \leq n, \\ i_j \neq l, j \in \{1, \dots, m\}}} \right) \times \bar{A}_m(x_{i_1}, \dots, x_{i_m}) \right)^2.$$

Following the proof in [3, p. 287] and similarly to (2.18) we have

$$\begin{aligned} E(A - C)^2 &\leq \frac{2(m-1)^2\sigma^2}{nm(n-m+1)\sigma_2^2} \\ &\leq \frac{2(m-1)^2(|J(0)| + c_0)^2\sigma_0^2}{nm(n-m+1)\sigma_2^2} \\ &\leq \frac{4m(|J(0)| + c_0)^2\sigma_0^2}{n^2\sigma_2^2}. \end{aligned}$$

To complete the proof of (2.11), it suffices to show that

$$E(B - D)^2 \leq \frac{30m^2c_0^2\sigma_0^2}{n^2\sigma_2^2}. \quad (2.24)$$

Let

$$\begin{aligned} L &= \int_{-\infty}^{+\infty} [\varphi(H_n(y)) - \varphi(H(y)) - (H_n(y) - H(y))J(H(y))] dy, \\ L_l &= \int_{-\infty}^{+\infty} [\varphi(H_{n,l}(y)) - \varphi(H(y)) - (H_{n,l}(y) - H(y))J(H(y))] dy. \end{aligned}$$

Similarly to (2.21), we have

$$\begin{aligned} &\frac{m^2\sigma_2^2}{n} E(B - D)^2 \\ &= E \left(\int_{-\infty}^{+\infty} [\varphi(H_n(y)) - \varphi(H_{n,l}(y)) - (H_n(y) - H_{n,l}(y))J(H_{n,l}(y))] dy \right. \\ &\quad \left. + \int_{-\infty}^{+\infty} (H_n(y) - H_{n,l}(y)) [J(H_{n,l}(y)) - J(H(y))] dy \right)^2 \\ &\leq c_0^2 E \left(0.5 \int_{-\infty}^{+\infty} (H_n(y) - H_{n,l}(y))^2 dy \right. \\ &\quad \left. + \int_{-\infty}^{+\infty} |H_n(y) - H_{n,l}(y)| |H_{n,l}(y) - H(y)| dy \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq 0.5c_0^2 E \left(\int_{-\infty}^{+\infty} (H_n(y) - H_{n,l}(y))^2 dy \right)^2 \\
&\quad + 2c_0^2 E \left(\int_{-\infty}^{+\infty} |H_n(y) - H_{n,l}(y)| |H_{n,l}(y) - H(y)| dy \right)^2 \\
&= 0.5c_0^2 L + 2c_0^2 L_l,
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
L &= E \left(\int_{-\infty}^{+\infty} (H_n(y) - H_{n,l}(y))^2 dy \right)^2, \\
L_l &= E \left(\int_{-\infty}^{+\infty} |H_n(y) - H_{n,l}(y)| |H_{n,l}(y) - H(y)| dy \right)^2.
\end{aligned}$$

We first consider L . By (1.10) and (2.9),

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (H_n(y) - H_{n,l}(y))^2 dy \\
&= \binom{n}{m}^{-2} \int_{-\infty}^{+\infty} \left(\sum_{1 \leq i_1 < \dots < i_{m-1} \leq n-1} (I[h(X_{i_1}, \dots, X_{i_{m-1}}, X_n) \leq y] - H(y)) \right)^2 dy \\
&= \binom{n}{m}^{-2} \sum_2 h_2(X_{i_1}, \dots, X_{i_{m-1}}, X_{j_1}, \dots, X_{j_{m-1}}, X_n),
\end{aligned}$$

where $\sum_2$ denotes the sum over all pairs of $m-1$ -tuples $(i_1, \dots, i_{m-1}), (j_1, \dots, j_{m-1}) \in C_{n-1, m-1}$ and

$$\begin{aligned}
&h_2(x_1, \dots, x_{m-1}, x_m, \dots, x_{2m-2}, x_n) \\
&= \int_{-\infty}^{+\infty} (I[h(x_1, \dots, x_{m-1}, x_n) \leq y] - H(y)) \\
&\quad \times (I[h(x_m, \dots, x_{2m-2}, x_n) \leq y] - H(y)) dy.
\end{aligned}$$

A direct calculation gives $\text{Card}(\sum_2) = \binom{n-1}{m-1}^2$. So, by (2.16) and the Minkowski inequality, we obtain

$$L \leq \frac{4m^4 \sigma_0^2}{n^4}. \tag{2.26}$$

As to L_l , note that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |H_n(y) - H_{n,l}(y)| |H_{n,l}(y) - H(y)| dy \\
&= \binom{n}{m}^{-2} \int_{-\infty}^{+\infty} \left| \sum_{1 \leq i_1 < \dots < i_{m-1} \leq n-1} (I[h(X_{i_1}, \dots, X_{i_{m-1}}, X_n) \leq y] - H(y)) \right| \\
&\quad \times \left| \sum_{1 \leq j_1 < \dots < j_m \leq n-1} (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right| dy
\end{aligned}$$

$$\leq \binom{n}{m}^{-2} \sum_{1 \leq i_1 < \dots < i_{m-1} \leq n-1} \int_{-\infty}^{+\infty} |I[h(X_{i_1}, \dots, X_{i_{m-1}}, X_n) \leq y] - H(y)| \\ \times \left| \sum_{1 \leq j_1 < \dots < j_m \leq n-1} (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right| dy.$$

Thus, by the Minkowski inequality again

$$L_l \leq \binom{n}{m}^{-4} \binom{n-1}{m-1}^2 E \left(\int_{-\infty}^{\infty} |I[h(X_1, \dots, X_{m-1}, X_n) \leq y] - H(y)| \right. \\ \times \left. \left| \sum_{1 \leq j_1 < \dots < j_m \leq n-1} (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right| dy \right)^2 \\ \leq L_{l,1} + L_{l,2}, \quad (2.27)$$

where

$$L_{l,1} = \frac{2m^2}{n^2} \binom{n}{m}^{-2} E \left(\int_{-\infty}^{\infty} |I[h(X_1, \dots, X_{m-1}, X_n) \leq y] - H(y)| \right. \\ \times \left. \left| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right| dy \right)^2, \\ L_{l,2} = \frac{2m^2}{n^2} \binom{n}{m}^{-2} E \left(\int_{-\infty}^{\infty} |I[h(X_1, \dots, X_{m-1}, X_n) \leq y] - H(y)| \right. \\ \times \left. \left| \sum_4 (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right| dy \right)^2,$$

$\sum_3$ denotes the sum over $(j_1, \dots, j_m) \in C_{n-1,m} - \{1, \dots, m-1\}$ and $\sum_4$ denotes the sum over all pairs $(i_1, \dots, i_{m-1}, n) \in C_{n-1,m-1}$ and $\{j_1, \dots, j_{m-1}\} \cap \{1, \dots, m-1\} \neq \emptyset$. Observing that

$$\text{Card} \left(\sum_4 \right) = \binom{n-1}{m} - \binom{n-m}{m} \leq \frac{2m}{n} \binom{n-1}{m}$$

we have

$$L_{l,2} \leq \frac{4m^3}{n^3} \binom{n}{m}^{-2} \binom{n-1}{m} \sum_4 E \left(\int_{-\infty}^{\infty} |I[h(X_1, \dots, X_{m-1}, X_n) \leq y] - H(y)| \right. \\ \times \left. |I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)| dy \right)^2 \\ \leq \frac{32m^4}{n^4} \binom{n}{m}^{-2} \binom{n-1}{m}^2 \sigma_0^2 \\ \leq \frac{32m^4}{n^4} \sigma_0^2 \leq \frac{8m^4 \sigma_0^2}{n^3}. \quad (2.28)$$

In the second inequality above, we used (2.16).

To estimate $L_{l,1}$, note that

$$\begin{aligned}
& E \left(\int_{-\infty}^{\infty} |I[h(X_1, \dots, X_{m-1}, X_n) \leq y] - H(y)| \right. \\
& \quad \times \left. \left| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right| dy \right)^2 \\
&= E \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |I[h(X_1, \dots, X_{m-1}, X_n) \leq x] - H(x)| \right. \\
& \quad \times |I[h(X_1, \dots, X_{m-1}, X_n) \leq y] - H(y)| \\
& \quad \times \left| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq x] - H(x)) \right| \\
& \quad \times \left. \left| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right| \right) dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(|I[h(X_1, \dots, X_{m-1}, X_n) \leq x] - H(x)| \\
& \quad \times |I[h(X_1, \dots, X_{m-1}, X_n) \leq y] - H(y)|) \\
& \quad \times E \left(\left| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq x] - H(x)) \right| \right. \\
& \quad \times \left. \left| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right| \right) dx dy \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \|I[h(X_1, \dots, X_{m-1}, X_n) \leq x] - H(x)\|_2 \\
& \quad \times \|I[h(X_1, \dots, X_{m-1}, X_n) \leq y] - H(y)\|_2 \\
& \quad \times \left\| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq x] - H(x)) \right\|_2 \\
& \quad \times \left\| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq y] - H(y)) \right\|_2 dx dy \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (H(x)(1-H(x)))^{1/2} (H(y)(1-H(y)))^{1/2} \\
& \quad \times \frac{3m^2}{n} \binom{n-m}{m}^2 (H(x)(1-H(x)))^{1/2} (H(y)(1-H(y)))^{1/2} dx dy \\
&\leq \frac{3m^2}{n} \binom{n-m}{m}^2 \sigma_0^2.
\end{aligned}$$

Here we used the following estimate:

$$\left\| \sum_3 (I[h(X_{j_1}, \dots, X_{j_m}) \leq x] - H(x)) \right\|_2^2 \leq \frac{3m^2}{n} \binom{n-m}{m}^2 H(x)(1-H(x)).$$

Thus, we obtain

$$L_{l,1} \leq \frac{6m^4\sigma_0^2}{n^3} \quad (2.29)$$

and hence

$$L_l \leq \frac{14m^4\sigma_0^2}{n^3}, \quad (2.30)$$

by (2.27) and (2.28).

This proves (2.24) by (2.25), (2.26), and (2.30), as desired. This completes the proof of Lemma 2.4. $\square$

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