On the Cohomology of Some Complex Hyperbolic Arithmetic 3-Manifolds

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Abstract Given an arithmetic lattice of the unitary group $U(3, 1)$ arising from a hermitian form over a CM-field, we show that all unitary representations of $U(3, 1)$ with nonzero cohomology contribute to the cohomology of the attached arithmetic complex 3-manifold, at least when we pass to a finite-index subgroup of the given arithmetic lattice.

Keywords Cohomology · Unitary representation · Theta lift · Complex hyperbolic 3-manifold

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1 Introduction

Let G be a connected reductive Lie group containing a compact Cartan subgroup. Let $K \subset G$ be a maximal compact subgroup and $X = G/K$ the global Riemannian symmetric space associated to G . Consider a lattice $\Gamma \subset G$. Let F be a finite dimensional representation of G and $\tilde{F}$ the local system on $\Gamma \backslash X$ determined by F .

The L^2 -cohomology space $H_{(2)}^*(\Gamma \backslash X, \tilde{F})$ is defined to be the cohomology of the complex of F -valued smooth differential forms ω such that ω and $d\omega$ are both square integrable. (In particular, $H_{(2)}^*(\Gamma \backslash X, \tilde{F})$ agrees with the usual cohomology

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$H^*(\Gamma \backslash X, \tilde{F})$ when $\Gamma \backslash G$ is compact.) This space is related to the decomposition of the regular representation of G on $L^2(\Gamma \backslash G)$ via a Matsushima type formula as follows. Write $L^2(\Gamma \backslash G)$ as the direct sum of the discrete part $L^2_d(\Gamma \backslash G)$ and its orthogonal complement, the continuous part $L^2_{\text{eis}}(\Gamma \backslash G)$ furnished by Eisenstein series (see [17]). Write

$$L^2_d(\Gamma \backslash G) \simeq \widehat{\bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) H_\pi},$$

where $\hat{G}$ is the unitary dual of G and the multiplicity $m(\pi, \Gamma)$ is a non-negative integer for each π . Since G has a compact Cartan, the space $H^*_{(2)}(\Gamma \backslash X, \tilde{F})$ is finite dimensional and one has the formula [3]

$$H^*_{(2)}(\Gamma \backslash X, \tilde{F}) \simeq \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) H^*_{\text{ct}}(G, H_\pi \otimes F), \quad (1.1)$$

where H^*_{ct} denotes continuous cohomology. As already noted, $H^*_{(2)}(\Gamma \backslash X, \tilde{F})$ agrees with the ordinary cohomology space $H^*(\Gamma \backslash X, \tilde{F})$ when Γ is cocompact, in which case (1.1) is the usual Matsushima formula, see [4, p. 223].

The representation π with $H^*_{\text{ct}}(G, H_\pi \otimes F) \neq 0$ turns out to be very special, and they have been completely classified by Vogan and Zuckerman [27]. Together with the formula (1.1), this implies a large amount of “automatic vanishing” for various parts of the L^2 -cohomology. The purpose of this note is to show that, for $G = \text{U}(3, 1)$ and Γ an arithmetic lattice arising from hermitian forms over CM-fields (see Sects. 4 and 5), these are essentially all the vanishing there is, at least when we pass to a finite-index subgroup of Γ .

Theorem 1.1 *Let $G = \text{U}(3, 1)$ and Γ an arithmetic lattice of G arising from hermitian forms over CM-fields. Let π be an irreducible unitary representation of G with non-zero cohomology. Then there exists a subgroup $\Gamma' \subseteq \Gamma$ of finite index, such that $m(\pi, \Gamma') \neq 0$. If π is not one-dimensional then it occurs in the space of cusp forms.*

It seems quite likely that the analogous result is valid for any $\text{U}(n, 1)$. On the other hand, vanishing results with respect to congruence subgroups arising from division algebras with involutions of the second kind [5, 23, 24] indicate that the above restriction on the type of arithmetic subgroups is probably necessary.

2 Unitary Representations with Non-zero Cohomology

Let G be a connected reductive Lie group containing a compact Cartan subgroup. We recall from [27, Sect. 5] the construction of irreducible unitary representations of G with non-zero cohomology.

Let $\mathfrak{g}_0$ be the Lie algebra of G with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. Let $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$ (and also for $\mathfrak{g}_0$ by our assumption). For $x \in i\mathfrak{t}_0$ the linear transformation $\text{ad}(x)$ of $\mathfrak{g}$ is diagonalizable with real eigenvalues. Let $\mathfrak{q}$ (resp. $\mathfrak{u}$, resp. $\mathfrak{l}$) be the sum of non-negative (resp. positive, resp. zero) eigenspaces of $\text{ad}(x)$.

Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, and $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$. Let K and L be the connected subgroups of G with Lie algebra $\mathfrak{k}_0$ and $\mathfrak{l}_0$, respectively. Let $\Delta(\mathfrak{g})$ denote the roots of $\mathfrak{t}$ in $\mathfrak{g}$. Fix a positive system $\Delta^+(\mathfrak{l})$ of the roots of $\mathfrak{t}$ in $\mathfrak{l}$. Then $\Delta^+(\mathfrak{g}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u})$ is a positive system for $\Delta(\mathfrak{g})$. Let ρ be half the sum of roots in $\Delta^+(\mathfrak{t}, \mathfrak{g})$, $\rho(\mathfrak{u} \cap \mathfrak{p})$ half the sum of roots of $\mathfrak{t}$ in $\mathfrak{u} \cap \mathfrak{p}$. Let $\lambda : \mathfrak{l} \rightarrow \mathbb{C}$ be a one-dimensional representation of $\mathfrak{l}$ such that

$$\begin{cases} \text{(a)} & \lambda \text{ is the differential of a unitary character of } L; \\ \text{(b)} & \text{if } \alpha \in \Delta(\mathfrak{u}), \text{ then } \langle \alpha, \lambda|_{\mathfrak{t}} \rangle \geq 0. \end{cases} \quad (2.1)$$

Let $\mu(\mathfrak{q}, \lambda)$ be the representation of K of highest weight $\lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p})$. In [27, Theorem 5.3], Vogan and Zuckerman prove the existence of a unique irreducible unitary representation $A_{\mathfrak{q}}(\lambda)$ of G with lowest K -type $\mu(\mathfrak{q}, \lambda)$ and infinitesimal character $\lambda + \rho$. In particular, the representation $A_{\mathfrak{q}}(\lambda)$ is in the discrete series if and only if L is compact, and all discrete series representations of G arise this way. Finally, every irreducible unitary representation with non-zero cohomology is of the form $A_{\mathfrak{q}}(\lambda)$.

Now we specialize to the case $G = \mathrm{U}(3, 1)$. We realize G as the group of linear transformations preserving the hermitian form on $\mathbb{C}^4$ with matrix $\mathrm{diag}(1, 1, 1, -1)$. We take $K = \mathrm{U}(3) \times \mathrm{U}(1)$, embedded in G as block diagonal matrices. Let $\mathfrak{t}_0$ be the Cartan subalgebra consisting of diagonal matrices and identify $\sqrt{-1}\mathfrak{t}_0$ with $\mathbb{R}^4$ in the obvious manner. Since G is of hermitian type, we have the usual decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ into holomorphic and anti-holomorphic tangent spaces. The roots of $\mathfrak{t}$ in $\mathfrak{g}$ are $\Delta(\mathfrak{g}) = \Delta(\mathfrak{k}) \cup \Delta(\mathfrak{p}^+) \cup \Delta(\mathfrak{p}^-)$, with

$$\begin{aligned} \Delta(\mathfrak{k}) &= \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq 3\}, \\ -\Delta(\mathfrak{p}^-) &= \Delta(\mathfrak{p}^+) = \{(\varepsilon_i - \varepsilon_4) \mid i = 1, 2, 3\}, \end{aligned}$$

where ε_i denotes evaluation on the i th coordinate.

Take

$$x = (x_1, x_2, x_3; x_4) \in \sqrt{-1}\mathfrak{t}_0$$

and consider the corresponding parabolic subalgebra $\mathfrak{q}$ as in the first paragraph of this page. Up to conjugation by K , we may assume $x_1 \geq x_2 \geq x_3$. The following lemma is obvious.

Lemma 2.1 *Assume that x is not in the center of $\mathfrak{g}$, i.e., its coordinates are not all equal.*

- (a) *If $x_3 \geq x_4$ then $\Delta(\mathfrak{u}) \subseteq \Delta(\mathfrak{p}^+)$, and the corresponding $A_{\mathfrak{q}}(\lambda)$'s are holomorphic (lowest weight) representations. Similarly if $x_4 \geq x_1$ then the corresponding $A_{\mathfrak{q}}(\lambda)$'s are anti-holomorphic representations.*
- (b) *Assume $x_i \neq x_4$ for $i = 1, 2, 3$. Then L is compact, and the corresponding $A_{\mathfrak{q}}(\lambda)$'s are discrete series representations.*

Excluding the situations considered in the preceding lemma, we have

$$x = (x_1, x_2, x_3; x_2), \quad x_1 > x_2 > x_3.$$

All such x 's define the same $\mathfrak{q}$. So from now on we fix one of them, say

$$x = x_0 = (1, 0, -1; 0) \quad (2.2)$$

and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the parabolic subalgebra defined by x_0 . This is the case of interest to us. Note that

$$L \simeq \mathrm{U}(1)^2 \times \mathrm{U}(1, 1), \quad \Delta(\mathfrak{u} \cap \mathfrak{p}) = \{\varepsilon_1 - \varepsilon_4, -\varepsilon_3 + \varepsilon_4\}.$$

Since G is of hermitian type, for any $(\mathfrak{g}, K)$ -module V we have a natural decomposition of cohomology spaces

$$H^n(\mathfrak{g}, \mathfrak{k}, V) = \bigoplus_{p+q=n} H^{p,q}(\mathfrak{g}, \mathfrak{k}, V),$$

corresponding to the Hodge structure on the cohomology of locally symmetric spaces via the Matsushima formula (1.1). By Proposition 6.19 of [27] we easily obtain the following result. Here we will abuse notations slightly to write $A_{\mathfrak{q}}(\lambda)$ also for the Harish-Chandra module underlying the unitary representation.

Proposition 2.2 *Let $\mathfrak{q}$ be defined by the element x_0 of (2.2), and let λ be a one-dimensional representation of $\mathfrak{l}$ satisfying (2.1). Let F be a finite dimensional irreducible representation of G .*

(a) *If the lowest weight of F is $-\lambda|_{\mathfrak{t}}$ then*

$$\dim H^{1,1}(\mathfrak{q}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda) \otimes F) = \dim H^{2,2}(\mathfrak{q}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda) \otimes F) = 1$$

and $H^{p,q}(\mathfrak{q}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda) \otimes F) = 0$ whenever $(p, q) \neq (1, 1)$ or $(2, 2)$.

(b) *If the lowest weight of F is not $-\lambda|_{\mathfrak{t}}$ then $H^i(\mathfrak{q}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda) \otimes F) = 0$ for all i .*

Since λ is one-dimensional, it is determined by its restriction to $\mathfrak{t}$. We will therefore abuse notations slightly to denote $\lambda|_{\mathfrak{t}}$ by λ . Then, any λ satisfying (2.1) is of the form

$$\lambda = \lambda_r = (a + r, r, -b + r; r) \quad (a, b, r \in \mathbb{Z}, a, b \geq 0). \quad (2.3)$$

In particular,

$$\lambda_0 = (a, 0, -b; 0) \quad (a, b \in \mathbb{Z}, a, b \geq 0). \quad (2.4)$$

Then clearly

$$A_{\mathfrak{q}}(\lambda) = A_{\mathfrak{q}}(\lambda_r) \simeq A_{\mathfrak{q}}(\lambda_0) \otimes \det^r, \quad (2.5)$$

where $\det$ denotes the determinant character of $\mathrm{U}(3, 1)$.

Consider now another unitary group $\mathrm{U}(1, 1)$. As is well known, $\mathrm{U}(3, 1)$ and $\mathrm{U}(1, 1)$ together form an example of what is called a reductive dual pair inside the symplectic group $\mathrm{Sp}_{16}(\mathbb{R})$. Let ω be the smooth oscillator representation of metaplectic cover of $\mathrm{Sp}_{16}(\mathbb{R})$ attached to the additive character $t \mapsto e^{iax}$, with some fixed $a > 0$. (For all this, see [10].) When restricted to the subgroups, ω can be considered

an ordinary representation of the product $U(3, 1) \times U(1, 1)$. More precisely, the splitting of the metaplectic cover over $U(3, 1) \times U(1, 1)$ depends on the choice of a pair of characters $\chi = (\chi_1, \chi_2)$ of $\mathbb{C}^\times$ with trivial restrictions to $\mathbb{R}^\times$ (cf. [14]). With this in mind, we speak of χ -theta liftings. We could in particular choose both χ_1 and χ_2 to be trivial, in such case we simply use the term “theta lifting”.

We recall the following result from [19], specialized to the present case.

Proposition 2.3 *The representation $A_q(\lambda_0)$ in (2.5) is the local theta lift from the holomorphic discrete series representation $\pi' = \pi'(a, b)$ of $U(1, 1)$ with lowest weight $(a + 2, -b - 2)$.*

Remark 2.4 As explained above the theta lift in the proposition uses the trivial splitting character. If instead we choose χ_2 to be trivial but let

$$\chi_1(c) = \left(\frac{c}{\bar{c}}\right)^r \quad (c \in \mathbb{C}^\times)$$

with r as in (2.3) and write $\chi_r = (\chi_1, 1)$ for the resulting pair, then $A_q(\lambda)$ is the χ_r -theta lift of $\pi'(a, b)$.

It is important to note that the theta lifting in the above proposition is explicitly constructed (in [19]) as follows. Let $\pi'^\vee$ be the contragredient of $\pi' = \pi'(a, b)$, and suppose it is realized on some Hilbert space $\mathcal{V}$. Let $\mathcal{V}^\infty \subset \mathcal{V}$ be the subspace of smooth vectors. Similarly let $\mathcal{W}$ be the space for the smooth oscillator representation ω . On the algebraic tensor product $\mathcal{W} \otimes \mathcal{V}^\infty$ we have the actions of $U(3, 1)$ and $U(1, 1)$, where $U(3, 1)$ acts on the first factor via ω , and $U(1, 1)$ acts by $\omega \otimes \pi'^\vee$. Consider the integrals

$$\int_{U(1,1)} \langle \omega \otimes \pi'^\vee(h)u, v \rangle dh \quad (u, v \in \mathcal{W} \otimes \mathcal{V}^\infty),$$

where dh denotes a fixed Haar measure on $U(1, 1)$. These integrals are absolutely convergent, and the resulting sesquilinear form on $\mathcal{W} \otimes \mathcal{V}^\infty$ is positive semi-definite. Let R be the radical of this sesquilinear form. Then $\mathcal{W} \otimes \mathcal{V}^\infty / R$ is a pre-Hilbert space on which $G = U(3, 1)$ acts unitarily, and the resulting unitary representation on the completed Hilbert space is precisely $A_q(\lambda_0)$.

For later use we also record the known theta correspondence for the dual pairs $U(4), U(2)$ and $U(4), U(1, 1)$, also with respect to the trivial splitting character.

Proposition 2.5 [12] (a) *The theta correspondence for the compact dual pair $U(4), U(2)$ is given as follows. An irreducible representation of $U(2)$ occurs in the correspondence if and only if its highest weight is of the form $(a + 2, b + 2)$ with $a \geq b \geq 0$. In such case the corresponding representation of $U(4)$ is of highest weight $(a + 1, b + 1, 1, 1)$.*

(b) *An irreducible representation of $U(1, 1)$ occurs in theta correspondence with $U(4)$ if and only if it is a holomorphic discrete series representation with lowest weight $(a + 2, -b - 2)$, where $a, b \geq 0$. In such case the corresponding representation of $U(4)$ is of highest weight $(a, 0, 0, -b)$.*

3 Non-vanishing of Certain Integrals of Matrix Coefficients

In this section only, assume that F is a non-Archimedean local field of characteristic zero. Let E be a separable F -algebra of dimension 2. In other words, E is either a quadratic field extension of F , or a product of two copies of F . Let V be a hermitian E -module, namely, it is a free E -module of finite rank, equipped with a non-degenerate F -bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow E$$

such that

$$\langle u, v \rangle = \langle v, u \rangle^t, \quad \langle au, v \rangle = a \langle u, v \rangle, \quad a \in E, u, v \in V.$$

Here “ t ” denotes the non-trivial automorphism of E (as an F -algebra). Likewise, let V' be a skew hermitian E -module. Then

$$W := V \otimes_E V'$$

is a symplectic space over F under the form

$$\langle u \otimes u', v \otimes v' \rangle := \text{tr}_{E/F}(\langle u, v \rangle \cdot \langle u', v' \rangle).$$

Denote by

$$H(W) := W \times F$$

the associated Heisenberg group, whose multiplication is given by

$$(w, t)(w', t') := (w + w', t + t' + \langle w, w' \rangle).$$

The product $U(V) \times U(V')$ of unitary groups (or general linear groups in the split case) acts on $H(W)$ as group automorphisms through its natural action on W . We define the associated Jacobi group to be the semi-direct product

$$J := (U(V) \times U(V')) \ltimes H(W).$$

Fix a non-trivial unitary character $\psi : F \rightarrow \mathbb{C}^\times$. Assume that V and V' have respective ranks 4 and 2, as free E -modules. Write ω for the smooth oscillator representation of J attached to ψ and the trivial splitting characters, that is, up to isomorphism, it is the unique smooth representation of J with the following properties:

- as a representation of $H(W)$, it is irreducible with central character ψ ;
- both the representation $\omega|_{U(V) \ltimes H(W)}$ and $\omega|_{U(V') \ltimes H(W)}$ have trivial Kudla characters (see [9, Sect. B.2] for the notion of Kudla characters).

Note that the contragredient $\omega^\vee$ of ω is isomorphic to the smooth oscillator representation of J attached to ψ^{-1} and the trivial splitting characters.

Let π' be a unitarizable irreducible smooth representation of $U(V')$, and write $\pi'^\vee$ for its contragredient.

Proposition 3.1 *The integrals in*

$$\omega \times \pi'^{\vee} \times \omega^{\vee} \times \pi' \rightarrow \mathbb{C},$$

$$(\phi, v^{\vee}, \phi^{\vee}, v) \mapsto \int_{U(V')} \langle g' \cdot \phi, \phi^{\vee} \rangle \langle g' \cdot v^{\vee}, v \rangle dg', \quad (3.1)$$

are absolutely convergent, where dg' denotes a Haar measure on $U(V')$. The map (3.1) is not identically zero if E splits, or V splits, or π' is not the trivial representation.

Proof Write V'^{-} for the space V' equipped with the form scaled by -1 . Put

$$V'^{\square} := V' \oplus V'^{-}.$$

It has a Lagrangian submodule

$$V'^{\Delta} := \{(v', v') \in V'^{\square}\}.$$

As before,

$$W^{\square} := V \otimes_E V'^{\square}$$

is a symplectic space over F . It has a Lagrangian subspace

$$W^{\Delta} := V \otimes_E V'^{\Delta},$$

and an orthogonal decomposition

$$W^{\square} := W \oplus W^{-}, \quad \text{where } W^{-} := V \otimes_E V'^{-}.$$

Denote by $\omega^{\square}$ the smooth oscillator representation of

$$J^{\square} := (U(V) \times U(V'^{\square})) \ltimes H(W^{\square}),$$

attached to ψ and the trivial splitting characters. Fix a non-zero linear functional λ_{Δ} on $\omega^{\square}$ which is invariant under

$$W^{\Delta} \subset W^{\square} \subset H(W^{\square}) \subset J^{\square}.$$

It is unique up to scalar multiplications. Moreover, for all p in the parabolic subgroup $P(V'^{\Delta})$ of $U(V^{\square})$ stabilizing V'^{Δ} , one has (cf. [9, Sect. B.2])

$$\lambda_{\Delta}(p\phi^{\square}) = |p|^2 \lambda_{\Delta}(\phi^{\square}), \quad \phi^{\square} \in \omega^{\square}. \quad (3.2)$$

Here $|p|$ denotes the value at p of the following positive character:

$$P(V'^{\Delta}) \xrightarrow{\text{restriction on } V'^{\Delta}} \text{GL}(V'^{\Delta}) \xrightarrow{\det} E^{\times} \xrightarrow{\text{normalized absolute value}} \mathbb{R}_{+}^{\times}.$$

Write

$$\eta : \omega \otimes \omega^{\vee} \xrightarrow{\sim} \omega^{\square}$$

for the unique linear isomorphism with the following two properties:

- it is equivariant with respect to the homomorphism

$$H(W) \times H(W) \xrightarrow{1 \times \varepsilon_W} H(W) \times H(W^-) \xrightarrow{\varsigma} H(W^\square),$$

where ε_W denotes the isomorphism $H(W) \rightarrow H(W^-)$, $(w, t) \mapsto (w, -t)$ (note that $W = W^-$ as vector spaces over F), and ς denotes the homomorphism $(w, t; w^-, t') \mapsto (w, w^-; t + t')$;

- $\lambda_\Delta \circ \eta$ coincides with the canonical pairing between ω and $\omega^\vee$.

Then η is also equivariant with respect to the inclusion (cf. [26, Sect. 3])

$$U(V') \times U(V') = U(V') \times U(V'^-) \subset U(V'^\square).$$

Consequently, the following diagram commutes:

$$\begin{array}{ccc} \omega \otimes \omega^\vee & \xrightarrow{\eta} & \omega^\square \\ c_\omega \downarrow & & \downarrow \Lambda_\Delta \\ C^\infty(U(V')) & \xleftarrow{r_1} & I, \end{array} \quad (3.3)$$

where

$$I := \{f \in C^\infty(U(V'^\square)) \mid f(px) = |p|^2 f(x), p \in P(V'^\Delta), x \in U(V'^\square)\}$$

is a representation of $U(V'^\square)$ under right translations; Λ_Δ denotes the map

$$\phi^\square \mapsto (g \mapsto \lambda_\Delta(g\phi^\square));$$

c_ω denotes the matrix coefficient map

$$\phi \otimes \phi^\vee \mapsto (g \mapsto \langle g \cdot \phi, \phi^\vee \rangle);$$

and r_1 denotes the restriction through the inclusion $U(V') \subset U(V'^\square)$.

Note that there is a commutative diagram

$$\begin{array}{ccc} C^\infty(U(V')) & \xleftarrow{r_1} & I \\ \downarrow & & \downarrow i_{V'} \\ D^\infty(U(V')) & \xleftarrow{r_2} & D^\infty(P(V'^\Delta) \backslash U(V'^\square)), \end{array} \quad (3.4)$$

where “ D^∞ ” stands for the space of smooth measures; $i_{V'}$ is a $U(V'^\square)$ -equivariant linear isomorphism; r_2 is the restriction map through the open embedding $U(V') \hookrightarrow P(V'^\Delta) \backslash U(V'^\square)$; and the left vertical arrow is the linear isomorphism of multiplying by the Haar measure dg' .

Since $P(V'^{\Delta}) \backslash U(V'^{\square})$ is compact, every element in the image of r_2 is integrable. Therefore the image of c_{ω} consists of only integrable functions. This proves the first assertion of the proposition.

If either E or V splits, then by [15, Theorem 1.2], the map Λ_{Δ} is surjective. Therefore, the image of c_{ω} contains the space of all Schwartz functions on $U(V')$, and the proposition follows.

Now assume that E is a field, and V does not split. Then [15, Theorem 1.2] implies that the image of $i_{V'} \circ \Lambda_{\Delta}$ equals the space of all smooth measures on $P(V'^{\Delta}) \backslash U(V'^{\square})$ whose integrals vanish. Therefore, the image of c_{ω} contains the space of all Schwartz functions on $U(V')$ whose integrals vanish. If π' is not the trivial representation, then π' has a bounded non-constant matrix coefficient, and the proposition follows easily. $\square$

We remark that when V splits, the dual pair $U(V), U(V')$ is in the so called stable range, and the assertion of the proposition also follows from [18].

4 Global Theta Lifting

From now on let F be a totally real number field, and E/F a totally imaginary quadratic extension of F . Let V be a hermitian space over E of dimension 4. We assume that

$$\begin{cases} \text{(a)} & \text{for one fixed real place } v_0, V \text{ has signature } (3, 1); \\ \text{(b)} & \text{at all other real places, } V \text{ is positive definite.} \end{cases} \quad (4.1)$$

View the isometry group $\mathbb{G} = U(V)$ as an algebraic group over F . By the above assumption, we see that $\mathbb{G}(F_v) = U(3, 1)$ for $v = v_0$, and $\mathbb{G}(F_v) = U(4)$ for any other real place v .

Let V' be a skew-hermitian space over E of dimension 2. Let $\mathbb{G}' = U(V')$ be the corresponding unitary group. Then for any real place v , $\mathbb{G}'(F_v)$ is isomorphic to either $U(2)$ or $U(1, 1)$. We assume that $\mathbb{G}'(F_{v_0})$ is isomorphic to $U(1, 1)$, i.e., V' splits at the place v_0 .

Set

$$W = V \otimes_E V'.$$

As in the preceding section, this is a symplectic space over F of dimension 16. Let $\mathrm{Sp}(W)$ be the corresponding symplectic group. We have a natural map

$$\iota : \mathbb{G} \times \mathbb{G}' \longrightarrow \mathrm{Sp}(W)$$

given by the obvious actions of $\mathbb{G}$ and $\mathbb{G}'$ on W . We remark that the reductive dual pair $\mathbb{G}, \mathbb{G}'$ is outside the range considered in [20].

Let $\mathbb{A} = \mathbb{A}_F$ be the ring of adeles of F . Fix a non-trivial character $\psi = \prod \psi_v$ of $\mathbb{A}/F$. We assume that for each real place v of F , there is a positive element $a \in F_v \simeq \mathbb{R}$, such that

$$\psi_v(x) = e^{iax} \quad (x \in F_v).$$

Let ω_ψ be the oscillator representation of the metaplectic group $\mathrm{Mp}(W)$ attached to ψ . We recall that $\mathrm{Mp}(W)$ fits into the short exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W)_{\mathbb{A}} \longrightarrow 1.$$

Let C_F and C_E be the idele class groups of F and E , respectively. Recall from [14] and [8] that for any pair of characters $\chi = (\chi_1, \chi_2)$ of C_E satisfying certain condition—which means trivial restriction to C_F in the present context—we have a homomorphism

$$\iota_\chi : \mathbb{G}(\mathbb{A}) \times \mathbb{G}'(\mathbb{A}) \longrightarrow \mathrm{Mp}(W)$$

lifting ι . In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{G}(\mathbb{A}) \times \mathbb{G}'(\mathbb{A}) & \xrightarrow{\iota_\chi} & \mathrm{Mp}(W) \\ \parallel & & \downarrow \\ \mathbb{G}(\mathbb{A}) \times \mathbb{G}'(\mathbb{A}) & \xrightarrow{\iota} & \mathrm{Sp}(W)_{\mathbb{A}}. \end{array}$$

Let $\omega_\chi = \omega_{\chi, \psi}$ be the composition of ι_χ with ω_ψ . In this paper, we shall just take χ to be trivial (i.e., both χ_1 and χ_2 are trivial). We write 1 for this trivial χ , and let $\omega = \omega_1 = \omega_{1, \psi}$. This is a restricted tensor product: $\omega = \otimes \omega_v$, where each ω_v is a local oscillator representation as discussed in Sects. 2 and 3. As is well known, the representation ω may be realized on the Schwartz space $S(X(\mathbb{A}))$, where X is any Lagrange subspace of the symplectic space W . The theta-distribution is defined by

$$\theta(\phi) = \sum_{\xi \in X(k)} \phi(\xi) \quad (\phi \in S(X(\mathbb{A})))$$

which satisfies

$$\theta(\omega(\gamma)\phi) = \theta(\phi) \quad (\gamma \in \mathrm{Sp}(W)_F).$$

For any fixed Schwartz function ϕ , the formula

$$\theta_\phi(g, h) = \theta(\omega(g, h)\phi) \quad (g \in \mathbb{G}(\mathbb{A}), h \in \mathbb{G}'(\mathbb{A}))$$

defines a smooth, slowly increasing function on $\mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A}) \times \mathbb{G}'(F) \backslash \mathbb{G}'(\mathbb{A})$. See [28, 29] for these facts.

Let $\sigma = \otimes \sigma_v$ be a cuspidal automorphic representation realized on a subspace $H_\sigma \subset L^2(\mathbb{G}'(F) \backslash \mathbb{G}'(\mathbb{A}))$. For each smooth $f \in H_\sigma$ the function

$$\theta_\phi^f(g) = \int_{\mathbb{G}'(F) \backslash \mathbb{G}'(\mathbb{A})} \theta_\phi(g, h) \overline{f(h)} dh \quad (g \in \mathbb{G}(\mathbb{A})) \quad (4.2)$$

is well defined, and determines a slowly increasing function on $\mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A})$. Note, however, that with a bar over $f(h)$ in the integrand, our definition is slightly different from the usual conventions, but is more compatible with local theta correspondence.

Take $\phi_1, \phi_2 \in S(X(\mathbb{A}))$ and $f_1, f_2 \in H_\sigma$, assumed to be smooth. We shall consider the inner product

$$\langle \theta_{\phi_1}^{f_1}, \theta_{\phi_2}^{f_2} \rangle = \int_{\mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A})} \theta_{\phi_1}^{f_1}(g) \overline{\theta_{\phi_2}^{f_2}(g)} dg. \quad (4.3)$$

Because of our assumption (4.1), the group $\mathbb{G}$ is anisotropic if and only if $F \neq \mathbb{Q}$. If $\mathbb{G}$ is anisotropic, then the integral in (4.3) is certainly absolutely convergent. If $\mathbb{G}$ is isotropic, then $F = \mathbb{Q}$ and the assumption (4.1) implies that V is of Witt index 1 over F . In this case the criterion of [29] again implies that the integral in (4.3) is absolutely convergent.

In any case, the right hand side of (4.3) can be computed via the Rallis inner product formula, which has been worked out in greater generality in [8] using Ichino's extended version of the Siegel–Weil formula for unitary groups [11]. We now explain the formula in the present setting.

In the formalism of L-groups [2, 16], the dual group of $\mathbb{G}$ is ${}^L\mathbb{G}^0 = \mathrm{GL}(4, \mathbb{C})$, and the non-trivial element of the Galois group $\mathrm{Gal}(E/F)$ acts on $\mathrm{GL}(4, \mathbb{C})$ by an outer automorphism. The semi-direct product $\mathrm{Gal}(E/F) \ltimes \mathrm{GL}(4, \mathbb{C})$ is (a quotient of) the L-group of $\mathbb{G}$ denoted by ${}^L\mathbb{G}$. Now first take the standard representation of $\mathrm{GL}(4, \mathbb{C})$ on $\mathbb{C}^4$ and then induce it to $\mathrm{Gal}(E/F) \ltimes \mathrm{GL}(4, \mathbb{C})$. This gives an irreducible eight-dimensional representation r of the L-group ${}^L\mathbb{G}$. We let $L(s, \sigma) = L(s, \sigma, r)$ be the Langlands L-function attached to the cusp form σ and the representation r .

Let $\varepsilon = \varepsilon_{E/F}$ be the quadratic character of C_F associated to the extension E/F , and $L(s, \varepsilon)$ the corresponding Hecke L-function. Let S_∞ be the set of Archimedean places of F (all real according to our assumption). Let S be a finite set of places containing S_∞ , such that σ_v is unramified for $v \notin S$. Then the partial L-functions

$$L^S(s, \sigma) = \prod_{v \notin S} L_v(s, \sigma), \quad L^S(s, \varepsilon) = \prod_{v \notin S} L_v(s, \varepsilon)$$

are well defined and entire, so they can be evaluated at any point.

Now suppose the vectors ϕ_i, f_i are all factorizable:

$$\phi_i = \otimes \phi_{iv}, \quad f_i = \otimes f_{iv}, \quad i = 1, 2.$$

Assume that for each place $v \notin S$ the following conditions are satisfied:

- (a) The groups $\mathbb{G}, \mathbb{G}'$ are unramified at v .
- (b) The vectors ϕ_{iv}, f_{iv} are all unramified for $i = 1, 2$.
- (c) One has the usual normalization

$$\langle \phi_{1v}, \phi_{2v} \rangle = 1 = \langle f_{1v}, f_{2v} \rangle.$$

Then we can express the Rallis inner product formula as

$$\langle \theta_{\phi_1}^{f_1}, \theta_{\phi_2}^{f_2} \rangle = I_S(\phi_1, \phi_2; f_1, f_2) \cdot \frac{L^S(\frac{3}{2}, \sigma)}{\prod_{k=1}^4 L^S(2+k, \varepsilon^k)}, \quad (4.4)$$

where

$$I_S(\phi_1, \phi_2; f_1, f_2) = \prod_{v \in S} I_v(\phi_1, \phi_2; f_1, f_2)$$

and

$$I_v(\phi_1, \phi_2; f_1, f_2) = \int_{\mathbb{G}'(F_v)} \langle \omega_v(h) \phi_{1v}, \phi_{2v} | \overline{\langle \sigma_v(h) f_{1v}, f_{2v} \rangle} dh. \quad (4.5)$$

We now discuss conditions under which the right hand side of (4.4) will not be identically zero. First observe that if σ were one-dimensional then $s = 3/2$ is precisely the first point where the product $\prod_{v \notin S} L_v(s, \sigma)$ fails to converge absolutely. On the other hand if σ is infinite dimensional (as will be the case for us) then any non-trivial estimate towards the Ramanujan conjecture on the Hecke eigenvalues imply that the product $\prod_{v \notin S} L_v(s, \sigma)$ converges absolutely at $s = 3/2$. Note that such estimates do exist—for example one could combine base change with the results of [6] or [21]. It follows that $L^S(\frac{3}{2}, \sigma) \neq 0$. Thus it suffices to consider the local integrals in (4.5) for places $v \in S$.

Let S_0 be the finite set of places where the group $\mathbb{G}$ fails to be quasi-split. By our assumption (4.1) and the conditions for S we have

$$S_\infty \subseteq S_0 \subseteq S,$$

and a finite place v belongs to S_0 precisely when the extension E/F does not split at v , and that V has Witt index 1 over F_v . From the local results of Sects. 2 and 3 we can now deduce the following result. Recall that $\mathbb{G}'(F_v)$ is isomorphic to either $U(1, 1)$ or $U(2)$ for any $v \in S_\infty$.

Theorem 4.1 *Suppose that*

- (a) *At the place v_0 (cf. (4.1)), σ_{v_0} is a holomorphic discrete series representation of $\mathbb{G}'(F_{v_0}) \simeq U(1, 1)$ that appears in Proposition 2.3 (called π' there).*
- (b) *For any other $v \in S_\infty$, the representation σ_v is one of the discrete series representations of $\mathbb{G}'(F_v)$ discussed in Propositions 2.5.*
- (c) *When $v \in S_0$ is finite and $\mathbb{G}'(F_v)$ is anisotropic, σ_v is not the trivial representation.*

Then the right hand side of (4.4) is not identically zero. The functions θ_ϕ^f with $\phi \in S(X(\mathbb{A}))$ and $f \in H_\sigma$ generate an irreducible automorphic representation $\pi = \otimes \pi_v$ which occurs discretely in $L^2(\mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A}))$. In particular, for each $v \in S_\infty$, the representation π_v is the local theta lift of σ_v described in Proposition 2.3 or 2.5.

Proof We need to make sure that for each $v \in S$ the local integrals (4.5) are not identically zero. Observe that if v is not in S_0 then the dual pair $\mathbb{G}(F_v), \mathbb{G}'(F_v)$ is in the so called stable range, and we obtain the result from Sect. 3 or [18]. If v is in S_0 and finite then there are two possibilities for $\mathbb{G}'(F_v)$. If $\mathbb{G}'(F_v)$ is not anisotropic then σ_v (as a local component of the automorphic cuspidal representation σ) must be infinite dimensional, and in particular non-trivial. So the results of Sect. 3 imply that

the local integrals (4.5) are not identically zero. Finally if $\mathbb{G}'(F_v)$ is anisotropic then our condition (c) and Sect. 3 give what we want.

The assertion about what happens at the Archimedean places follows from the remarks after Proposition 2.3. $\square$

If $F \neq \mathbb{Q}$ then $\mathbb{G}$ is anisotropic over F , and the automorphic representation π is automatically cuspidal. Suppose $F = \mathbb{Q}$. Then our assumption (4.1) implies that V has Witt index 1. Write $V = H \oplus V_0$ where H is a hyperbolic plane and V_0 is anisotropic. Let $\mathbb{G}_0 = \mathrm{U}(V_0)$. A general principle [22] says that π is cuspidal if and only if the theta lift of σ to $\mathbb{G}_0$ is zero. This last condition can be arranged as follows. Fix a finite place v_1 where the extension E/F does not split (we could choose $v_1 \in S_0$). Let π'_{v_1} be a supercuspidal representation of $\mathbb{G}'(F_{v_1})$. Up to equivalence there are exactly two hermitian forms of dimension 2 over E_w , where w is the place of E lying over v . Consequently there are two unitary groups, one of them being $\mathbb{G}_0(F_v)$. Now theta dichotomy [7] says that π'_{v_1} has a non-zero theta lift to exactly one (not both) of these two groups. We choose π'_{v_1} so that it does not lift to $\mathbb{G}_0(F_v)$. There exists a cuspidal automorphic representation σ of $\mathbb{G}'$ satisfying the conditions of Theorem 4.1, and such that $\sigma_{v_1} \simeq \pi'_{v_1}$. It is then clear that σ does not lift to $\mathbb{G}_0$, and hence π is cuspidal.

5 Conclusions

We are now in a position to prove Theorem 1.1. Let $\mathbb{G}$ be as in Sect. 4. Then

$$\mathbb{G}_\infty \stackrel{\text{def}}{=} \prod_{v \in S_\infty} \mathbb{G}(F_v) \simeq \mathrm{U}(3, 1) \times \mathrm{U}(4)^t,$$

where $t + 1 = [F : \mathbb{Q}]$ is the degree of F . According to Margulis, the projection of an arithmetic subgroup of $\mathbb{G}(F)$ to $\mathrm{U}(3, 1)$ is called an arithmetic subgroup of $\mathrm{U}(3, 1)$: this is what we mean by “an arithmetic subgroup of $\mathrm{U}(3, 1)$ arising from hermitian forms over CM-fields” in the statement of Theorem 1.1. Similarly, the projection of a congruence subgroup of $\mathbb{G}(F)$ to $\mathrm{U}(3, 1)$ is called a congruence subgroup of $\mathrm{U}(3, 1)$. Note that if Γ is a congruence subgroup of $\mathrm{U}(3, 1)$ then

$$\det(\Gamma) = \{\det(\gamma) \mid \gamma \in \Gamma\}$$

is a congruence subgroup of $\mathrm{U}(1)$ and must be finite. Hence $\Gamma \cap \mathrm{SU}(3, 1)$ is of finite index in Γ .

Let Γ be an arithmetic subgroup of $\mathrm{U}(3, 1)$, of the type defined above. Let π_{v_0} be an irreducible unitary representation of $\mathrm{U}(3, 1)$ with non-zero cohomology.

We first assume that π_{v_0} is the representation $A_q(\lambda)$ described by Proposition 2.2 and (2.3)–(2.4). For each $v \in S_\infty$, different from v_0 , we let π_v be the one-dimensional representation of $\mathrm{U}(4)$ discussed in Proposition 2.5. Then by the results of Sect. 4, there exists a cuspidal automorphic representation π of $\mathbb{G}(\mathbb{A})$ such that (i) its factor at v_0 is the representation $A_q(\lambda_0)$ (cf. Proposition 2.3), and (ii) the factors at the other Archimedean places are the π_v ’s already given above.

Let $\mathbb{A}_f$ be the finite adeles of F and let K_f be an open compact subgroup of $\mathbb{G}(\mathbb{A}_f)$, such that π has a non-zero vector fixed by K_f . There is a finite set $\{x_1, \dots, x_m\}$ of elements of $\mathbb{G}(\mathbb{A}_f)$ such that

$$\mathbb{G}(\mathbb{A}) = \bigcup_{j=1}^m \mathbb{G}(F)\mathbb{G}_{\infty}x_jK_f$$

(a disjoint union). Set $K_j = x_jK_fx_j^{-1}$ and $K = \cap K_j$. Then K is open compact in $\mathbb{G}(\mathbb{A}_f)$. Let $\Gamma_0 = \mathbb{G}(F) \cap (\mathbb{G}_{\infty} \times K)$. Then $\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K$ is covered by a disjoint union of finite copies of $\Gamma_0\backslash\mathbb{G}_{\infty}$, and pull-back of functions respects the action of $\mathbb{G}_{\infty}$. It follows that the representation

$$\pi_{\infty}^0 = A_q(\lambda_0) \otimes \left(\bigotimes_{v \in S_{\infty} \setminus \{v_0\}} \pi_v \right)$$

occurs in $L^2(\Gamma_0\backslash\mathbb{G}_{\infty})$. Replacing Γ_0 by a finite-index subgroup if necessary, we may assume that all elements of Γ_0 has determinant 1. Then since π_v is one-dimensional for each $v \in S_{\infty} \setminus \{v_0\}$, the space of π_{∞}^0 can be identified with a space of functions on $\Gamma_0\backslash\mathbb{U}(3, 1)$. Thus $A_q(\lambda_0)$ occurs in $L^2(\Gamma_0\backslash\mathbb{U}(3, 1))$. Let $\Gamma_1 = \Gamma \cap \Gamma_0$. Then Γ_1 is of finite index in Γ and Γ_0 , and $A_q(\lambda_0)$ occurs in $L^2(\Gamma_1\backslash\mathbb{U}(3, 1))$. Finally, since $A_q(\lambda) = A_q(\lambda_0) \otimes \det^r$ (cf. (2.5)) and elements of Γ_1 have determinant 1, we see that $A_q(\lambda)$ also occurs in $L^2(\Gamma_1\backslash\mathbb{U}(3, 1))$.

We remark that in the above we have tried to handle theta liftings from various possible forms of $\mathbb{U}(V')$. But for the purpose of proving Theorem 1.1 we could have just assumed from the very beginning that V' is split over F . Then the π_v discussed above would be the trivial representation for each $v \in S_{\infty} \setminus \{v_0\}$, and that condition (c) in Theorem 4.1 would be vacuous.

We now return to our proof. From Lemma 2.1 we see that in any other situation π_{v_0} would be either holomorphic or anti-holomorphic, or in the discrete series (or one-dimensional, which is the trivial case). But for discrete series representations the result is well known and valid for all arithmetic lattices—not just those arising from hermitian forms, see for example [25]. If π_{v_0} is holomorphic the result is again known [1, 4, 13]. Alternatively it can be proved in the same way as above, by making different choices of the hermitian space V' , positive definite at the place v_0 . This concludes our discussion.

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