Detecting Quaternionic Maps Between Hyperkähler Manifolds

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Abstract For a harmonic map between two hyperkähler manifolds, we prove a Weitzenböck type formula for the defining quantity of quaternionic maps, and apply it to harmonic morphisms. We also provide a sufficient and necessary condition for a smooth map being quaternionic.

Keywords Harmonic maps · Hyperkähler manifolds · Quaternionic maps · Weitzenböck formula

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1 Introduction

A Riemannian manifold (M, g) is called hyperkähler if it possesses covariant constant, in the covariant differentiation ∇_g , complex structures J^1, J^2, J^3 which satisfy the quaternionic relation: $J^3 = J^1 J^2$. The covariant constant complex structures

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$aJ^1 + bJ^2 + cJ^3$ with $(a, b, c) \in \mathbb{S}^2$ is referred to the hyperkähler $\mathbb{S}^2$. A hyperkähler manifold is Ricci-flat and its dimension is a multiple of 4.

Let M and N be two hyperkähler manifolds with complex structures J^Q and $\mathcal{J}^P$, respectively, for $P, Q = 1, 2, 3$ which satisfy the quaternionic relation. A smooth map $f : M \rightarrow N$ is called a quaternionic map if

$$\sum_{P, Q} A_{PQ} \mathcal{J}^P \circ df \circ J^Q = df \quad (1.1)$$

where A_{PQ} denote the entries of a constant matrix A in $SO(3)$. Arising from a dimension reduction procedure in higher dimensional gauge theory, these maps are nonlinear generalization of the Cauchy–Riemann–Fueter equations for functions between the quaternions, and they are called triholomorphic maps in [6]. It is known that a quaternionic map is harmonic, in fact, it minimizes the Dirichlet energy in its homotopy class when M is compact [6] (also see [2, 9]). Also known is that a holomorphic map is quaternionic but the converse is not necessarily true since the rank of the differential of such a map can be odd [3]. In fact, there is a homotopy condition that determines whether a quaternionic map is holomorphic (in any complex structures in the hyperkähler $\mathbb{S}^2$) or not (see [3, Theorem 3.1]).

In this note, we explore various conditions which force harmonic maps or smooth maps to be quaternionic. The first result is a Weitzenböck type formula useful for detecting when a harmonic map is quaternionic.

Theorem 1.1 *Let M^{4m} and N^{4n} be two hyperkähler manifolds with covariant constant complex structures $J^1, J^2, J^3 = J^1 J^2$ and $\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3 = \mathcal{J}^1 \mathcal{J}^2$, respectively. Suppose that $f : M \rightarrow N$ is a harmonic map. Let $\phi = df - \sum_{P, Q=1}^3 A_{PQ} \mathcal{J}^P \circ df \circ J^Q$ where $(A_{PQ}) \in SO(3)$. Then*

$$\Delta|\phi|^2 = 2|\nabla\phi|^2 - 8 \sum_{i,j=1}^{4m} R^N(\phi \cdot e_i, df \cdot e_j, df \cdot e_i, df \cdot e_j)$$

where R^N is the curvature tensor of N and $\{e_1, \dots, e_{4m}\}$ is an orthonormal basis at the point under consideration on M . Moreover, if M is compact and f is harmonic morphism, then f satisfies (1.1) on M if it does so at one point on M .

Here we recall that a map from a manifold to another is called a harmonic morphism if it preserves the Laplace equation, i.e., the composition of the map with any local harmonic function on the domain is harmonic on the target.

In the second result, we find a necessary and sufficient condition, in terms of quantities that depend only on homotopy classes of smooth maps from M to N , for a smooth map being quaternionic.

Theorem 1.2 *Suppose that $u : M \rightarrow N$ is a smooth map where M, N are hyperkähler manifolds as above and M is compact. Let B be the matrix with entries*

$$B_{PQ} = - \int_M \langle J^P, u^* \mathcal{J}^Q \rangle dV \quad \text{for } P, Q = 1, 2, 3.$$

Then u is a quaternionic map if and only if

$$\det B \geq 0 \quad \text{and} \quad \operatorname{tr} \sqrt{BB^t} = E(u) \quad (1.2)$$

where $E(u)$ is the Dirichlet energy of u .

Using this criterion we can easily determine which homotopy class of differentiable maps from a flat torus $\mathbb{T}^{4m}$ to $\mathbb{T}^{4n}$ contains a quaternionic representative.

Throughout this note, M, N are hyperkähler manifolds of dimension $4m, 4n$, with J^1, J^2, J^3 and $\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3$ which satisfy the quaternionic relation, respectively. The following ranges for indices are adapted:

$$\begin{aligned} 1 \leq i, j, k, l &\leq 4m, \\ 1 \leq \alpha, \beta, \gamma, \delta, \eta &\leq 4n, \\ 1 \leq P, Q, S, T, U, V &\leq 3. \end{aligned}$$

Summation is taken over indices according to their appropriate range.

2 A Weitzenböck Formula for Harmonic Maps

In this section, we will prove Theorem 1.1.

In local coordinates, we write $f_i^\alpha = \partial f^\alpha / \partial x^i$ and the complex structures

$$J^P \frac{\partial}{\partial x^k} = \sum_i (J^P)_k^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \mathcal{J}^P \frac{\partial}{\partial y^\alpha} = \sum_\beta (\mathcal{J}^P)_\alpha^\beta \frac{\partial}{\partial y^\beta}.$$

It follows that

$$\sum_l (J^P)_j^l g_{kl} = \left\langle \frac{\partial}{\partial x^k}, J^P \frac{\partial}{\partial x^j} \right\rangle = - \left\langle J^P \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \right\rangle = - \sum_i (J^P)_k^i g_{ij}.$$

Equation (1.1) reads

$$\sum_{j, \beta} A_{PQ} (J^P)_i^j (\mathcal{J}^Q)_\beta^\alpha f_j^\beta = f_i^\alpha. \quad (2.1)$$

We introduce

$$\phi = df - \sum_{P, Q} A_{PQ} \mathcal{J}^P \circ df \circ J^Q. \quad (2.2)$$

Proof of Theorem 1.1 We calculate by using the moving frames as in [4, (1.11)]:

$$\begin{aligned} f_{ijj}^\alpha &= f_{jji}^\alpha \\ &= f_{jji}^\alpha + \sum_k f_k^\alpha R_{kji}^M - \sum_{\beta, \gamma, \delta} f_j^\beta f_i^\gamma f_j^\delta R_{\alpha\beta\gamma\delta}^N \\ &= f_{jji}^\alpha + \sum_k f_k^\alpha Ric_{ki}^M - \sum_{\beta, \gamma, \delta} f_j^\beta f_i^\gamma f_j^\delta R_{\alpha\beta\gamma\delta}^N, \end{aligned}$$

where $R_{\alpha\beta\gamma\eta}^N$ is the curvature tensor of N and Ric_{ki}^M is the Ricci tensor of M . Because M is Ricci-flat and f is harmonic, we have

$$f_{ijj}^\alpha = - \sum_{\beta, \gamma, \delta} f_j^\beta f_i^\gamma f_j^\delta R_{\alpha\beta\gamma\delta}^N. \quad (2.3)$$

Since matrices in $SO(3)$ preserves the quaternionic relation, namely, if $J^3 = J^1 J^2$ and $A \in SO(3)$, then $A(J^3) = A(J^1)A(J^2)$. We may therefore assume, for simplicity, that the matrix $(A_P Q)$ in (1.1) is the identity matrix $I_{3 \times 3}$. In the local coordinates, we write

$$\phi_i^\alpha = f_i^\alpha - \sum_{k, \beta, P} (\mathcal{J}^P)_\beta^\alpha f_k^\beta (J^P)_i^k.$$

From (2.3) and that the complex structures are covariant constant, it follows that

$$\begin{aligned} \phi_{ijj}^\alpha &= f_{ijj}^\alpha - \sum_{k, \beta, P} (\mathcal{J}^P)_\beta^\alpha f_{kjj}^\beta (J^P)_i^k \\ &= - \sum_{\beta, \gamma, \delta} f_j^\beta f_i^\gamma f_j^\delta R_{\alpha\beta\gamma\delta}^N + \sum_{k, \beta, \gamma, \delta, \eta, P} (\mathcal{J}^P)_\beta^\alpha f_j^\gamma f_k^\delta f_j^\eta R_{\beta\gamma\delta\eta}^N (J^P)_i^k. \end{aligned}$$

We set

$$\psi_i^\alpha = \sum_{k, \beta, P} (\mathcal{J}^P)_\beta^\alpha f_k^\beta (J^P)_i^k.$$

Then we have

$$\begin{aligned} \sum_{i, j, \alpha} \phi_i^\alpha \phi_{ijj}^\alpha &= \sum_{i, j, \alpha} \left(f_i^\alpha - \sum_{k, \beta, P} (\mathcal{J}^P)_\beta^\alpha f_k^\beta (J^P)_i^k \right) \left(- \sum_{\beta, \gamma, \delta} f_j^\beta f_i^\gamma f_j^\delta R_{\alpha\beta\gamma\delta}^N \right. \\ &\quad \left. + \sum_{k, \beta, \gamma, \delta, \eta, P} (\mathcal{J}^P)_\beta^\alpha f_j^\gamma f_k^\delta f_j^\eta R_{\beta\gamma\delta\eta}^N (J^P)_i^k \right) \\ &= - \sum_{i, j, \alpha, \beta, \gamma, \delta} f_i^\alpha f_j^\beta f_i^\gamma f_j^\delta R_{\alpha\beta\gamma\delta}^N + \sum_{i, j, \alpha, \beta, \gamma, \delta} \psi_i^\alpha f_j^\beta f_i^\gamma f_j^\delta R_{\alpha\beta\gamma\delta}^N \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i,j,k,\alpha,\beta,\gamma,\delta,\eta,P} f_i^\alpha f_j^\gamma f_k^\delta f_j^\eta (\mathcal{J}^P)_\beta^\alpha (J^P)_i^k R_{\beta\gamma\delta\eta}^N \\
 & - \sum_{i,j,k,\alpha,\beta,\gamma,\delta,\eta,P} f_j^\gamma f_k^\delta f_j^\eta \psi_i^\alpha (\mathcal{J}^P)_\beta^\alpha (J^P)_i^k R_{\beta\gamma\delta\eta}^N.
 \end{aligned} \quad (2.4)$$

In orthonormal frames,

$$(J^P)_k^i = -(J^A)_i^k \quad \text{and} \quad (\mathcal{J}^P)_\alpha^\beta = -(\mathcal{J}^P)_\beta^\alpha.$$

We have

$$\begin{aligned}
 & \sum_{i,j,k,\alpha,\beta,\gamma,\delta,\eta,P} f_i^\alpha f_j^\gamma f_k^\delta f_j^\eta (\mathcal{J}^P)_\beta^\alpha (J^P)_i^k R_{\beta\gamma\delta\eta}^N \\
 & = \sum_{i,j,k,\alpha,\beta,\gamma,\delta,\eta,P} f_i^\alpha f_j^\gamma f_k^\delta f_j^\eta (-\mathcal{J}^P)_\alpha^\beta (-J^P)_k^i R_{\beta\gamma\delta\eta}^N \\
 & = \sum_{j,k,\beta,\gamma,\delta,\eta} f_j^\gamma f_k^\delta f_j^\eta \psi_k^\beta R_{\beta\gamma\delta\eta}^N.
 \end{aligned} \quad (2.5)$$

Furthermore, when $P \neq Q$,

$$\begin{aligned}
 \sum_{i,\alpha,P,Q} (\mathcal{J}^Q)_\tau^\alpha (\mathcal{J}^P)_\beta^\alpha (J^Q)_i^l (J^P)_i^k & = \sum_{i,\alpha,P,Q} (\mathcal{J}^Q)_\tau^\alpha (-\mathcal{J}^P)_\alpha^\beta (J^Q)_i^l (-J^P)_k^i \\
 & = \sum_{i,\alpha,P,Q} (\mathcal{J}^Q)_\tau^\alpha (\mathcal{J}^P)_\alpha^\beta (J^P)_k^i (J^Q)_i^l \\
 & = \sum_S (\mathcal{J}^S)_\tau^\beta (-J^S)_k^l \\
 & = - \sum_S (\mathcal{J}^S)_\tau^\beta (J^S)_k^l
 \end{aligned} \quad (2.6)$$

where $S \neq P, Q$ and we have used the quaternionic relation for the two triples of complex structures. By grouping the terms with $P = Q$ together and using (2.6) for the rest, we have

$$\begin{aligned}
 & \sum_{i,j,k,\alpha,\beta,\gamma,\delta,\eta,P} f_j^\gamma f_k^\delta f_j^\eta \psi_i^\alpha (\mathcal{J}^P)_\beta^\alpha (J^P)_i^k R_{\beta\gamma\delta\eta}^N \\
 & = \sum_{i,j,k,\alpha,\beta,\gamma,\delta,\eta} f_j^\gamma f_k^\delta f_j^\eta \left\{ \sum_{l,\tau,Q} (\mathcal{J}^Q)_\tau^\alpha f_l^\tau (J^Q)_i^l \right\} (\mathcal{J}^P)_\beta^\alpha (J^P)_i^k R_{\beta\gamma\delta\eta}^N \\
 & = 3 \sum_{j,k,\beta,\gamma,\delta,\eta} f_j^\gamma f_k^\delta f_j^\eta \psi_k^\beta R_{\beta\gamma\delta\eta}^N
 \end{aligned}$$

$$\begin{aligned}
& -2 \sum_{j,k,l,\beta,\gamma,\delta,\eta,\tau,Q} f_j^\gamma f_k^\delta f_j^\eta (\mathcal{J}^Q)_\tau{}^\beta f_l^\tau (J^Q)_k{}^l R_{\beta\gamma\delta\eta}^N \\
& = 3 \sum_{j,k,\beta,\gamma,\delta,\eta} f_j^\gamma f_k^\delta f_j^\eta f_k^\beta R_{\beta\gamma\delta\eta}^N - 2 \sum_{j,k,\beta,\gamma,\delta,\eta} f_j^\gamma f_k^\delta f_j^\eta \psi_k{}^\beta R_{\beta\gamma\delta\eta}^N. \quad (2.7)
\end{aligned}$$

Insert (2.5), (2.7) into (2.4):

$$\begin{aligned}
\sum_{i,j,\alpha} \phi_i^\alpha \phi_{ijj}^\alpha & = -4 \sum_{i,j,\beta,\gamma,\delta,\eta} f_i^\gamma f_i^\delta f_j^\beta f_j^\eta R_{\gamma\beta\delta\eta}^N + 4 \sum_{i,j,\beta,\gamma,\delta,\eta} \psi_i^\gamma f_i^\delta f_j^\beta f_j^\eta R_{\gamma\beta\delta\eta}^N \\
& = -4 \sum_{i,j,\beta,\gamma,\delta,\eta} \phi_i^\gamma f_i^\delta f_j^\beta f_j^\eta R_{\gamma\beta\delta\eta}^N.
\end{aligned}$$

This completes the proof of the first part of the theorem.

Now assume f is a harmonic morphism. A result in [7] and [8] asserts that f is harmonic and weakly horizontal conformal, where the latter means: for each $p \in M$, either $df_p = 0$ or df_p maps the horizontal space $(\ker df_p)^\perp$ conformally onto $T_{f(p)}N$, i.e., df_p is surjective and there exists a number $\kappa(p) \neq 0$ such that

$$h(df_p(X), df_p(Y)) = \kappa(p)^2 g(X, Y), \quad \forall X, Y \in (\ker df_p)^\perp.$$

Then at any point $p \in M$, either $|\nabla f|^2(p) = 0$ hence $R^N(\phi \cdot e_i, df \cdot e_j, df \cdot e_i, df \cdot e_j) = 0$, or there is an orthonormal set of tangent vectors $e_1, \dots, e_{4n} \in (\ker df_p)^\perp$ such that

$$df \cdot e_i = |\nabla f|^2 w_i$$

form an orthonormal basis for $T_{f(p)}N$. In this case,

$$\sum_j R^N(\phi \cdot e_i, df \cdot e_j, df \cdot e_i, df \cdot e_j) = |\nabla f|^2 \text{Ric}^N(\phi \cdot e_i, df \cdot e_i) = 0$$

as N is Ricci-flat. Therefore,

$$\Delta|\phi|^2 = 2|\nabla\phi|^2.$$

When M is compact, the subharmonic function $|\phi|^2$ is constant, hence vanishes identically on M if it does so at one point. $\square$

By a similar argument, we can show

Corollary 2.1 *Let M and N be two hyperkähler manifolds. Assume that M is compact and N is flat. Then f is a quaternionic map if it satisfies (1.1) at one point on M .*

3 A Homotopy Condition for Smooth Mappings Being Quaternionic

In this section, we will prove Theorem 1.2.

The energy functional on $C^\infty(M, N)$ is

$$E(u) = \frac{1}{2} \|du\|^2 = \frac{1}{2} \int_M \sum_{i,j,\alpha,\beta} g^{ij} h_{\alpha\beta} \partial_i u^\alpha \partial_j u^\beta dV$$

where g, h are the Riemannian metrics on M, N respectively and dV is the volume element on M . Define a functional $E_T : C^\infty(M, N) \times SO(3) \rightarrow \mathbb{R}$ by

$$E_T(u, A) = \sum_{P,Q} A_{PQ} \int_M \langle J^P, u^* \mathcal{J}^Q \rangle dV.$$

Fix two sets of complex structures $J^1, J^2, J^3 = J^1 J^2$ on M and $\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3 = \mathcal{J}^1 \mathcal{J}^2$ on N and define a functional $I : C^\infty(M, N) \times SO(3) \rightarrow \mathbb{R}$ by

$$I(u, A) = \frac{1}{2} \int_M \left| du - \sum_{P,Q} A_{PQ} \mathcal{J}^P \circ du \circ J^Q \right|^2 dV.$$

It is clear that $I(u) = 0$ if and only if u is a quaternionic map.

Each J^P or $\mathcal{J}^Q$ determines a Kähler form on M or N , and it is well-known [1, 5] that $\int_M \langle J^P, u^* \mathcal{J}^Q \rangle dV$ only depends on the homotopy class of u . Hence, for any given $A \in SO(3)$, $E_T(u, A)$ is homotopy invariant which means that, if u is homotopy to v then $E_T(u, A) = E_T(v, A)$. The following proposition is first proved in [6], see also [2] and [9].

Proposition 3.1 *For any smooth map $u : M \rightarrow N$ and $A \in SO(3)$, we have*

$$E(u) + E_T(u, A) = \frac{1}{4} I(u, A). \quad (3.1)$$

If u is a quaternionic map, then it minimizes energy in its homotopy class.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 By Proposition 3.1 and $I(u, A) \geq 0$ for all $A \in SO(3)$, it is clear that if u solves the quaternionic equation (1.1) with $A \in SO(3)$, then

$$\begin{aligned} E(u) &= \min_{A \in SO(3)} - \sum_{P,Q} A_{PQ} \int_M \langle J^P, u^* \mathcal{J}^Q \rangle dV \\ &= \min_{A \in SO(3)} \sum_{P,Q} A_{PQ} B_{PQ} \\ &= \min_{A \in SO(3)} \operatorname{tr} AB^t. \end{aligned}$$

For the constraint $A \in SO(3)$, the Lagrange multiplier method yields: for extreme values

$$\frac{\partial}{\partial A_{PQ}} \left(\sum_{S,T} A_{ST} B_{ST} + \sum_{S,T} \lambda_{ST} (\delta_{ST} - A_{SK} A_{TK}) \right) = 0 \quad (3.2)$$

and for local minima

$$\frac{\partial^2}{\partial A_{PQ} \partial A_{UV}} \left(\sum_{S,T} A_{ST} B_{ST} + \sum_{S,T} \lambda_{ST} \left(\delta_{ST} - \sum_K A_{SK} A_{TK} \right) \right) \geq 0. \quad (3.3)$$

By (3.2), we get

$$B_{PQ} = \sum_K \lambda_{PK} A_{KQ} + \sum_K \lambda_{KP} A_{KQ}.$$

Let λ be the 3×3 matrix with entries λ_{PQ} and let $\Lambda = \lambda + \lambda^t$. The last identity can be rewritten as

$$B = \Lambda A.$$

The inequality (3.3) implies that Λ is nonnegative definite, in turn, it implies that $\det B \geq 0$. In summary, u is a quaternionic map only if

$$\begin{cases} \Lambda A = B, \\ \operatorname{tr} \Lambda = E(u), \\ \det B \geq 0. \end{cases} \quad (3.4)$$

It follows that for a quaternionic map u , we have by (3.4)

$$BB^t = \Lambda A A^t \Lambda = \Lambda^2,$$

therefore,

$$\operatorname{tr} \sqrt{BB^t} = \operatorname{tr} \Lambda = E(u).$$

We have shown that a quaternionic map satisfies (1.2).

Conversely, assume (1.2) holds for a smooth map u . There exists $S \in SO(3)$ diagonalizing the symmetric matrix

$$BB^t = S^t \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} S \quad (3.5)$$

where μ_1, μ_2, μ_3 are nonnegative since BB^t is semi-positive definite. Then define

$$\Lambda = S^t \begin{pmatrix} \sqrt{\mu_1} & 0 & 0 \\ 0 & \sqrt{\mu_2} & 0 \\ 0 & 0 & \sqrt{\mu_3} \end{pmatrix} S. \quad (3.6)$$

We claim that there exists $A \in SO(3)$ which solves the equation

$$\Lambda A = B. \quad (3.7)$$

To see this, we consider two cases.

First, assume $\det \Lambda \neq 0$. Take $A = \Lambda^{-1}B$ and we check directly from (3.5) and (3.6) that $A \in SO(3)$: $AA^t = \Lambda^{-1}BB^t(\Lambda^{-1})^t = I_{3 \times 3}$ and $\det A > 0$ since $\det B > 0$ and $\det \Lambda > 0$.

Second, assume $\det \Lambda = 0$. There are three cases.

Case (i): $\mu_3 = 0, \mu_1 \neq 0, \mu_2 \neq 0$. Equation (3.7) is equivalent to

$$\begin{pmatrix} \sqrt{\mu_1} & 0 & 0 \\ 0 & \sqrt{\mu_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} SA = SB. \quad (3.8)$$

Set $SB = (C_{PQ})_{3 \times 3}$. By (3.5) and $\mu_3 = 0$, we have

$$C_{3P} = 0 \quad (3.9)$$

and

$$CC^t = SBB^tS^t = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.10)$$

We then define

$$D_{1P} = \frac{1}{\sqrt{\mu_1}}C_{1P} \quad \text{and} \quad D_{2P} = \frac{1}{\sqrt{\mu_2}}C_{2P},$$

and it follows from (3.10) that $(D_{11}, D_{12}, D_{13}), (D_{21}, D_{22}, D_{23})$ are mutually orthogonal unit vectors and we can choose D_{3P} as the third row together with D_{1P}, D_{2P} to form a matrix D in $SO(3)$. It then follows from (3.9) that $SA = D$ solves (3.8).

Case (ii): $\mu_1 \neq 0, \mu_2 = \mu_3 = 0$. Observe (3.8) is further simplified by setting $\mu_2 = 0$, and consequently $C_{2P} = C_{3P} = 0$ and we can choose D_{2P}, D_{3P} , together with $D_{1P} = C_{1P}/\sqrt{\mu_1}$ to form $D \in SO(3)$ and then solve (3.8) by $SA = D$.

Case (iii): $\mu_1 = \mu_2 = \mu_3 = 0$. In this case, $\Lambda = B = 0$ and any A solves (3.7).

Therefore, we have shown that the above claim holds.

The solvability of (3.7) and the equality in the assumption (1.2) assert

$$\begin{aligned} E_T(u, A) &= - \sum_{P, Q} A_{PQ} B_{PQ} \\ &= - \sum_{P, Q} (A^t)_{QP} B_{PQ} \\ &= - \sum_{P < Q} (A^{-1})_{QP} B_{PQ} \\ &= - \sum_P \Lambda_{PP} \\ &= - \operatorname{tr} \Lambda \\ &= -E(u). \end{aligned}$$

Now it follows from Proposition 3.1 that $I(u, A) = 0$, hence u is a solution to the quaternionic equation (1.1) with the above A . $\square$

We recall the following criterion [3] which detects when a quaternionic map becomes holomorphic with respect to some complex structures in the hyperkähler $\mathbb{S}^2$.

Theorem 3.1 *Suppose u is a quaternionic map. Let B be as in Theorem 1.2. Then*

$$(\operatorname{tr} B)^2 \geq \max\{\text{eigenvalues of } BB^t\}$$

and the equality holds if and only if u is a holomorphic map with respect to some complex structures in the hyperkähler $\mathbb{S}^2$ on M and on N .

Combining Theorems 1.2 and 3.1, we obtain a homotopy condition which detects which homotopy class admits a holomorphic map between hyperkähler manifolds.

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