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Partial C^0-Estimate for Kähler–Einstein Metrics

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Abstract In this short note, we give a proof of our partial C^0-estimate for Kähler–Einstein metrics. Our proof uses a compactness theorem of Cheeger–Colding–Tian and L^2-estimate for $\bar\partial$-operator.

Keywords Kähler–Einstein metrics · Gromov–Hausdorff limit · Partial C^0-estimate

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1 Introduction

In this paper, we give a proof of our conjecture on the partial C^0-estimate for Kähler–Einstein metrics with positive scalar curvature. As a corollary, as we already pointed out in [9], the Gromov–Hausdorff limits of Kähler–Einstein manifolds are projective varieties.

Let M be a compact Kähler manifold with positive first Chern class $c_1(M)$. We denote by $\mathcal{K}(M)$ the set of all Kähler metrics ω with Kähler class $[\omega] = c_1(M)$. Consider its set

$$\mathcal{K}(M, t_0) = \big\{\omega \in \mathcal{K}(M) \mid \mathrm{Ric}(\omega) \geq t_0\omega\big\}.$$

Clearly, $\mathcal{K}(M, t_0)$ is empty unless $t_0 \leq 1$ and $\mathcal{K}(M, 1)$ is the set of Kähler–Einstein metrics on M with Kähler class $c_1(M)$.

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By the Kodaira embedding theorem, for ℓ sufficiently large, any basis of $H^0(M, K_M^{-\ell})$ embeds M into a projective space $\mathbb{C}P^N$. For any $\omega \in \mathcal{K}(M, t_0)$, choose a Hermitian metric h with ω as its curvature form and any orthonormal basis $\{\sigma_i\}_{0\le i\le N}$ of each $H^0(M, K_M^{-\ell})$ with respect to the induced inner product by h and ω. Put

$$\rho_{\omega,\ell}(x) = \sum_{i=0}^{N} \|\sigma_i\|_h^2(x). \tag{1.1}$$

This is independent of the choice of h and the orthonormal basis $\{\sigma_i\}$.

Conjecture 1.1 [8] *There are uniform constants $c_k = c(k, n) > 0$ for $k \ge 1$ and $\ell_i \to \infty$ such that for any $\omega \in \mathcal{K}(M, t_0)$ and $\ell = \ell_i$ for each i,*

$$\rho_{\omega,\ell} \ge c_\ell > 0. \tag{1.2}$$

Remark 1.2 In fact, I expect the stronger version of Conjecture 1.1: There are uniform constants $c_k = c(k, n) > 0$ for $k \ge 1$ and $\ell_0 = \ell_0(n)$ such that for any $\omega \in \mathcal{K}(M, t_0)$, and $\ell \ge \ell_0$, $\rho_{\omega,\ell} \ge c_\ell$.

The resolution of this conjecture will lead to a proof of the Yau–Tian–Donaldson conjecture: If M is K-stable for a sufficiently large ℓ, then M admits a Kähler–Einstein metrics. For recent progress, see [11].

If ω_i is a sequence of Kähler metrics on M with $[\omega_i] = c_1(M)$ and their Ricci curvature greater than or equal to $t_0 > 0$, then by taking a subsequence if necessary, we may assume that (M, ω_i) converge to a length space (M_∞, d_∞). On the other hand, for ℓ sufficiently large, we have embeddings $\psi_i : M \hookrightarrow \mathbb{C}P^N$ by an orthonormal basis of $H^0(M, K_M^{-\ell})$ with respect to ω_i. By taking a subsequence if necessary, we may assume that $\psi_i(M) \subset \mathbb{C}P^N$ converge to a holomorphic cycle $\bar{M}_\infty \subset \mathbb{C}P^N$. It was known (see [9]) that the irreducibility of $\bar{M}_\infty$ implies Conjecture 1.1.

We have expected since the early 1990s:

Conjecture 1.3 *The Gromov–Hausdorff limit M_∞ coincides with the complex limit $\bar{M}_\infty$. In particular, $\bar{M}_\infty$ is irreducible.*

Our main theorem of this paper is to confirm Conjecture 1.1 for $\mathcal{K}(M, 1)$, precisely.

Theorem 1.4 *There are a positive constant $\epsilon = \epsilon(n) > 0$ and sufficiently large $\ell = \ell(n)$ such that $\rho_{\omega,\ell} \ge \epsilon$ for all $\omega \in \mathcal{K}(M, 1)$.*

It is known that for each n, there are only finitely many family of compact Kähler manifolds of complex dimension n and with positive first Chern class. Hence, Theorem 1.4 holds for all Kähler–Einstein manifolds of dimension n.

Let (M_i, ω_i) be any sequence of Kähler–Einstein manifolds with $\mathrm{Ric}(\omega_i) = \omega_i$ and which converges to (M_∞, d_∞) in the Gromov–Hausdorff topology. Theorem 1.4 follows from the following:

Theorem 1.5 *There are a positive constant $c > 0$ and sufficiently large $\ell = \ell(n)$ such that $\rho_{\omega_i,\ell} \geq c$ for all (M_i, ω_i).*

It follows from [1] that there is a closed subset $\mathcal{S} \subset M_\infty$ of Hausdorff codimension at least 4 such that $M_\infty \backslash \mathcal{S}$ is a smooth Kähler manifold and d_∞ is induced by a Kähler–Einstein metric ω_∞ outside $\mathcal{S}$ with $\mathrm{Ric}(\omega_\infty) = \omega_\infty$. Moreover, ω_i converges to ω_∞ in the C^∞-topology outside $\mathcal{S}$.

A consequence of the above theorem implies (as indicated in [9]):

Theorem 1.6 *The Gromov–Hausdorff limit M_∞ is a variety embedded in some $\mathbb{C}P^N$ and $\mathcal{S}$ is a subvariety.*

In particular, this theorem affirms Conjecture 1.3 for Kähler–Einstein metrics. The proof of this theorem is based on the same arguments as those in the proof of Theorem 1.4, that is, constructing holomorphic sections which separate points for the limit of embeddings ψ_i. A proof based on Theorem 1.4 was already given by Chi Li in his thesis. We will omit the details in this note (cf. [11, Sect. 5]).

The main theorems above were announced with an outlined proof in our expository paper [10] submitted for a proceeding last June. I also lectured on these theorems in several conferences in last June and July. Meanwhile, I learned that Donaldson and Sun had posted a preprint [4] last June in which they also gave an independent proof of these two theorems. This note just provides a proof of the first theorem following the arguments in [10]. The second theorem follows easily as we indicated above. The first version of this note was completed last July. In this newer version, I made some simplifications and fixed some notations.

It is also worth mentioning that I gave a complete solution for the YTD conjecture in the case of Fano manifolds (see [11]) last October. My solution relies on establishing the partial C^0-estimate for conic Kähler–Einstein metrics.

2 Proving Theorem 1.5

The proof of Theorem 1.5 is essentially a localized version of the proof for the following:

Proposition 2.1 *By taking a subsequence if necessary, for each ℓ, we see that $H^0(M_i, K_{M_i}^{-\ell})$ converges to $H^0(M_\infty, K_{M_\infty}^{-\ell})$ as i tends to ∞ in the sense: There are orthonormal bases $\{\sigma_a^i\}_{0\leq a\leq N}$ of $H^0(M_i, K_{M_i}^{-\ell})$ with respect to h_i such that σ_a^i converges to σ_a^∞ $(0 \leq a \leq N)$ as i tends to ∞ and $\{\sigma_a^\infty\}$ forms an orthonormal basis of $H^0(M_\infty, K_{M_\infty}^{-\ell})$.*

As in [7], we prove this by using the L^2-estimate for $\bar\partial$-operator and the theory for elliptic equations. Next we recall from [8]:

Lemma 2.2 *For each i and any $\sigma \in H^0(M_i, K_{M_i}^{-\ell})$, we have the following identities:*

$$\Delta_{\omega_i} \|\sigma\|^2 = \|\nabla\sigma\|^2 - n\ell\|\sigma\|^2 \tag{2.1}$$

and

$$\Delta_{\omega_i}\|\nabla\sigma\|^2 = \left\|\nabla^2\sigma\right\|^2 - \big((n+2)\ell - 1\big)\|\nabla\sigma\|^2. \tag{2.2}$$

Corollary 2.3 *There is a uniform constant C_0 such that for any $\sigma \in H^0(M_i, K_{M_i}^{-\ell})$ ($\ell > 0$), we have*

$$\sup_{M_i}\|\sigma\| \le C_0\ell^{\frac{n}{2}}\left(\int_{M_i}\|\sigma\|^2\omega_i^n\right)^{\frac{1}{2}} \tag{2.3}$$

and

$$\sup_{M_i}\|\nabla\sigma\| \le C_0\ell^{\frac{n+1}{2}}\left(\int_{M_i}\|\sigma\|^2\omega_i^n\right)^{\frac{1}{2}}. \tag{2.4}$$

It follows from Lemma 2.2 and the standard Moser iteration since the Sobolev constants of (M_i, ω_i) are uniformly bounded due to some results of Croke and Li (see [3, 5, 6]).

It follows that by taking a subsequence if necessary, we may assume σ_a^i converges to a σ_a^∞ as i tends to ∞. Furthermore, one can show that $\rho_{\omega_i,\ell}$ are uniformly continuous and converge to $\rho_{\omega_\infty,\ell}$ which is also continuous on M_∞.

Thus, in order to prove Theorem 1.5, we only need to show

$$\inf_x \rho_{\omega_\infty,\ell}(x) > 0. \tag{2.5}$$

Since $\rho_{\omega_\infty,\ell}$ is continuous and M_∞ is compact, it suffices to show that for any $x \in M_\infty$, there is $\ell = \ell_x$ such that

$$\rho_{\omega_\infty,\ell}(x) > 0. \tag{2.6}$$

This can be achieved by using the L^2-estimate and the structure results on M_∞ from [1].

According to [1], for any $r_i \mapsto 0$, by taking a subsequence if necessary, we have a tangent cone $\mathcal{C}_x$ of $(M_\infty, \omega_\infty)$ at x, where $\mathcal{C}_x$ is the limit $\lim_{i\to\infty}(M_\infty, r_i^{-2}\omega_\infty, x)$ in the Gromov–Hausdorff topology, satisfying:

(i) $\mathcal{C}_x$ is a Kähler cone with vertex o;
(ii) Each $\mathcal{C}_x$ is regular outside a closed subcone $\mathcal{S}_x$ of complex codimension at least 2. Such a $\mathcal{S}_x$ is the singular set of $\mathcal{C}_x$;
(iii) There is a natural Kähler Ricci-flat metric g_x on $\mathcal{C}_x \backslash \mathcal{S}_x$ which is also a cone metric.

Since g_x is a Kähler cone metric, its Kähler form ω_x is equal to $\sqrt{-1}\partial\bar\partial\rho_x^2$ on the regular part of $\mathcal{C}_x$, where ρ_x denotes the distance function from the vertex of $\mathcal{C}_x$, denoted by x for simplicity. In other words, the trivial bundle $L_x = \mathcal{C}_x \times \mathbb{C}$ over $\mathcal{C}_x$ admits a Hermitian metric $e^{-\rho_x^2}|\cdot|^2$ whose curvature is ω_x.

Without loss of generality, we may choose r_i such that $k_i = r_i^{-2}$ are integers.

Now we fix some notations: For any $r, \delta, \epsilon > 0$, we denote by $V(x;\delta,\epsilon,r)$ the set

$$\left\{y \in \mathcal{C}_x \mid \delta < d(o,y) < r,\ d(y,\mathcal{S}_x) > \epsilon\right\},$$

where $B_R(o, g_x)$ denotes the geodesic ball of $(\mathcal{C}_x, g_x)$ centered at the vertex o and with radius R.

If $\mathcal{C}_x$ has isolated singularity, i.e., $\partial B_1(x)$ is smooth, then we can drop ϵ and write

$$V(x; \delta, r) = \{y \in \mathcal{C}_x \mid \delta < d(o, y) < r\}.$$

Let k_i be the above sequence such that $(M_\infty, k_i\omega_\infty, x)$ converges to $(\mathcal{C}_x, g_x, x)$. By [1], for any given $\delta, \epsilon > 0$, whenever i is sufficiently large, there are diffeomorphisms $\phi_i : V(x; \delta, \epsilon, 2) \mapsto M_\infty \backslash \mathcal{S}$, where $\mathcal{S}$ is the singular set of M_∞, satisfying:

(i) $d(x, \phi_i(V(x; \delta, \epsilon, 2))) < \delta r_i$ and $\phi_i(V(x; \delta, \epsilon, 2)) \subset B_{3r_i}(x)$, where $B_R(x)$ the geodesic ball of $(M_\infty, \omega_\infty)$ with radius R and center at x;
(ii) If g_∞ is the Kähler metric with the Kähler form ω_∞ on $M_\infty \backslash \mathcal{S}$, then

$$\lim_{i\to\infty} \|r_i^{-2}\phi_i^* g_\infty - g_x\|_{C^6(V(x;\delta/2,\epsilon/2,3))} = 0, \tag{2.7}$$

where the norm is defined in terms of the metric g_x.

Lemma 2.4 *There is an integer $a > 0$ such that for i sufficiently large, there are $\delta_i \mapsto 0$*[1] *and isomorphisms ψ_i from the trivial bundle $\mathcal{C}_x \times \mathbb{C}$ onto $K_{M_\infty}^{-ak_i}$ over $V(x; \delta, \epsilon, 2)$ commuting with ϕ_i satisfying*:

$$\|\psi_i(1)\|^2 = e^{-\rho_x^2} \quad \text{and} \quad \|D\psi_i\|_{C^4} \le \delta_i, \tag{2.8}$$

where $\|\cdot\|$ denotes the induced norm on $K_{M_\infty}^{-k_i}$ by g_x, D denotes the covariant derivative with respect to the norms $\|\cdot\|$ and $e^{-\rho_x^2/2}|\cdot|$.

Proof First we note that by a recent result of Colding–Naber [2], the regular part $\mathrm{Reg}(\mathcal{C}_x)$ of $\mathcal{C}_x$ is geodesically convex, so its universal cover $\widetilde{\mathrm{Reg}}(\mathcal{C}_x)$ is a finite cover, say of the order $a \ge 1$. Since g_x is a Kähler Ricci-flat metric on $\mathcal{C}_x$, the induced metric on $\widetilde{\mathrm{Reg}}(\mathcal{C}_x)$ has its holonomy group in $\mathrm{SU}(n)$ and consequently, $K_{\mathcal{C}_x}^{-a}$ admits a parallel section. It follows that for i sufficiently large, $K_{M_\infty}^{-a}$ is trivial over $\phi_i(V)$ for any open subset V whose closure is contained in $\mathrm{Reg}(\mathcal{C}_x)$. This makes it plausible to construct the isomorphisms ψ_i. For simplicity, we assume $a = 1$ or equivalently, $\mathrm{Reg}(\mathcal{C}_x)$ is simply connected, otherwise, we do the following on its universal cover $\widetilde{\mathrm{Reg}}(\mathcal{C}_x)$. We cover $V(x; \delta, \epsilon, 2)$ by finitely many geodesic balls $B_{s_\alpha}(y_\alpha)$ $(1 \le \alpha \le N)$ such that the closure of each $B_{2s_\alpha}(y_\alpha)$ is strongly convex and contained in $\mathrm{Reg}(\mathcal{C}_x)$. Now we construct ψ_i. For simplicity of notations, we fix a sufficiently large i and write $r = r_i$, $k = k_i$ etc. First we construct $\tilde{\psi}_\alpha$ over each $B_{2s_\alpha}(y_\alpha)$. For any $y \in B_{2s_\alpha}(y_\alpha)$, let $\gamma_y \subset B_{2s_\alpha}(y_\alpha)$ be the unique minimizing geodesic from y_α to y. We define $\tilde{\psi}_\alpha$ as follows: First we define $\tilde{\psi}_\alpha(1) \in L_i|_{\phi(y_\alpha)}$, where $L_i = K_{M_\infty}^{-k}$, such that $\|\psi(1)\|^2 = e^{-\rho_x^2(y_\alpha)}$. Next, for any $y \in U_\alpha$, where $U_\alpha = B_{2s_\alpha}(y_\alpha)$, define $\tilde{\psi}_\alpha : \mathbb{C} \mapsto L_i|_y$ by $\tilde{\psi}_\alpha(a(y)) = \tau(\phi(y))$, where $a(y)$ is the parallel transport of 1 along γ_y with respect to the standard norm above and $\tau(\phi(y))$ is the parallel transport of $\psi(1)$

[1] In fact, δ_i can be chosen depending only on $\|k_i\phi^* g_\infty - g_x\|_{C^6(V(x;\delta/2,\epsilon/2,3))}$.

along $\phi \cdot \gamma_y$. Clearly, we have the first equation in (2.8). The estimates on derivatives can be done as follows: If $a : U_\alpha \mapsto U_\alpha \times \mathbb{C}$ and $\tau : U_\alpha \mapsto \phi^* L|_{U_\alpha}$ are two sections such that $\tilde{\psi}_\alpha(a) = \tau$, then we have the identity: $D\tau = D\tilde{\psi}_\alpha(a) + \tilde{\psi}_\alpha(Da)$, where D denotes the covariant derivatives with respect to various norms on line bundles. By the definition, one can easily see that $D\tilde{\psi}_\alpha(y_\alpha) \equiv 0$. To estimate $D\tilde{\psi}_\alpha$ at y, we differentiate along γ_y to get

$$D_T D_X \tau = D_T \big(D_X \tilde{\psi}_\alpha(a)\big) + \tilde{\psi}_\alpha(D_T D_X a),$$

where T is the unit tangent of γ_y and X is a vector field along γ_y with $[T, X] = 0$. Here we have used the fact that $D_T \tilde{\psi}_\alpha = 0$ which follows from the definition. Using the curvature formula, we see that it is the same as

$$k_i \phi^* \omega_\infty(T, X) \tilde{\psi}_\alpha(a) = D_T \big(D_X \tilde{\psi}_\alpha(a)\big) + \omega_x(T, X) a.$$

Using

$$\lim_{i \to \infty} k_i \phi^* \omega_\infty = \lambda_i \phi^* \omega_\infty = \omega_x,$$

we deduce from the above that $D_T(D_X \tilde{\psi}_\alpha(a))$ converges to 0 as i tends to ∞. Since $D_X \tilde{\psi}_\alpha = 0$ at y_α, we see that $\|D\tilde{\psi}_\alpha\|_{C^0(U)}$ can be made sufficiently small. The higher derivatives can be bounded in a similar way.

Next we want to modify each $\tilde{\psi}_\alpha$ to get the required $\psi = \psi_i$. For any α, β, we set

$$\theta_{\alpha\beta} = \tilde{\psi}_\alpha^{-1} \circ \tilde{\psi}_\beta : U_\alpha \cap U_\beta \mapsto S^1.$$

Clearly, $\theta_{\alpha\gamma} = \theta_{\alpha\beta} \cdot \theta_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, so we have a closed cycle $\{\theta_{\alpha\beta}\}$. Using the fact that $K_{M_\infty}^{-k}$ is trivial over $\phi_i(V)$ for any $V \Subset \mathrm{Reg}(\mathcal{C}_x)$, one can deduce from the simply connectedness that this cycle has to be exact, moreover, by replacing each U_α by $B_{s_\alpha}(x_\alpha)$, we can construct $\zeta_\alpha : B_{s_\alpha}(x_\alpha) \mapsto S^1$ satisfying:

(i) $\tilde{\psi}_\alpha \cdot \zeta_\alpha = \tilde{\psi}_\beta \cdot \zeta_\beta$ on $B_{s_\alpha}(x_\alpha) \cap B_{s_\beta}(x_\beta)$;

(ii) $\|D\zeta_\alpha\|_{C^3(B_{s_\alpha}(x_\alpha))}$ is dominated by $\|D\tilde{\psi}_\alpha\|_{C^3(U)}$.

Then we can get ψ by setting $\psi = \tilde{\psi}_\alpha \cdot \zeta_\alpha$ on $V(x; \delta, \epsilon, 2) \cap B_{s_\alpha}(x_\alpha)$. It is easy to show that this ψ has all the properties we asked. □

We also have the following extension property:

Lemma 2.5 *Given any $c > 0$, there are small $\delta > \epsilon > 0$, which may depend on c and the cone $\mathcal{C}_x$,*[2] *such that for any holomorphic function f on $V(x; \delta, \epsilon, 2)$ with $|f|, |df| \le c$ and $\mathrm{Re}(f) \ge 1$ on $\partial B_1(o, g_x) \cap V(x; 1/2, \delta^{\frac{1}{4n(n+1)}}, 2)$, we have*

$$\mathrm{Re}(f)(\sqrt{\delta} v) \ge \frac{3}{4},$$

where $v \in \partial B_1(o, g_x)$ satisfying $d(v, \mathcal{S}_x \cap \partial B_1(x, g_x)) \ge 1/100$.

[2] It is possible to remove the dependence on $\mathcal{C}_x$ by using known estimates on Green function over manifolds with nonnegative Ricci curvature.

Proof This is proved by using the Green function on $B_1(o, g_x)$ with boundary value 0. Let η be a cut-off function satisfying: $\eta(t) = 0$ for $t \leq 1$, $\eta(t) = 1$ for $t \geq 2$ and $|\eta'(t)| \leq 1$. Put

$$\zeta_1(y) = \eta\big(\delta^{-1} d(o, y)\big) \quad \text{and} \quad \zeta_2 = \eta\big(\epsilon^{-1} d(y, \mathcal{S}_x)\big).$$

Define

$$F = \zeta_1 \zeta_2 \mathrm{Re}(f).$$

Then F vanishes near the singularity of $\mathcal{C}_x$ and smooth on $\mathrm{Reg}(\mathcal{C}_x)$. Moreover,

$$\Delta F = 2\nabla(\zeta_1 \cdot \zeta_2) \cdot \nabla \mathrm{Re}(f) + \Delta(\zeta_1 \cdot \zeta_2)\mathrm{Re}(f).$$

Clearly, it implies that ΔF supports in $V(x; \delta, \epsilon, 2) \backslash V(x; 2\delta, 2\epsilon, 2)$ and for some uniform constant C', we have

$$|\Delta F| \leq C'\big(\delta^{-2} + \epsilon^{-2}\big) \quad \text{on } V(x; \delta, \epsilon, 2) \backslash V(x; 2\delta, 2\epsilon, 2).$$

If $G(\cdot, \cdot)$ denotes the Green function on $B_1(o, g_x)$ with boundary value 0, then we have

$$F(y) = \int_{\partial B_1(o, g_x)} F(z) \frac{\partial G(y, z)}{\partial \nu} \, dz - \int_{B_1(o, g_x)} \Delta F(z) G(y, z) \, dz. \tag{2.9}$$

Note that if δ is sufficiently small, the first integral on the right can be bounded from below by

$$1 - c \int_{\partial B_1(o, g_x) \backslash V(x; 1/2, \delta^{\frac{1}{4n(n+1)}}, 2)} \frac{\partial G(y, z)}{\partial \nu} \, dz \geq 1 - \frac{1}{8}.$$

On the other hand, if $y = \sqrt{\delta} v$ as above, we have

$$\big|G(y, z)\big| \leq C'' \delta^{-(n-1)}, \quad \forall z \in V(x; \delta, \epsilon, 2) \backslash V(x; 2\delta, 2\epsilon, 2),$$

where C'' is a uniform constant which may depend on $\mathcal{C}_x$. Since $\mathcal{S}_x$ is codimension at least 4,[3] by choosing ϵ sufficiently small, we can deduce from the above

$$\left| \int_{B_1(o, g_x)} \Delta F(z) G(y, z) \, dz \right| \leq \frac{1}{8}.$$

Then the lemma follows easily from (2.9). $\square$

Now we can apply the L^2-estimate to proving (2.6), and consequently, Theorem 1.5. Choose $0 < \epsilon < \delta$ small such that Lemma 2.5 holds. For each i sufficiently large, there is a section $\tau_i = \psi_i(e)$ of $K_{M_i}^{-k_i}$ on $\phi_i(V(x; \delta, \epsilon, 2))$ such that $\|\tau_i\|^2 = e^{1-\rho_x^2}$. Clearly, $\|\tau_i\|$ is greater than 1 inside $B_1(x)$ and less than 1 outside

[3] This is not crucial, by choosing ζ_2 more carefully, we can prove this lemma under the assumption that $\mathcal{C}_x$ arises from a limit of conic Kähler–Einstein metrics and has $\mathcal{S}_x$ of codimension 2.

$B_1(x)$. By Lemma 2.4, $\|\bar\partial\tau_i\| \le C\delta_i$ for some uniform constant C. Let η be the cut-off function satisfying: $\eta(t)=0$ for $t\le 1$, $\eta(t)=1$ for $t\ge 2$ and $|\eta'(t)|\le 1$. We define for any $y\in V(x;\delta,\epsilon,2)$,

$$\tilde\tau_i\big(\phi_i(y)\big)=\eta\big(7/2-d(o,y)\big)\eta\big(\delta^{-1}d(o,y)\big)\eta\big(\epsilon^{-1}d(y,\mathcal{S}_x)\big)\tau_i^l\big(\phi_i(y)\big). \tag{2.10}$$

Here l is a large integer which depends only on δ. Clearly, $\tilde\tau_i$ extends to a Lipschitz section of $K_{M_\infty}^{-lk_i}$ on M_∞ satisfying:

(i) $\tilde\tau_i$ coincides with τ_i^ℓ on $\phi_i(V(x;2\delta,2\epsilon,3/2))$;
(ii) $\int_{M_\infty}\|\bar\partial\tilde\tau_i\|^2\omega_\infty^n\le Cr_i^{2n-2}$.

Note that C, C' et al., always denote uniform constants. Set $\ell=lk_i$. By the L^2-estimate for $\bar\partial$-operator,[4] we get a section v_i of $K_{M_\infty}^{-\ell}$ such that $\bar\partial v_i=\bar\partial\tilde\tau_i$ and

$$\int_{M_\infty}\|v_i\|^2\omega_\infty^n\le\frac{1}{\ell}\int_{M_\infty}\|\bar\partial\tilde\tau_i\|^2\omega_\infty^n\le Cr_i^{2n-2}\ell^{-1}.$$

Then $\sigma_i=\tilde\tau_i-v_i$ is a holomorphic section of $K_{M_\infty}^{-\ell}$. By (i),

$$\bar\partial v_i=0\quad\text{on }\phi_i\big(V(x;2\delta,2\epsilon,3/2)\big),$$

then by applying the standard elliptic estimates, we can get

$$\|v_i\|^2(y)\le C\big(\delta^{\frac{1}{4n(n+1)}}r_i\big)^{-2n}\int_{M_\infty}\|v_i\|^2\omega_\infty^n\le C'\delta^{-\frac{1}{2(n+1)}}r_i^{-2}\ell^{-1}$$

for any $y\in\phi_i(V(x;1/2,\delta^{\frac{1}{4n(n+1)}},3/2))\cap\partial B_1(o,g_x)$.

We can make $\|v_i\|$ restricted to $\phi_i(V(x;1/2,\delta,3/2))\cap\partial B_1(o,g_x)$ as small as we want on so long as $(\delta l^{2(n+1)})^{-1}$ is small enough. On the other hand, since $\|\sigma_i\|$ are uniformly bounded, as i tends to ∞, σ_i restricted to $\phi_i(B_{3/2}(o,g_x))\backslash\mathcal{S}$ converges to a holomorphic function f on $B_{3/2}(o,g_x)$. By our construction, $\tilde\tau_i$ converges to 1 on $B_{3/2}(o,g_x)$. This implies that $\mathrm{Re}(f)\ge 1/2$ on $\partial B_1(o,g_x)$. Then by Lemma 2.5, we can show that if i is sufficiently large, $\|\sigma_i\|(\phi_i(\sqrt{\delta}v))\ge 1/4$ for some $v\in\partial B_1(o,g_x)$ with $d(v,\mathcal{S}_x)\ge 1/10$.

By applying the second estimate in Corollary 2.3 to σ_i, we get

$$\sup_{M_\infty}\|\nabla\sigma_i\|\le C''\ell^{\frac{n+1}{2}}\bigg(\int_{M_\infty}\|\sigma_i\|^2\omega_\infty^n\bigg)^{\frac12}\le\tilde C\delta^{-\frac14}r_i^{-1}.$$

Since the distance $d(x_i,\phi_i(\sqrt{\delta}v))$ is less than $2\sqrt{\delta}r_i$ for i sufficiently large, we deduce from the above estimates

$$\|\sigma_i\|(x_i)\ge 1/4-2\tilde C\delta^{\frac14}.$$

[4]We can do it directly on M_∞ by using the L^2-estimate on M_i and Proposition 2.1.

Therefore, if we choose δ sufficiently small, then $\rho_{\omega_\infty,\ell}(x) > 1/8$. The theorem is proved.

Remark 2.6 There is another way of constructing $\tilde{\tau}_i$. Instead of transplanting the constant function 1 on $V(x;\delta,\epsilon,2)$, one can do the same on $V(x;\delta,\epsilon,R)$ for a very large R. Then one can use the L^2-estimate to construct σ_i even without using Lemma 2.5.

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