Uniruled Symplectic Divisors

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Abstract In this article, we consider the problem of lifting the GW theory of a symplectic divisor to that of the ambient manifold in the context of symplectic birational geometry. In particular, we generalize Maulik–Pandharipande’s relative/absolute correspondence to relative-divisor/absolute correspondence. Then, we use it to lift a minimal uniruled invariant of a divisor to that of the ambient manifold.

Keywords Birational symplectic geometry · Gromov–Witten invariants · Symplectic divisor · Uniruled invariant

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1 Introduction

The relation between the ambient manifold Gromov–Witten theory and that of a smooth divisor is a basic problem in geometry and physics. It was clear from the history that this problem is a very complicated one and it is unlikely any simple formula can be obtained. However, there are a number of beautiful results computing the divisor invariants in terms of the ambient space invariants (see [2, 4, 5, 12, 25, 32]). In this article, we consider the opposite direction lifting an invariant of divisor to that of ambient manifold. On such a general ground one expects no good solution. Therefore, to have any reasonable answer, we have to choose our invariant

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carefully. Our choice is motivated by symplectic birational geometry.

In symplectic birational geometry, uniruledness is a central notion defined via GW invariants. More precisely, a symplectic manifold is called uniruled if there is a non-vanishing genus zero GW-invariant involving a point insertion and with a non-trivial class curve. In [9], we proved that uniruledness is invariant under symplectic birational cobordism. The next step of symplectic birational geometry program is to study various surgery operations such as contraction, flip and flop. The main perspective comes from the basic fact in the projective birational program that various birational surgery operations have a common feature: the subset being operated on is necessarily uniruled. Therefore, in our program we also need to understand uniruled symplectic submanifolds. In this article and its sequel, we focus on symplectic uniruled divisors. Our key observation is that, as in the projective birational program, such a divisor admits a dichotomy depending on the positivity of its normal bundle. If the normal bundle is non-negative in certain sense, it will force the ambient manifold to be uniruled. If the normal bundle is negative in certain sense, we can contract it to obtain a simpler symplectic manifold. In this article, we treat the case of non-negative normal bundles. In [21] the negative case will be dealt with in dimension 6. The non-negative case is essentially a problem to lift a uniruled GW-invariant of a divisor to that of the ambient space. It is surprising that this case does have a nice answer. It is still an interesting question if any other type of invariants can be thus lifted.

To state our main theorem we need the notion of a minimal uniruled class, which is a uniruled class with minimal symplectic area among all uniruled classes. Suppose that $\iota : D \rightarrow (X, \omega)$ is a symplectic submanifold of codimension 2, i.e., a smooth divisor. Let N_D be the normal bundle of D in X . Notice that N_D is a 2-dimensional symplectic vector bundle and hence has a well defined first Chern class. We will often use N_D to denote the first Chern class.

Theorem 1.1 *Suppose D is uniruled and A is a minimal uniruled class of D such that*

$$\langle \iota^* \alpha_1, \dots, \iota^* \alpha_l, [pt], \beta_2, \dots, \beta_k \rangle_{0,A}^D \neq 0 \quad (1)$$

for $k \leq N_D(A) + 1$. Then (X, ω) is uniruled.

In particular, we have the following corollary.

Corollary 1.2 *Suppose D is uniruled and the normal bundle N_D is non-negative on a minimal uniruled class. Then X is uniruled if either*

- *D is homologically injective or*
- *D is projectively uniruled.*

Another consequence of Theorem 1.1 is as follows.

Corollary 1.3 *Suppose that X is a non-uniruled manifold containing a uniruled divisor satisfying (1). Then, $k > N_D(A) + 1$. In particular, $N_D(A) < 0$ if $k = 1$.*

The above corollary is the first step towards the constructions of symplectic divisorial contraction in the third paper of the series [21].

The idea of proof is similar to that of [9]. We partition insertions of D into two types, global insertions $\iota^* \beta_i$ and local insertions α_j . The relative/absolute correspondence of Maulik–Pandharipande interchanges relative GW-invariants with certain admissible absolute GW-invariants having a similar partition of insertions. We extend the correspondence to include certain super-admissible GW-invariants. In addition, as in [4], in this case invariants of the divisor enters the “extended” relative/absolute correspondence in a non-trivial way. In fact, this is our main strategy to lift a minimal uniruled invariant of divisor to a uniruled invariant of ambient manifold.

We are informed by Aleksey Zinger that he is writing up a geometric argument, without the relative Gromov–Witten theory, for comparing GW invariants of a divisor and of the ambient space ([33]).

As a by product our main theorem also gives a rather general from divisor to ambient space inductive construction of uniruled symplectic manifolds.

Theorem 1.1 can be easily applied in a variety of situations, giving a comprehensive generalization of some early results of McDuff. For instance, if D is a Fano manifold with $b_2 = 1$ and non-negative normal bundle, then X must be uniruled. The case of $D = \mathbb{P}^{n-1}$ is studied by McDuff in [26, 27] (see Sect. 2 in this paper, specifically, Theorem 2.13). We list many more examples in Appendix. We should mention that another obvious inductive construction is from fiber to total space. In particular, a uniruled and homologically injective fiber in a symplectic fibration will force the total space to be uniruled (Corollary 2.10). Although it might be possible to derive this result as a consequence of our main theorem, it actually can be established by a simpler and classical argument.

The paper is organized as follows. In Sect. 2, we first review basic properties of uniruled symplectic manifolds. Then using direct geometric arguments we describe the fiber-to-total space approach, as well as some early examples of McDuff which motivate our divisor-to-ambient strategy. In Sect. 3, we sketch a relative-divisor/absolute correspondence to connect absolute invariants with relative invariants and invariants of divisor. The new ingredient is the appearance of invariants of the divisor corresponding to the additional super-admissible absolute invariants. We actually need to study a more delicate version of our correspondence involving a point insertion. We establish it in Sect. 4. This requires several powerful techniques in Gromov–Witten theory and extensive knowledge of relative invariants of $\mathbb{P}^1$ -bundles, which we summarize in the Appendix. The main theorem is also proved in Sect. 4. The applications will be studied in Sect. 5.

2 Uniruled Symplectic Manifolds

2.1 Definition and Basic Properties

Definition 2.1 Let $A \in H_2(X; \mathbb{Z})$ be a nonzero class. A is said to be a uniruled class if there is a nonzero GW invariant

$$([pt], \alpha_2, \dots, \alpha_k)_A^X \quad (2)$$

with a point insertion.

Let A be a uniruled class. A is said to be a strongly uniruled class if k can be chosen to be 3. A is said to be minimal if it has the smallest symplectic area among all uniruled classes.

Definition 2.2 X is said to be (symplectically) uniruled if there is a uniruled class, and strongly uniruled if there is a strongly uniruled class.

The notion of strongly uniruled is studied e.g., in [22–24, 28].

Remark 2.3 If the insertions in (2) are all of even degree, then the class is called an evenly uniruled class. This notion is studied in [28] and there is provided a beautiful characterizations of uniruledness in terms of units of the quantum cohomology ring.

It is easy to see that we could well use the more general disconnected GW invariants to define this concept. Moreover, we have the following basic property proved in [9].

Theorem 2.4 *A symplectic manifold X is uniruled if there is a nonzero disconnected genus zero descendant GW invariant involving a point constraint.*

This flexibility is important for the proof of another basic property also proved in [9].

Theorem 2.5 *Being uniruled is a birational cobordism property. In particular, if $\tilde{X}$ is a blow-up of X , then X is uniruled if and only if $\tilde{X}$ is uniruled.*

As mentioned in the introduction, the notion of symplectically uniruled is a natural extension of the fundamental notion in algebraic geometry. More precisely, for projective manifold, we have the following result of Kollár and the second author, strengthened by an observation of McDuff.

Theorem 2.6 *A projectively uniruled manifold is symplectically uniruled. Moreover, a minimal uniruled class with respect to an ample class is strongly uniruled with both additional insertion being powers of a Kähler form.*

In dimension 4 it follows from [16–18, 26] that the converse of Theorem 2.6 is essentially true.

Fano manifolds are (projectively) uniruled. The analogue of a Fano manifold in the symplectic category is a monotone symplectic manifold where $C_1 = \lambda\omega$ for $\lambda > 0$. It would be a challenging problem to show that any monotone symplectic manifold is indeed uniruled.

Another important class of examples is provided by the following beautiful result in [28].

Theorem 2.7 *Hamiltonian S^1 -manifolds are uniruled.*

Remark 2.8 It would be interesting to see whether there are uniruled manifolds such that every uniruled invariant must have odd degree insertions. Interestingly, such a manifold cannot be projective by Theorem 2.6. We are not aware of such manifolds. It seems that Hamiltonian S^1 -manifolds are a good case to investigate.

A rich source of uniruled manifolds comes from uniruled fibrations. Suppose that $\pi : X \rightarrow B$ is a fibration (with possibly singular fibers) where X and B are symplectic manifolds. We call it an almost complex fibration if there are tamed J, J' for X, B such that π is almost complex. For example, by the famous Thurston construction, if a symplectic fiber bundle over a symplectic manifold has fiber (F, σ) and $[\sigma]$ is a restriction class, then the total space X has a symplectic form Ω that restricts to σ on the fibers and hence is almost complex. Lefschetz fibrations, or more generally, locally holomorphic fibrations studied in [6] are also almost complex.

Let $\iota : \pi^{-1}(b) \rightarrow X$ be the embedding for a generic fiber over $b \in B$. We have the following.

Proposition 2.9 *Suppose that $\pi : X \rightarrow B$ is an almost complex fibration between symplectic manifolds X, B . Then, for $A \in H_2(\pi^{-1}(b); \mathbb{Z})$ and $\alpha_2, \dots, \alpha_k \in H^*(X; \mathbb{R})$,*

$$\langle [pt], \iota^* \alpha_2, \dots, \iota^* \alpha_k \rangle_A^{\pi^{-1}(b)} = \langle [pt], \alpha_2, \dots, \alpha_k \rangle_A^X. \quad (3)$$

Proof First of all, (3) makes sense as both invariants are well-defined at the same time. Choose J, J' such that π is almost complex. Suppose that $f : C \rightarrow X$ is a genus 0 stable map with homology class A . Then, $\pi \circ f$ is holomorphic with zero homology class. Therefore, $\text{im}(\pi \circ f)$ is a point. Namely, $\text{im}(f)$ is contained in a fiber. Choose a point $pt \in \pi^{-1}(b)$. Then we have the identification of the moduli spaces of k -marked genus zero stable curves with the first marked point going to pt ,

$$\overline{\mathcal{M}}_{0,k}^X(A, pt) = \overline{\mathcal{M}}_{0,k}^{\pi^{-1}(b)}(A, pt).$$

Furthermore, they have the same virtual fundamental cycles. As π is almost complex we have the splitting

$$f^*TX = f^*T\pi^{-1}(b) \oplus \underline{\mathbb{C}}^l,$$

where l is the codimension of a fiber and $\underline{\mathbb{C}}^l$ is the trivial complex bundle of dimension l . As C has genus zero, we have

$$H^1(C, \underline{\mathbb{C}}^l) = 0.$$

It implies that

$$[\overline{\mathcal{M}}_{0,k}^X(A, pt)]^{\text{vir}} = [\overline{\mathcal{M}}_{0,k}^{\pi^{-1}(b)}(A, pt)]^{\text{vir}}.$$

By integrating $\alpha_2, \dots, \alpha_k$ against the virtual fundamental cycles, we obtain (3). $\square$

Consequently, we have the following corollary.

Corollary 2.10 *Suppose that $\pi : X \rightarrow B$ is an almost complex fibration between symplectic manifolds X, B . If a smooth fiber is uniruled and homologically injective (over $\mathbb{R}$), then X is uniruled.*

The homologically injective assumption could be a strong one. Notice that for a fiber bundle, the Leray–Hirsch theorem asserts that, under the homologically injective assumption, the homology group of the total space is actually isomorphic to the product of the homology group of the fiber and the base. However, Corollary 2.10 can still be applied for all product bundles, and all projective space fibrations (more generally, if the rational cohomology ring of a smooth uniruled fiber is generated by the restriction of $[\omega]$).

Moreover, we were informed by McDuff that a Hamiltonian bundle is homologically injective (or equivalently, cohomologically split) if (cf. [19])

- (a) the base is S^2 (Lalonde–McDuff–Polterovich), and more generally, a complex blow-up of a product of projective spaces,
- (b) the fiber satisfies the hard Lefschetz condition (Blanchard), or its real cohomology is generated by H^2 .

Here is another variation. As in the case of a projective space, for a uniruled manifold up to dimension 4, insertions of a uniruled class can all be assumed to be of the form $[\omega]^i$, thus we also have the following corollary.

Corollary 2.11 *If the general fibers of a possibly singular uniruled fibration are 2-dimensional or 4-dimensional, then the total space is uniruled.*

This in particular applies to a 2-dimensional symplectic conic bundle. A symplectic conic bundle is a conic hypersurface bundle in a smooth $\mathbb{P}^k$ bundle. Holomorphic conic bundles are especially important in the theory of 3-folds. It is conjectured that a projective uniruled 3-fold is either birational to a trivial $\mathbb{P}^1$ -bundle or a conic bundle.

2.2 Some Motivating Examples

In this subsection, we present examples of uniruled manifolds from the divisor-to-ambient construction. These examples motivate Theorem 1.1 and generalize some early results of McDuff in a slightly different context. The common feature is that the geometric situation is simple enough that a direct geometric argument can be applied.

We start from the simplest situation of trivial normal bundles. Let $\iota : D \rightarrow X$ be a symplectic divisor. McDuff treated the case that D is a standard projective space and X is semi-positive, see Theorem 2.13. In general, we have the following theorem.

Theorem 2.12 *Suppose the normal bundle $N_D := N_{D|X}$ is trivial. If there is a nonzero invariant $\langle [pt], \iota^* \alpha_2, \dots, \iota^* \alpha_k \rangle_A^D$, then X is uniruled and in fact,*

$$\langle [pt], \alpha_2, \dots, \alpha_k \rangle_A^X = \langle [pt], \iota^* \alpha_2, \dots, \iota^* \alpha_k \rangle_A^D.$$

Proof The argument is parallel to that of Proposition 2.9. First notice that the triviality of the normal bundle implies that the

$$\mathrm{vir} \dim \overline{\mathcal{M}}_{0,k}^X(A) = \mathrm{vir} \dim \overline{\mathcal{M}}_{0,k}^D(A) + 2.$$

On the other hand, $\deg_X([pt]) = \deg_D([pt]) + 2$. Hence $\langle [pt], \tilde{\alpha}_2, \dots, \tilde{\alpha}_k \rangle_A^X$ is also defined.

We choose an ω -compatible almost structure j on D and extend it to an ω -compatible almost complex structure on X . Furthermore, we choose J in such a fashion that D has an almost complex product neighborhood.

Choose a point $pt \in D$. Let $\overline{\mathcal{M}}_{0,k}^X(A, pt)$ and $\overline{\mathcal{M}}_{0,k}^D(A, pt)$ be the moduli spaces of genus zero stable maps of homology class A such that $e(x_1) = pt$. Suppose that $f : C \rightarrow X$ is a genus zero stable map in $\overline{\mathcal{M}}_{0,k}^X(A, pt)$. It is well-known that any component of $\mathrm{im}(f)$ either lies in D or intersects D positively. One the other hand, $D \cdot A = 0$ by the assumption and $f(x_1) \in D$. Therefore, $\mathrm{im}(f)$ lies completely inside D . Namely, $\overline{\mathcal{M}}_{0,k}^X(A, pt) = \overline{\mathcal{M}}_{0,k}^D(A, pt)$.

Furthermore, $f^*TX = f^*TD \oplus \mathbb{C}$ and $H^1(C, \mathbb{C}) = 0$ imply that

$$[\overline{\mathcal{M}}_{0,k}^X(A, pt)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{0,k}^D(A, pt)]^{\mathrm{vir}}.$$

By integrating $\alpha_2, \dots, \alpha_k$ against the virtual fundamental cycles, we obtain

$$\langle [pt], \alpha_2, \dots, \alpha_k \rangle_A^X = \langle [pt], \iota^* \alpha_2, \dots, \iota^* \alpha_k \rangle_A^D.$$

Therefore X is uniruled. □

When the normal bundle is not trivial, the situation is more complicated. But the identification between appropriate GW-invariants of the divisor and the ambient manifold still remains to be valid in some cases. The following is a particular important example in [27] established in the early 1990s.

Theorem 2.13 *Let (X, ω) be a semi-positive symplectic $2n$ -manifold which contains a submanifold P symplectomorphic to $\mathbb{P}^{n-1}$ whose normal Chern number m is non-negative. Then a certain blow-up of X is uniruled, and if $0 \leq m \leq 2$ or $n = 2$, X itself is uniruled.*

As a consequence of Theorem 2.5, X itself is still uniruled even if $n > 1$ and $m \geq 3$.

The case of $D = \mathbb{P}^1$ was first proved in [26], generalizing a result of [7]. Moreover, in that case, X is shown to be the connected sum of either $\mathbb{P}^2$ or an S^2 -bundle over a Riemann surface with a number of $\mathbb{P}^2$. This is a fundamental result in symplectic 4-manifold theory.

It is instructive to examine her argument in the case of $D = \mathbb{P}^2$ geometrically.

Remark 2.14 Notice that here we are in the semi-positive territory, thus we can directly compute invariants via cutting down a generic moduli space by generic cycles. We will not mention this explicitly.

Let us first consider the case of trivial normal bundle. We pick a point x in D and a surface F intersecting D at one point $y \neq x$. Let l be the line class. By the previous proposition,

$$\langle [x], F \rangle_l^X = \langle [x], [y] \rangle_l^D = 1.$$

In the case of normal bundle $\mathcal{O}(1)$, we choose two points x and y in M . Any A -curve outside D can intersect D only at one point, and there is a unique line in D through x and y . Namely,

$$\langle [x], [y] \rangle_l^X = \langle [x], [y] \rangle_l^D = 1.$$

In the case of normal bundle $\mathcal{O}(2)$, we choose two points x and y and a line L in D away from x and y . Namely,

$$\langle [x], [y], [L] \rangle_l^X = \langle [x], [y], [L] \rangle_l^D = 1.$$

In each of above cases, we show that the GW invariant of the ambient manifold X is equal to the corresponding invariant of D .

The next case of normal bundle $\mathcal{O}(3)$ is different. The simple relation between GW invariants of D and X is no longer true. McDuff's strategy is to blow up a line of $\mathbb{P}^2$ in X to reduce it to the previous situation. Here, we give a different argument which motivates the correspondence in the next section.

Now, for the invariant of X to be well defined we need two points and two lines, or three points, or one point and four lines. We choose two points x and y in D . We also pick two lines L_1 and L_2 in D . Let z be the intersection point of L_1 and L_2 . We assume that z is not in the unique line through x and y . We claim that

$$1 = \langle [x], [y] \rangle_l^D = \langle [x], [y], [L_1], [L_2] \rangle_l^X - \langle [x], [y], [z] \rangle_l^X. \quad (4)$$

The point is that any curve through x, y, z also intersects (x, y, L_1, L_2) . Let C be a curve intersecting (x, y, L_1, L_2) . If C is not inside D , then C has to intersect z because C has at most three intersection points with D . If C is inside D , then C must be the unique line through x and y .

We remark that the above proof is just a sketch. To do this calculation we realize the constraints x, y, L_1, L_2 in a very non-generic way, and for a complete proof we would have to prove that this is justified.

It follows that one of the invariants on the right hand side of (4) is not zero and hence X is uniruled. We want to emphasize that in this case the invariant of D cannot be identified with a single invariant of X . However, it was pointed out to us by Zinger that if X is the projectivization of $\mathcal{O} \oplus \mathcal{O}(3)$ over $\mathbb{P}^2$ and D is the section with normal bundle $\mathcal{O}(3)$, then the invariant $\langle [x], [y], [z] \rangle_l^X$ does vanish.

When $\mathcal{O}(m)$ increases, we can similarly express the relevant invariant of D as a more and more complicated combination of invariants of the ambient space X . Our main idea is that such a process is best cast into the language of the relative-divisor/absolute correspondence established in the following three sections.

3 Degeneration Formula and Correspondence

A powerful tool in GW theory is the degeneration formula. To explore its power systematically, a very useful “relative/absolute correspondence” was constructed by Maulik–Pandharipande. It has been extended to the situation of blow-up by the authors and Hu [9] to relate absolute invariants of a manifold and relative invariants of the blow-up manifold. Such an extension is crucial to prove the birational invariance of uniruledness.

However, only a subset set of colored absolute invariants appears in the relative/absolute correspondence. They are admissible in the sense that the multiplicity of relative insertions is exactly $D \cdot A$, where D is the divisor and A is the curve class. It is natural to consider non-admissible absolute invariants. If the absolute invariant is sub-admissible in the sense that the multiplicity of relative insertions is less than $D \cdot A$, we can always use the divisor axiom to add more insertions to obtain an equivalent admissible invariants. The interest is on super-admissible invariants where the multiplicity is bigger than $D \cdot A$.

3.1 Symplectic Cut and the Degeneration Formula

3.1.1 Symplectic Cut

Let (X, ω) be a closed symplectic manifold. Let S be a hypersurface having a neighborhood with a free Hamiltonian S^1 -action. For instance, if there is a symplectic submanifold in X , then hypersurfaces corresponding to sphere bundles of the normal bundle have this property. Let Z be the symplectic reduction at the level S , then Z is the S^1 -quotient of S and is a symplectic manifold of dimension 2 less.

We can cut X along S to obtain two closed symplectic manifolds $(\bar{X}^+, \omega^+)$ and $(\bar{X}^-, \omega^-)$ each containing a smooth copy of Z , and satisfying $\omega^+|_Z = \omega^-|_Z$ (see [11]).

In particular, the pair (ω^+, ω^-) defines a cohomology class of $\bar{X}^+ \cup_Z \bar{X}^-$, denoted by $[\omega^+ \cup_Z \omega^-]$. Let p be the continuous collapsing map

$$p : X \rightarrow \bar{X}^+ \cup_Z \bar{X}^-.$$

It is easy to observe that

$$p^*([\omega^+ \cup_Z \omega^-]) = [\omega]. \quad (5)$$

3.1.2 Degeneration Formula

Given a symplectic cut, there is a basic link between absolute invariants of X and relative invariants of $(\bar{X}^\pm, Z)$ in [20] (see also [10, 15]). We now describe such a formula.

Let $B \in H_2(X; \mathbb{Z})$ be in the kernel of

$$p_* : H_2(X; \mathbb{Z}) \longrightarrow H_2(\bar{X}^+ \cup_Z \bar{X}^-; \mathbb{Z}).$$

By (5) we have $\omega(B) = 0$. Such a class is called a vanishing cycle. For $A \in H_2(X; \mathbb{Z})$ define $[A] = A + \text{Ker}(p_*)$ and

$$\langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_{g, [A]}^X = \sum_{B \in [A]} \langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_{g, B}^X. \quad (6)$$

Notice that ω has constant pairing with any element in $[A]$. It follows from the Gromov compactness theorem that there are only finitely many such elements in $[A]$ represented by J -holomorphic stable maps. Therefore, the summation in (6) is finite.

At this stage we need to assume that each cohomology class α_i is of the form

$$\alpha_i = p^*(\alpha_i^+ \cup_Z \alpha_i^-). \quad (7)$$

Here $\alpha_i^\pm \in H^*(\overline{X}^\pm; \mathbb{R})$ are classes with $\alpha_i^+|_Z = \alpha_i^-|_Z$ so that they give rise to a class $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{X}^+ \cup_Z \overline{X}^-; \mathbb{R})$.

The degeneration formula expresses $\langle \tau_{d_1} \alpha_1, \dots, \tau_{d_k} \alpha_k \rangle_{g, [A]}^X$ as a sum of products of relative invariants of $(\overline{X}^+, Z)$ and $(\overline{X}^-, Z)$, possibly with disconnected domains. In each product of relative invariants, what is relevant for us are the following conditions:

- the union of two domains along relative marked points is a stable genus g curve with k marked points,
- the total curve class is equal to $p_*(A)$,
- the relative insertions are dual to each other,
- if α_i^+ appears for i in a subset of $\{1, \dots, k\}$, then α_j^- appears for j in the complementary subset of $\{1, \dots, k\}$.

We first modify Definition 5.5 in [9].

Definition 3.1 A class $A \in H_2(\tilde{X}; \mathbb{Z})$ is called (ω, E) -effective if A is represented by a pseudo-holomorphic stable map for all ω -tamed almost complex structures, and $A \cdot E \geq 0$.

Let $\theta_1 = \mathbf{1}, \dots, \theta_{m_S}$ be a self dual basis of $H^*(S; \mathbb{R})$ with respect to the intersection pairing, where $\mathbf{1}$ is the distinguished degree 0 class. E is a $\mathbb{P}^{k-1}$ -bundle over S , so it has a basis of the form

$$\{\pi_S^* \theta_i \cup [E]^j\}, \quad 1 \leq i \leq m_S, \quad 0 \leq j \leq k-1. \quad (8)$$

Here $[E]$ is understood to be the first Chern class of the tautological line bundle over the projective bundle $\mathbb{P}(N_{S|X})$. It is well known that

$$E^k = - \sum_{1 \leq i \leq k} c_i(N_{S|X}) E^{k-i}. \quad (9)$$

Definition 3.2 For $\delta = \pi^* \theta \cup [E]^j \in H^*(E; \mathbb{R})$, we define

$$\deg_S(\delta) = \deg(\theta), \quad \deg_f(\delta) = 2j.$$

Denote this basis of E by $\Theta = \{\delta_t\}$, $1 \leq t \leq km_S$ with

- $\delta_1 = \mathbf{1}$,
- If $\deg(\delta_t) < \deg(\delta_{t'})$, then $t < t'$,
- If $\deg(\delta_t) = \deg(\delta_{t'})$ and $\deg_f(\delta_t) < \deg_f(\delta_{t'})$, then $t < t'$,
- The S -parts of δ_{km_S-t} and δ_{t+1} are S -dual to each other,
- The f -degrees of δ_{km_S-t} and δ_{t+1} sum to $k-1$.

Clearly, the basis Θ is still self dual if either $k=1$, i.e., when S is a symplectic hypersurface, or $E^k=0$, which is the case if the normal bundle $N_{S|X}$ is trivial.

In general, let $\check{\Theta} = \{\check{\delta}_t\}$ denote the dual basis.

Lemma 3.3

$$\check{\delta}_{km_S-t} = \delta_{t+1} + \sum_{u \leq t} c_u \delta_u, \quad (10)$$

where the correction term comes from $c(E)^{-1}$.

Proof First observe that

$$(\pi^* \theta_i \cup E^p) \cup (\pi^* \theta_j \cup E^q) = 0$$

whenever $\deg \theta_i + \deg \theta_j > \dim S$, or $\deg \theta_i + \deg \theta_j = \dim S$ but θ_i is not S -dual to θ_j (recall that $\{\theta_i\}$ is a self dual basis of S).

This implies the pairing between δ_{t+1} and δ_x with $x > km_S - t$ is trivial. Then the claim can be proved by induction on $km_S - t$. $\check{\delta}_{km_S}$ is supposed to vanish on each δ_i with $i \neq km_S$ and has pairing 1 with δ_{km_S} . Clearly the dual of $\check{\delta}_{km_S}$ is $\delta_1 = \mathbf{1}$. $\square$

What is important is that the leading term (the term with the highest f -degree) of $\check{\delta}_{km_S-t}$ is δ_{t+1} .

Definition 3.4 Denote the map $\check{\Theta} \rightarrow \Theta$ sending $\check{\delta}_{km_S-t}$ to δ_{t+1} by *lead*.

It follows from Lemma 3.3.

Corollary 3.5 This map *lead* is bijective. Denote the inverse map by *lead*⁻¹.

Definition 3.6 A dual standard $(\tilde{X}, E)$ -weighted partition $\check{\mu}$ with length $l(\check{\mu})$ is a partition weighted by classes of E from $\check{\Theta}$, i.e.,

$$\check{\mu} = \{(\mu_1, \check{\delta}_{K_1}), \dots, (\mu_{\ell(\check{\mu})}, \check{\delta}_{K_{\ell(\check{\mu})}})\}, \quad (11)$$

where μ_i and k_i are positive integers with $K_i \leq km_S$.

Similarly we define a standard $(\tilde{X}, E)$ -weighted partition as a partition weighted by classes of E from Θ .

Definition 3.7 For a dual standard $(\tilde{X}, E)$ -weighted partition $\check{\mu}$ as in (11), we associate the following standard $(\tilde{X}, E)$ -weighted partition by taking the leading terms:

$$\text{Lead}(\check{\mu}) = \{(\mu_1, \delta_{km_S+1-K_1}), \dots, (\mu_{\ell(\mu)}, \delta_{km_S+1-K_{\ell(\mu)}})\}. \quad (12)$$

An important consequence of Corollary 3.5 is as follows.

Lemma 3.8 *The map Lead in (12) induces a natural bijection between the set of dual standard $(\tilde{X}, E)$ -weighted partitions and the set of standard $(\tilde{X}, E)$ -weighted partitions.*

We then first partially order standard $(\tilde{X}, E)$ -weighted partitions.

Definition 3.9

$$\Gamma(\varpi')|_{\mu'} \overset{\circ}{<} \Gamma(\varpi)|_{\mu}$$

if one of the conditions below holds

- (1) $\omega_X(p_*A') < \omega_X(p_*A)$,
- (2) equality in (1) and the arithmetic genus satisfies $g' < g$,
- (3) equality in (1)–(2) and $\|\varpi'\| < \|\varpi\|$,
- (4) equality in (1)–(3) and $\deg_S(\mu') > \deg_S(\mu)$, where

$$\deg_S(\mu) = \sum_{i=1}^{l(\mu)} \deg_S(\delta_{K_i}),$$

- (5) equality in (1)–(4) and $\mu' \overset{l}{>} \mu$,

where $\|\varpi\|$ denotes the number of insertions of ϖ .

We partially order dual standard $(\tilde{X}, E)$ -weighted graphs in the following way.

Definition 3.10

$$\Gamma(\varpi')|_{\check{\mu}'} \overset{\circ}{<} \Gamma(\varpi)|_{\check{\mu}}$$

if

$$\Gamma(\varpi')|_{\text{Lead}(\check{\mu}')} \overset{\circ}{<} \Gamma(\varpi)|_{\text{Lead}(\check{\mu})},$$

where the latter partial order is given in the previous definition.

The proof of Lemma 5.10 in [9] is modified as follows.

Proof Let A be any (ω, E) -effective class of the standard relative invariant. It suffices to show that there are only finitely many (ω, E) -effective classes A' with $\omega_X(p_*A') \leq \omega_X(p_*A)$.

By the Gromov compactness theorem, there are only finitely many effective classes of X with ω_X -area bounded by $\omega_X(p_*A)$. We just need to show that each such class comes from finitely many (ω, E) -effective classes of $\tilde{X}$.

Observe that classes of $\tilde{X}$ map to the same class of X only if they differ by kL where L is the fiber class. Suppose A' is (ω, E) -effective: (1) if k is too negative, then $A' + kL$ has negative $\omega_{X'}$ area as $\omega(L) > 0$, so not effective; (2) if k is too positive, then $A + kL$ has negative pairing with the exceptional divisor E as $E \cdot L = -1$, so the relative invariant is not defined. $\square$

3.2 Trading Insertions

We still use Definition 5.11 in [9].

Definition 3.11 Given a standard $(\tilde{X}, E)$ -weighted partition μ , let

$$d_i(\mu) = d(\mu_i, \delta_{K_i}) = k\mu_i - k + \frac{1}{2} \deg_f(\delta_{K_i}),$$

and

$$\tilde{\mu} = \{\tau_{d_1(\mu)}(\tilde{\delta}_{K_1}), \dots, \tau_{d_{l(\mu)}(\mu)}(\tilde{\delta}_{K_{l(\mu)}})\}. \quad (13)$$

However, we change Definition 5.12 in [9].

Definition 3.12 The absolute descendent invariant associated to a dual standard $(\tilde{X}, E)$ -relative invariant

$$\langle \Gamma^\bullet(\varpi) | \check{\mu} \rangle^{\tilde{X}, E}$$

is then

$$\langle \tilde{\Gamma}^\bullet(\varpi, \widetilde{\text{Lead}(\check{\mu})}) \rangle^X.$$

It follows from Lemma 5.14 in [9] and Lemma 3.8 that we have the following crucial observation.

Lemma 3.13 *If $\check{\mu} \neq \check{\mu}'$, then $\widetilde{\text{Lead}(\check{\mu})} \neq \widetilde{\text{Lead}(\check{\mu}')}$. Therefore there is a natural bijection between the set of colored weighted absolute graphs relative to S and the set of dual weighted relative graphs in $\tilde{X}$ relative to the exceptional divisor E in the blow-up $\tilde{X}$ of X along S in the case $k > 1$, and it is still true in the case $k = 1$ if we restrict to the admissible ones.*

The inverse correspondence sends

$$\langle \tilde{\Gamma}(\varpi, \check{\mu}) \rangle^X$$

to

$$\Gamma(\varpi) | \text{Lead}^{-1}(\mu).$$

3.3 Full Correspondence

In this subsection we prove the relative/absolute correspondence in the following form.

We apply the degeneration formula to this connected absolute invariant

$$\langle \tilde{\Gamma}(\varpi, \tilde{\mu}) \rangle^X$$

of X to show that the dual relative graph

$$\Gamma(\varpi) | \text{Lead}^{-1}(\mu)$$

turns out to be the largest weighted relative graph appearing in the linear combination.

Notice that the correction term in (10) does not intersect the constraints of the leading term in the splitting formula. Thus we can still reduce to the relative invariant (21) in Step II.

3.4 Sup-admissible Graphs

Let $\iota : D \rightarrow X$ be a smooth connected symplectic divisor. As mentioned, we can cut along D , or precisely, cut along a small circle bundle S over D inside X .

In this case, as a smooth manifold, $\overline{X}^+ = X$, which we will denote by $\tilde{X}$. Denote the symplectic reduction of S in $\tilde{X}$ still by D . Notice, however, the symplectic structure is different from the original divisor. And $\overline{X}^- = \mathbb{P}(N_D \oplus \mathbb{C})$, the projectivization of $N_D \oplus \mathbb{C}$.¹ We will often denote it simply by P_D or P . Notice that $\mathbb{P}(N_D \oplus \mathbb{C})$ has two natural sections,

$$D_0 = \mathbb{P}(0 \oplus \mathbb{C}), \quad D_\infty = \mathbb{P}(N_D \oplus 0).$$

The symplectic reduction of S in P_D is the section D_∞ .

In summary, in this case, X degenerates into $(\tilde{X}, D)$ and (P_D, D_∞) . We also denote ω^- by ω_P .

We also observe that the section D_0 actually has the same symplectic structure and same neighborhood as the original divisor. We denote the inclusion of D_0 in P_D still by ι .

Definition 3.14 A class $A \in H_2(X; \mathbb{Z})$ is called effective for the symplectic cut along D if either

- A is represented by a pseudo-holomorphic map to X for all ω -tamed almost complex structures, or
- A is represented by a pseudo-holomorphic map to X^- for all ω_P -tamed almost complex structures, or
- A is in the image of ι_* and is represented by a pseudo-holomorphic map to $\tilde{X}$ for all ω^+ -tamed almost complex structures.

Here we allow the domain to be disconnected.

¹Notice that our convention here is opposite to that in [9].

Notice that the zero class $A = 0$ is considered to be effective here as a constant map is pseudo-holomorphic.

Definition 3.15 A connected colored graph Γ consists of one vertex decorated by (g, A) with g an integer, A an effective class in $H_2(X; \mathbb{Z})$, and two sets of colored tails, X -tails and D -tails.

We further weight each X -tail by a class $\alpha_i \in H^*(X; \mathbb{R})$, called an X -insertion. We also weight each D -tail by a pair (μ_i, β_i) , where μ_i is a non-negative integer, and β_i is a class in $H^*(D; \mathbb{R})$ called a D -insertion. We call the resulting graph a connected colored weighted graph and denote it by

$$\Gamma((\alpha_i) | ((\mu_i, \beta_i))).$$

The collection of pairs, $\mu = ((\mu_i, \beta_i))$, is called a weighted partition.

There is also the distinguished graph, the empty graph $\Gamma(\emptyset | \emptyset)$.

Definition 3.16 The dimension of the empty graph is defined to be zero. For a nonempty graph $\Gamma((\alpha_i) | ((\mu_i, \beta_i)))$, its dimension is defined to be

$$\begin{aligned} \dim \Gamma((\alpha_i) | ((\mu_i, \beta_i))) \\ = 2[-K_\omega(A) + (n-3)(1-g) + D \cdot A] \\ + \left[\sum (2 - 2\mu_i - \deg(\mu_i)) \right] + \left[\sum (2 - \deg(\alpha_i)) \right]. \end{aligned} \quad (14)$$

Remark 3.17 We can also consider the disjoint union $\Gamma^\bullet$ of several such graphs and use $A_{\Gamma^\bullet}$, $g_{\Gamma^\bullet}$ to denote the total homology class. Here the total arithmetic genus is $1 + \sum (g_i - 1)$.

Definition 3.18 A connected colored weighted graph with

$$\varpi = (\alpha_i) \quad \text{and} \quad \mu = ((\mu_i, \beta_i)),$$

written simply as $\Gamma(\varpi | \mu)$, is called

- admissible if $\sum_j \mu_j = D \cdot A$,
- strictly sup-admissible if

$$\sum_j \mu_j > D \cdot A \quad \text{and} \quad A \in \text{im}[\iota_* : H_2(D; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z})], \quad (15)$$

- strictly sub-admissible if $\sum_j \mu_j < D \cdot A$.

A possibly disconnected graph is called sup-admissible if it is a connected strictly sup-admissible graphs or the disjoint union of one or more connected admissible graphs.

Notice that every strictly sub-admissible absolute invariant can be made admissible by adding an appropriate number of D -insertions.

These graphs will be used to describe the structure of the components appearing in the decomposition formula; cf. Eq. (20).

The strictly sup-admissible graphs are connected because they correspond to curves that lie entirely in P_D . The other sup-admissible graphs describe the part of the curve lying in $\tilde{X}$ and hence need not be connected.

Definition 3.19 Suppose X is of dimension $2n \geq 4$. Let $\Theta = \{\delta_i\}$ be a self dual basis of $H^*(D; \mathbb{R})$ with respect to the cup product of D . Let $\mathcal{E} = \{\gamma_i\}$ be a basis of $H^*(X; \mathbb{R})$.

We will fix Θ and $\mathcal{E}$ in the rest of this paper. Notice that we do not require any compatibility of Θ and $\mathcal{E}$.

Definition 3.20 A Θ -standard weighted partition μ is a partition weighted by classes of D from Θ , i.e.,

$$\mu = \{(\mu_1, \delta_{K_1}), \dots, (\mu_{\ell(\mu)}, \delta_{K_{\ell(\mu)}})\}.$$

$\varpi = (\alpha_i)$ is called $\mathcal{E}$ -standard if each $\alpha_i \in \mathcal{E}$.

Definition 3.21 $\Gamma(\varpi|\mu)$ is called standard if

- the class of each vertex is a nonzero effective class,
- ϖ is $\mathcal{E}$ -standard,
- μ is Θ -standard,
- its dimension is zero.

Remark 3.22 It will become clear in the proof of Lemma 3.29 and Theorem 1.1 why such a definition is made.

3.4.1 Ordering the Graphs

Let I be the set of possibly disconnected sup-admissible standard colored weighted graphs. We will order I following [31]. The partial order is defined in terms of several preliminary partial orders.

Definition 3.23 The set of pairs (m, δ) where $m \in \mathbb{Z}_{>0}$ and $\delta \in H^*(D; \mathbb{R})$ is partially ordered by the following size relation:

$$(m, \delta) > (m', \delta') \tag{16}$$

if $m > m'$ or if $m = m'$ and $\deg(\delta) > \deg(\delta')$.

We may place the pairs of μ in decreasing order by size, i.e., by (16).

Definition 3.24 A lexicographic ordering on weighted partitions is then defined as follows:

$$\mu \overset{l}{>} \mu'$$

if, after placing μ and μ' in decreasing order by size, the first pair for which μ and μ' differ in size is larger for μ .

Next we introduce a relevant partial order on the effective curve classes of X (see Definition 3.1).

Definition 3.25

For effective classes A and A' in $H_2(X; \mathbb{Z})$, we say that $A' < A$ if $A - A' \in H_2(X; \mathbb{Z})$ has positive pairing with the symplectic form on X .

We partially order such weighted graphs in the following way.

Definition 3.26 The empty graph is smaller than any other graph. For any two non-empty admissible graphs $\Gamma(\varpi'|\mu')$ and $\Gamma(\varpi|\mu)$,

$$\Gamma(\varpi'|\mu') \overset{\circ}{<} \Gamma(\varpi|\mu)$$

if one of the conditions below holds

- (1) $A' < A$,
- (2) equality in (1) and the arithmetic genus satisfies $g' < g$,
- (3) equality in (1)–(2) and $\|\varpi'\| < \|\varpi\|$,
- (4) equality in (1)–(3) and $\deg(\mu') > \deg(\mu)$,
- (5) equality in (1)–(4) and $\mu' \overset{l}{>} \mu$,

where $\|\varpi\|$ denotes the number of X -tails, and $\deg(\mu)$ is the sum of $\deg(\mu_i)$.

If $\Gamma(\varpi'|\mu')$ is admissible and $\Gamma(\varpi|\mu)$ is connected and strictly sup-admissible,

$$\Gamma(\varpi'|\mu') \overset{\circ}{<} \Gamma(\varpi|\mu)$$

if $A' \leq A$.

The inequalities (3)–(5) are designed so that the dimension of the moduli space satisfying the larger constraint/condition is larger. This explains the seemingly strange conditions (4) and (5) where the inequalities are reversed.

Remark 3.27 It is easy to observe that this extended partial order $\overset{\circ}{<}$ is preserved under disjoint union of admissible graphs. Notice that we do not compare strictly sup-admissible graphs.

Remark 3.28 If we are only interested in genus zero invariants, then we can replace $g' < g$ in (2) by the inequality of the number of connected components, $n' > n$.

A crucial property of the ordering of I is that it is lower bounded. Here we define a partially order set to be lower bounded if there are only finitely many elements lower than a given element.

Lemma 3.29 *The partially ordered set I is lower bounded. In particular, there is a minimal standard invariant with nonzero curve class and nonzero value.*

Proof Given a standard colored weighted graph, we need to show that there are only finitely many standard colored weighted graphs lower in the partial ordering.

As a strictly sup-admissible graph is not smaller than any other graph, we only need to bound the number of admissible graphs.

First of all, the number of effective classes with area bounded above is finite due to the Gromov compactness. In particular, there is a minimal area $c(X, \omega)$ among all nonzero effective classes. As each vertex is a nonzero effective class, this implies the number of components of a standard graph with bounded area is bounded by $\frac{\omega(A)}{c(X, \omega)}$.

As the number of ϖ insertions is bounded from above and the ϖ insertions are chosen from a finite generating set $\mathcal{E}$, there is only a finite number of choices of ϖ .

Finally, the number of μ insertions and the total multiplicity of μ are both bounded by the intersection number $D \cdot A$. As the μ insertions are chosen from a finite generating set Θ and the multiplicities are positive, there is only a finite number of choices of μ . $\square$

3.5 Invariants Associated to Graphs

In this subsection, we associate to a sup-admissible standard colored weighted graph certain GW invariants of the symplectic cut. We just give the definition for connected graphs, the extension to disconnected graphs is straightforward.

3.5.1 Absolute Invariants

Definition 3.30 For a relative insertion (m, δ) , we associate to it the absolute descendant insertion $\tau_{m-1}(\tilde{\delta})$ on X supported on D , where $\tilde{\delta} = \delta[D]$. Given a standard (relative) weighted partition μ , let

$$d_i(\mu) = \mu_i - 1 \quad (17)$$

and

$$\tilde{\mu} = \{\tau_{d_1(\mu)}(\tilde{\delta}_{K_1}), \dots, \tau_{d_{l(\mu)}(\mu)}(\tilde{\delta}_{K_{l(\mu)}})\}. \quad (18)$$

It is convenient to view $[D]$ as the class of a Thom form supported near the symplectic divisor D . Then class $\tilde{\delta} = \delta[D]$ is represented by the wedge product of the pull back of a form representing δ in a neighborhood of D with the compactly supported Thom form of D . In terms of homology constraints, $\tilde{\delta}$ and δ correspond to the same cycle lying inside D .

Definition 3.31 The absolute descendant invariant associated to a connected standard colored weighted graph $\Gamma(\varpi|\mu)$ is

$$\langle \Gamma(\varpi; \tilde{\mu}) \rangle^X.$$

The invariant associated to the empty graph is called the empty absolute invariant and its value is defined to be 1.

Notice for such an absolute descendant invariant of X all the descendant insertions are supported on D . Such an invariant is colored in the sense that the insertions are divided into two collections, the X -insertions ϖ and the D -insertions $\tilde{\mu}$, with each insertion in ϖ being of the form γ_L , and each insertion in $\tilde{\mu}$ being of the form $\tau_d \tilde{\delta}_K$.

3.5.2 Relative Invariants

Definition 3.32 Let $\Gamma(\varpi|\mu)$ be a connected standard colored weighted graph.

If it is admissible, the relative invariant of the symplectic cut associated to it is

$$\langle \Gamma(\varpi|\mu) \rangle^{\tilde{X}, D}.$$

If it is strictly sup-admissible, the relative invariant of the symplectic cut associated to it is

$$\langle \pi^* \iota^* \varpi, \tilde{\mu} | \emptyset \rangle_A^{P, D_\infty},$$

where we view P as a bundle over its zero section D_0 and $\pi : P \rightarrow D_0$ is the projection, and $\tilde{\mu}$ here is given by

$$(\tau_{d_1(\mu)}(\delta_{K_1}[D_0]), \dots, \tau_{d_{l(\mu)}(\mu)}(\delta_{K_{l(\mu)}}[D_0])).$$

Finally, the invariant associated to the empty graph is the empty relative invariant and its value is defined to be 1.

3.5.3 Disconnected Graphs

Finally we look at possibly disconnected admissible graphs. Notice that the invariant of the disjoint union of two graphs is the product of invariants. We have also remarked that if Γ_1 is bigger than Γ'_1 and Γ_2 is bigger than Γ'_2 , then the union of Γ_1 and Γ_2 is bigger than the union of Γ'_1 and Γ'_2 .

3.6 Sup-admissible Correspondence

Consider the infinite-dimensional vector space $\mathbb{R}^I$ whose coordinates are ordered in the way compatible with the partial order of I . From the relative invariants in Definition 3.31 we can form a vector

$$v_{\text{rel}} \in \mathbb{R}^I$$

given by the numerical values. We also have the vector

$$v_{\text{abs}} \in \mathbb{R}^I$$

given by the numerical values of the sup-admissible invariants of X relative to D in Definition 3.12.

Theorem 3.33 *There is an invertible lower triangular linear transformation*

$$\Upsilon : \mathbb{R}^I \rightarrow \mathbb{R}^I$$

such that

- (i) *the coefficients of A are local in the sense of being dependent on D only;*
- (ii) $\Upsilon(v_{\text{rel}}) = v_{\text{abs}}$.

In particular, v_{rel} and v_{abs} determine each other.

Finally, if $I_0 \subset I$ denotes the subset of genus zero invariants with $\varpi = \emptyset$, then Υ further restricts to an invertible lower triangular transformation from $\mathbb{R}^{I_0}$ to $\mathbb{R}^{I_0}$.

Proof The idea is as follows. We start with a connected colored weighted graph $\Gamma(\varpi|\mu)$. The associated absolute invariant is $\langle \Gamma(\varpi; \tilde{\mu}) \rangle^X$.

We apply the degeneration formula to this connected absolute invariant to express it as a linear combination of relative invariants of $(\tilde{X}, D)$ with the coefficients being essentially certain relative invariants of the $\mathbb{P}^1$ -bundle. In the strictly sup-admissible case there is an additional term being the associated relative invariant of (P, D_∞) .

This is possible because the homomorphism p_* is obviously injective.

Of course we also need to first split the ϖ and $\tilde{\mu}$ insertions as in (7). Recall that each μ insertion is of the form $\gamma = \tau_d(\delta[D])$ for $\delta \in \Theta$. Then we set

$$\gamma^+ = 0, \quad \gamma^- = \tau_d(\delta[D_0]).$$

In other words, we distribute all the $\tilde{\mu}$ insertions to the $\mathbb{P}^1$ -bundle side. With this preferred distribution of insertions, the original graph $\Gamma(\varpi|\mu)$ turns out to be the largest weighted relative graph appearing in the linear combination. For a ϖ insertion τ we can take the $+$ class to be itself and the $-$ part to be the class of the $\mathbb{P}^1$ -bundle over the cycle of intersection, i.e.,

$$\tau^+ = \tau, \quad \tau^- = \pi^* i^* \tau.$$

The arguments for I_0 and I are similar, we just treat the case of I_0 , i.e., genus 0 and $\varpi = \emptyset$.

The absolute invariant associated to $\Gamma(\emptyset|\mu) \in I_0$ is of the form

$$\langle \tau_{d_1(\mu)}(\tilde{\delta}_{K_1}), \dots, \tau_{d_{l(\mu)}(\mu)}(\tilde{\delta}_{K_{l(\mu)}}) \rangle_A^X. \quad (19)$$

With all insertions distributed to the $\mathbb{P}^1$ -bundle side, (19) is expressed as the following sum:

$$\sum \langle \Gamma^-(\emptyset|\eta) \rangle^{\tilde{X}, D} \Delta(\eta) \langle \Gamma^+(\tau_{d_1(\mu)}(\delta_{K_1}[D_0]), \dots, \tau_{d_{l(\mu)}(\mu)}(\tilde{\delta}_{K_{l(\mu)}}[D_0])|\check{\eta}) \rangle^{P, D_\infty} \quad (20)$$

over appropriate pairs of weighted graphs. Here $\Delta(\eta)$ is a nonzero combinatorial constant depending on the multiplicities of η .

Case I: Suppose the graph $\Gamma(\emptyset|\mu)$ is strictly sup-admissible, i.e., $\sum_j \mu_j > D \cdot A$ and A is in the image of $\iota_* : H_2(D; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z})$.

- Ia. Let us first examine the term with $\check{\eta} = \eta = \emptyset$ in (20). In this term the relative invariant of (P, D_∞) is the relative invariant associated to the given graph in I_0 , and the relative invariant of $(\tilde{X}, D)$ is associated to the empty graph and hence is equal to 1.
- Ib. In any other term, $\eta \neq \emptyset$. We have a relative invariant of $(\tilde{X}, D)$ associated to a possibly disconnected admissible graph $\Gamma^-(\emptyset|\eta) \in I_0$. Regard the relative invariant

$$\langle \Gamma^+(\tau_{d_1(\mu)}(\delta_{K_1}[D_0]), \dots, \tau_{d_{l(\mu)}(\mu)}(\tilde{\delta}_{K_{l(\mu)}}[D_0])|\check{\eta}) \rangle^{P, D_\infty}$$

of (P, D_∞) as the coefficient of the graph of $\Gamma^-(\emptyset|\eta)$. The coefficient is nonzero only if the class of Γ^- is at most A . Thus we have $\Gamma^-(\emptyset|\eta) \overset{\circ}{\prec} \Gamma(\emptyset|\mu)$, according to our extended order in Definition 3.9.

Case II: Suppose the graph $\Gamma(\emptyset|\mu)$ is admissible.

- IIa. For the term with $\eta = \emptyset$, the relative invariant of (P, D_∞) is not associated to any graph in I_0 , as $\Gamma(\emptyset|\mu)$ is not strictly sup-admissible. Instead the relative invariant of (P, D_∞) is considered to be the coefficient of the empty graph in I_0 . But empty graph is certainly smaller than $\Gamma(\emptyset|\mu)$ according to Definition 3.9.
- IIb. For all other terms, as our order of admissible graphs agrees with that in [31], it follows from [31] that the largest graph Γ^- appearing in (20) with nonzero coefficient is the graph $\Gamma(\emptyset|\mu)$ itself.

For a disconnected graph by Sect. 3.5.3 we still have the leading term being the given graph. Thus the correspondence is lower triangular with nonzero diagonal entries. Such a correspondence is actually invertible as I_0 is lower bounded by Lemma 3.29. $\square$

Remark 3.34 When $\sum_j \mu_j < D \cdot A$, we have $l(\eta) - l(\mu) > 0$. The largest η is μ followed by $D \cdot A - \sum_j \mu_j$ pairs of $(1, D)$. In the extreme case all $\mu_j = 0$, the largest invariant has η with $A \cdot E$ pairs of $(1, D)$. Notice that when $\sum_j \mu_j < D \cdot A$, then the relative invariant $\langle [pt], \varpi|\mu \rangle_{g, A}^{\tilde{X}, D}$ is zero by definition. What Theorem 3.33 says in this case is that $\langle [pt], \varpi, \tilde{\mu} \rangle_{g, p_*(A)}^X$ is expressed as the sum of standard relative invariants whose weighted graph is lower than $\langle [pt], \varpi|\mu \rangle_{g, A}^{\tilde{X}, D}$.

3.7 Correspondence with a Point D -Insertion

With the application to uniruledness in mind, we require graphs have a point D -insertion. This is different from [9] where the point insertion is always an X -insertion.

In this subsection we still use P to denote $\mathbb{P}(N_D \oplus \mathbb{C})$.

3.7.1 Statement

From now on ϖ denotes an ordered collection

$$\{\alpha_1, \dots, \alpha_l\}, \quad \alpha_i \in H^*(X; \mathbb{R}). \quad (21)$$

Recall $\iota : D \rightarrow X$ is the embedding. Let

$$V = \min \{ \omega_D(A) > 0 \mid A \in H_2(D), \langle \iota^* \varpi, \tau_{i_1}([pt]), \tau_{i_2}(\beta_2), \dots, \tau_{i_k}(\beta_k) \rangle_A^D \neq 0 \}. \quad (22)$$

By linearity we can assume that each β_i is in Θ and each α_j is in $\mathcal{E}$. Such an invariant determines a standard colored weighted graph

$$\Gamma_0(\varpi | \mu) \quad (23)$$

in I with

$$\mu = ((i_1 + 1, [pt]), (i_2 + 1, \beta_2), \dots, (i_k + 1, \beta_k)).$$

Remark 3.35 Moreover, according to [9, 13] V is achieved by invariants with no descendants. Finally, V is finite if and only if D is uniruled.

Definition 3.36 We consider the subset $I_{D\text{-pt}} \subset I$ of colored standard graphs of X , satisfying the following conditions:

- (1) $g(\Gamma) = 0$,
- (2) the class A is nonzero and $\omega(A) \leq V$,
- (3) if the graph is admissible it has a D -point insertion,
- (4) if the graph is strictly sup-admissible then it is of the form Γ_0 just defined above, so $A \in \text{im}[\iota_* : H_2(D; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z})]$ and it has a D -point insertion. Moreover,

$$\omega(A) = \omega_D(A) = V, \quad \sum_{t=1}^k (i_t + 1) = D \cdot A + 1, \quad (24)$$

- (5) empty graph excluded.

If such a graph is not strictly sup-admissible we call it a restricted graph.

Let $\mathbb{R}_{D\text{-pt}}^I$ be the vector subspace spanned by the partially ordered set $I_{D\text{-pt}}$ of graphs, and $v_{D\text{-pt}}^{\text{abs}}$ be the vector of associated absolute invariants.

Notice that all the associated absolute invariants have a point insertion (possibly descendant).

We also have the vector of associated relative invariants $v_{D\text{-pt}}^{\text{rel}}$ of the symplectic cut, including all the relative invariants of (P, D_∞) with class $A \in \text{im}[\iota_* : H_2(D; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z})]$ satisfying $\omega(A) = V$, and insertions of the form $(\tilde{v}|\emptyset)$.

Theorem 3.37 $\mathcal{T}$ restricts to an invertible lower triangular linear transformation

$$T : \mathbb{R}_{D\text{-pt}}^I \rightarrow \mathbb{R}_{D\text{-pt}}^I$$

such that

$$T(v_{D\text{-pt}}^{\text{rel}}) = v_{D\text{-pt}}^{\text{abs}}.$$

Moreover, there is also an $I_{D\text{-pt},0}$ version.

In the remaining we provide the proof using the following vanishing results on the relative invariants of (P, D_∞) .

Theorem 3.38 Suppose A is a non-fiber class, i.e., $\omega_{D_0}(\pi_*(A)) > 0$.

(i) If $\omega_{D_0}(\pi_*(A)) < V$, then,

$$\langle \pi^* \iota^* \varpi, \tau_{i_1}([pt]_{D_0}[D_0]), \tau_{i_2}(\beta_2[D_0]), \dots, \tau_{i_k}(\beta_k[D_0]) | \mu \rangle_A^{P, D_\infty} = 0.$$

(ii) If $\omega_{D_0}(\pi_*(A)) = V$ and

$$m = \sum_i (i_i + 1) \leq D_0 \cdot A$$

is admissible or sub-admissible, then,

$$\langle \pi^* \iota^* \varpi, \tau_{i_1}([pt]_{D_0}[D_0]), \tau_{i_2}(\beta_2[D_0]), \dots, \tau_{i_k}(\beta_k[D_0]) | \emptyset \rangle_A^{P, D_\infty} = 0.$$

Theorem 3.38 will be proved in the next section.

3.7.2 Proof of Theorem 3.37

As in the proof of Theorem 3.33 we only prove the version with $\varpi = \emptyset$, i.e., the $I_{D\text{-pt},0}$ version.

We can assume that the graph $\Gamma(\emptyset|\mu)$ in $I_{D\text{-pt},0}$ is connected.

Case I: Let us first look at the case that $\Gamma(\emptyset|\mu)$ is restricted, or in other words, admissible.

Apply the degeneration formula to it as in the proof of Theorem 3.33. Notice that in this case $\delta_{K_1} = [pt]$.

Definition 3.39 The P -graphs $\Gamma^+(\cdots)$ in (20) are divided into 3 types.

- (i) The special graph $\Gamma^+(\tilde{\mu}|\emptyset)$. In this case the entire curve lies on the $\mathbb{P}^1$ -bundle side.²
- (ii) The connected component of Γ^+ containing the point insertion is not a fiber curve.
- (iii) The connected component of Γ^+ containing the point insertion is a fiber curve (possibly multiply covered).

The type of a $\Gamma^-(\emptyset|\eta)$ -graph in (20) is the type of the company P -graph. In particular, the type (i) $\Gamma^-(\emptyset|\eta)$ -graph is just the empty graph.

Now let us fix a term in (20).

I.1—Neither Γ^+ nor Γ^- is the Empty Graph. Then there are associated classes $B^+ \in H_2(P; \mathbb{Z})$ and $B^- \in H_2(\tilde{X}; \mathbb{Z})$ respectively with

$$B^+ + B^- = p_*(B).$$

It follows from (5) and Definition 3.36,

$$\omega_{\tilde{X}}(B^-) + \omega_P(B^+) = \omega(B) \leq V. \quad (25)$$

Since B is not the zero class, and the graph $\Gamma(\emptyset|\mu)$ is connected, we have $B^+ \neq 0$ and $B^- \neq 0$. Hence it follows from (25):

$$0 < \omega_P(B^+) < V, \quad 0 < \omega_{\tilde{X}}(B^-) < V. \quad (26)$$

We will show in this case as follows.

Proposition 3.40 *If the Γ^- -graph contributes then it is restricted.*

Proof This will be proved by a series of lemmas.

Lemma 3.41 *For a type (iii) P -graph each relative insertion on the fiber curve containing the absolute point insertion must be of the form $(s, [D])$. Consequently, for each type (iii) $\Gamma^-(\emptyset | \eta)$ -graph there is a point relative insertion, i.e., $\eta = \{(s, [pt]), \dots\}$.*

Proof Otherwise the fiber curve cannot meet both the point and the relative cycle. $\square$

²In this case it is tempting to think that the relative invariant $\langle [pt], \alpha_2, \dots, \alpha_k | \emptyset \rangle_A^{P, D_\infty}$ is the same as the absolute invariant $\langle [pt], \alpha_2, \dots, \alpha_k \rangle_A^P$. But in general this is not true.

Lemma 3.42 *Let $\pi : P \rightarrow D_0$ be the projection. Then*

$$B^+ = \pi_* B + (B \cdot D_\infty)F.$$

In particular,

$$\omega_{D_0}(\pi_*(B^+)) \leq \omega_P(B^+)$$

if $B^+ \cdot D_\infty \geq 0$, and the inequality is strict if $B^+ \cdot D_\infty$ is positive.

Proof $H_2(P; \mathbb{Z})$ is generated by $H_2(D_0; \mathbb{Z})$ and F , so we can write

$$B^+ = B_0 + mF$$

for some class B_0 of D_0 . Since $\pi_* B_0 = B_0$ we have

$$\pi_* B^+ = \pi_* B_0 + 0 = B_0.$$

On the other hand, since $B_0 \cdot D_\infty = 0$, we have $m = B^+ \cdot D_\infty \geq 0$.

Since $\omega_P(F) > 0$ and $\omega_{D_0} = \omega_D$, if $m = B^+ \cdot D_\infty \geq 0$, then

$$\begin{aligned} \omega_P(B^+) &= \omega_P(\pi_* B^+) + \omega_P(mF) \geq \omega_P(\pi_* B^+) \\ &= \omega_{D_0}(\pi_* B^+) = \omega_D(\pi_* B^+). \end{aligned} \quad \square$$

Lemma 3.43 *Type (ii) P -graph invariants vanish. Hence there are no contributing type (ii) $\Gamma^-(\emptyset|\eta)$ graphs in (20).*

Proof Observe that in this case the class B^+ is not a fiber class. And by Lemma 3.42 and (26) we have

$$\omega_D(\pi_*(B^+)) \leq \omega_P(B^+) < V.$$

The conclusion then follows from part 1 of Theorem 3.38. $\square$

Now it follows from Lemmas 3.41–3.43 that the contributing $\Gamma^-(\emptyset|\eta)$ -graphs in (20) are still restricted. $\square$

I.2— Γ^+ Is Empty. In this case Γ^- is simply the given graph $\Gamma(\emptyset|\mu)$.

I.3— Γ^- Is Empty. In this case we have the following lemma.

Lemma 3.44 *Suppose the given graph $\Gamma(\emptyset|\mu)$ is admissible. Either the type (i) P -graph $\Gamma^+(\tilde{\mu}|\emptyset)$ is not allowed or its invariant vanishes. Hence the empty Γ^- -graph is not a contributing graph in (20).*

Proof The type (i) P -graph $\Gamma(\emptyset|\mu)$ does not appear if the class B is not in the image ι_* .

Suppose $B = B^+$ is in the image of ι and we denote the class of D still by B . By our assumption, either $\omega(B) = \omega_D(B) < V$ or $\omega(B) = \omega_D(B) = V$. Since B^+ is not a fiber class, in either case the vanishing is given by Theorem 3.38. $\square$

Case II: Now consider the case that $\Gamma(\emptyset|\mu)$ is of the form Γ_0 satisfying (24).

We again apply the degeneration formula to the corresponding invariant of X distributing all the insertions to the $\mathbb{P}^1$ -bundle side. Then as argued in Theorem 3.33 the leading term in (20) is the special graph.

We only need to show that the remaining Γ^- -graphs in (20) are restricted.

By Lemma 3.44 the type (i) Γ^- -graph does not appear. Since A is a minimal uniruled class of D , by Theorem 3.38, type (ii) Γ^- -graphs do not appear either. In particular, $\check{\eta}$ is constrained by (iii). For the remaining Γ^- -graphs, the corresponding P -graphs are of type (iii), hence they are restricted.

Thus we have completed the proof of Theorem 3.37. $\square$

Remark 3.45 In fact, we have shown there is a sub-correspondence for the admissible graphs in $I_{D\text{-pt},0}$.

4 Uniruled Divisor/Ambient Space Correspondence

In this section, we first prove Theorem 3.38 in Sect. 4.1. Using a similar argument we derive a non-vanishing result in Sect. 4.2. Our uniruled divisor/ambient space correspondence is then easily established in Sect. 4.3.

4.1 Two Vanishing Theorems for Invariants with a Descendant Point Insertion

Relative GW invariants of $\mathbb{P}^1$ -bundle have been studied in [4, 31]. We will use both their techniques and results extensively. In [31] it is shown using virtual localization that relative invariants of a projective $\mathbb{P}^1$ -bundle are determined by absolute invariants of the base projective manifold. We need to apply the symplectic relative virtual localization theorem of Chen-Li [1] in our more general setting. However, for a $\mathbb{P}^1$ -bundle, the theorem is the same as the corresponding algebro-geometric case in [8]. We conveniently use the notations from the algebro-geometric case. Compared to [31], a new ingredient of our case is the point insertion for which we have to keep track at each induction step.

Let L be a complex line bundle over D and $P = \mathbb{P}(L \oplus \mathbb{C})$. Then $\pi : P \rightarrow D$ is a $\mathbb{P}^1$ -bundle with zero section D_0 and infinity section D_∞ . In this section, we calculate certain relative GW invariants of P relative to the infinity section D_∞ . First of all, there are two kinds of cohomology classes: $\pi^*\alpha$ and $\beta[D_0]$. As previously remarked, the class $\beta[D_0]$, which is the cup product of the pull-back of β with the Poincaré dual of D_0 , corresponds to the Poincaré dual of β viewed as a homology cycle of D_0 .

Let $A \in H_2(D_0; \mathbb{Z})$. We view it as a homology class of P . We shall consider relative invariants of the form

$$\langle \pi^* \iota^* \varpi, \tau_{i_1}([pt]_{D_0}[D_0]), \tau_{i_2}(\beta_2[D_0]), \dots, \tau_{i_k}(\beta_k[D_0]) | \mu \rangle_{0,A}^{P,D_\infty}, \quad (27)$$

where ϖ is as defined in (21).

To evaluate such an invariant we will need to study other types of relative invariants, twisted rubber invariants, as well as twisted invariants of the divisor.

Our interest here is to relate relative invariants with a descendant point insertion to invariants of D with a descendant point insertion.

4.1.1 Virtual Localization and Generalized Twisted Invariants

To simplify the computation, we consider the following invariant subset of the moduli space $\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)$ of relative stable maps. Without loss of generality, we choose a point, denoted by $S_1 = pt$, and choose submanifolds S_i , $2 \leq i \leq k$, of D_0 representing $\beta_1 = [pt]_{D_0}$ and $\beta_i[D_0]$, $2 \leq i \leq k$, respectively. We here assume that S_i , $i \geq 1$ intersect transversely. Let

$$\begin{aligned} \overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A, pt, S_2, \dots, S_k) \\ = \{f \in \overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A); f(x_0) = pt, f(x_i) \in S_i, 1 \leq i \leq k\}. \end{aligned} \quad (28)$$

We can construct a virtual fundamental cycle for this space which will give the desired relative invariants. Of course we need to modify the deformation-obstruction complex. The linearization of the Cauchy–Riemann operator is the complex

$$\mathcal{D}_{f,\{S_i\}}^P : \{v \in \Omega^0(f^*TP), v(x_i) \in TS_i\} \rightarrow \Omega^{0,1}(f^*TP). \quad (29)$$

As each S_i is in the fixed locus D_0 , $\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A, pt, S_2, \dots, S_k)$ also carries a natural S^1 -action. Thus we still can apply the localization formula to this moduli space. The only difference here is that we need to calculate the fixing-moving part of our new complex to determine the contributions.

Recall that fixed point loci are indexed by bipartite graphs. For each bipartite graph the corresponding component of the fixed point loci is the fiber product of a stable map moduli space of D_0 and a rubber stable map moduli space. Each vertex of the bipartite graph either corresponds to a connected component of stable maps into D_0 or a connected component of rubber stable maps into D_0 .

Let us analyze the following case: suppose that our bipartite graph consists of no edges and only one D_0 -vertex. Since there are no rubber constant vertices, there are no other constant vertices either. In this case, the fixed point set is simply

$$\overline{\mathcal{M}}_{0,k+l}^D(A, pt, S_2, \dots, S_k).$$

It is clear that the fixed part of the complex (29) is

$$\mathcal{D}_{f,\{S_i\}}^{D_0} : \{v \in \Omega^0(f^*TD_0), v(x_i) \in TS_i, 1 \leq i \leq k\} \rightarrow \Omega^{0,1}(f^*TD_0).$$

The moving part is the following subcomplex of $\mathcal{D}_f^L$ in the [Appendix](#).

$$\mathcal{D}_{f,\{x_i\}}^L : \{v \in \Omega^0(f^*L), v(x_i) = 0, i \leq k\} \rightarrow \Omega^{0,1}(f^*L). \quad (30)$$

Therefore, as the fixed point set, its virtual fundamental agrees with

$$[\overline{\mathcal{M}}_{0,k+l}^D(A, pt, S_2, \dots, S_k)]^{\text{virt}}.$$

Furthermore, $N^{\text{virt}} = \ker \mathcal{D}_{f,\{x_i\}}^L - \text{coker } \mathcal{D}_{f,\{x_i\}}^L$.

The virtual bundle $\text{coker } \mathcal{D}_{f, \{x_i\}}^L - \ker \mathcal{D}_{f, \{x_i\}}^L$ is of rank

$$k - c_1(L)(A) - 1. \quad (31)$$

Let $e_{L, \{x_i\}}$ be its Euler class. Then, consider the following GW invariant:

$$\begin{aligned} & \langle \iota^* \varpi, \tau_{i_1}([pt]), \tau_{i_2}(\beta_2), \dots, \tau_{i_k}(\beta_k) \rangle_{0, A}^{D, e_{L, \{x_i\}}} \\ &= \int_{[\overline{\mathcal{M}}_{0, k+l}^D(A, pt, S_2, \dots, S_k)]^{\text{vir}}} \prod_s \iota^* \alpha_s \prod \psi_j^{i_j} \wedge e_{L, \{x_i\}}. \end{aligned} \quad (32)$$

We call it a generalized L -twisted invariant.

There is a short exact sequence

$$0 \rightarrow \ker \mathcal{D}_{f, \{x_i\}}^L \rightarrow H^0(f^*L) \rightarrow \bigoplus_x L_{f(x_i)} \rightarrow \text{coker } \mathcal{D}_{f, \{x_i\}}^L \rightarrow H^1(f^*L) \rightarrow 0.$$

As a virtual bundle,

$$\text{coker } \mathcal{D}_{f, \{x_i\}}^L - \ker \mathcal{D}_{f, \{x_i\}}^L = H^1(f^*L) - H^0(f^*L) - \bigoplus_i L_{f(x_i)}. \quad (33)$$

Observe that we can use (33) to express $e_{L, \{x_i\}}$ in terms of the Euler class e_L of the virtual bundle (56) and the insertion $c_1(L)$. It is then not hard to see that Proposition A.1 applies to generalized L -twisted invariants of the form (32) as well.

4.1.2 The First Vanishing Theorem

By keeping track of the reduction scheme of [31] we have the following vanishing theorem.

Theorem 4.1 *Suppose that a non-fiber class B has the property that $\pi_*(B)$ is not a sum of a uniruled class and an effective class of D . Then any relative invariant of the form*

$$\langle \pi^* \iota^* \varpi, \tau_{i_1}([pt]_{D_0}[D_0]), \tau_{i_2}(\beta_2[D_0]), \dots, \tau_{i_k}(\beta_k[D_0]) | \mu \rangle_B^{P, D_\infty} \quad (34)$$

is equal to zero.

Proof Since π is an equivariant map, each $\pi^* \iota^* \alpha_i$ is already an equivariant class, namely, its equivariant extension is itself hence it contains no equivariant parameter. Applying the localization formula (60), the relative invariant is given by

$$\sum_F \int_{[F]^{\text{virt}}} \frac{\prod_j \tilde{\psi}_j^{i_j} \varpi}{e(N^{\text{virt}})}, \quad (35)$$

where each F is indexed by a bipartite graph and $\tilde{\psi}_j$ is the equivariant extension of ψ_j .

Choose a self dual basis $\{\alpha_i\}$ of $H^*(D; \mathbb{R})$ and let $\{\check{\alpha}_i\}$ be its dual basis. Then the diagonal class of D is simply

$$[\Delta] = \sum_i \alpha_i \otimes \check{\alpha}_i. \quad (36)$$

Fix a bipartite graph Λ corresponding to a component F . Let E be the set of oriented edges of Λ . For each edge of Λ , we insert the diagonal class $[\Delta]$ in (36) to split $[F]^{\text{virt}}$ as a disjoint union of products of virtual fundamental cycles, where each product has factors indexed by vertices of Λ . The union is over the space T of the maps from E to the set of ordered pairs $\{(\alpha_i, \check{\alpha}_i)\}$ such that two edges with opposite orientations map to opposite pairs. For each $P \in T$ we define P_λ to be the set of elements of $\{\alpha_i\}$, each coming from P and an edge originating at λ .

We thus can write explicitly the contribution of each bipartite graph as a sum

$$\sum_{P \in T} \prod_{\text{vertices}} V_{\lambda, P|_\lambda}, \quad (37)$$

where $V_{\lambda, P|_\lambda}$ is the twisted or rubber invariant at λ with added marked points constrained by $P|_\lambda$.

For each vertex λ , we denote its curve class by $B(\lambda)$. It is clear that $\pi_*(B) = \sum_\lambda \pi_*(B(\lambda))$.

Lemma 4.2 *Each $V_{\lambda, P|_\lambda}$ can be expressed as*

$$\sum_{\pi_*(B(\lambda))=B_1+\dots+B_k} a \langle \cdot, \cdot \rangle_{B_1}^D \cdots \langle \cdot, \cdot \rangle_{B_i}^D \cdots \langle \cdot, \cdot \rangle_{B_k}^D, \quad (38)$$

where a is a combinatoric constant, each insertion of $V_{\lambda, P|_\lambda}$ appears as an insertion of a factor invariant, and the factor invariants are linked by dual insertions from the diagonal class.

Proof There are two types of vertices, D_0 -vertices and rubber vertices.

If λ is a D_0 -vertex, $V_{\lambda, P|_\lambda}$ is a generalized L -twisted GW invariant of the form (32). We simply apply Farber–Pandharipande’s (see [3]) argument reviewed in Sect. A.1. The basic idea is that by marking an additional point we can express $V_{\lambda, P|_\lambda}$ as an ordinary GW invariant of D with an additional marked point plus the boundary contributions. By Proposition A.1, it can thus be written as (38), where the term with $k = 1$ is the principal term, and all other terms are boundary contributions.

When λ is a rubber vertex, it follows from Sect. A.3.2 that $V_{\lambda, P|_\lambda}$ is a twisted rubber invariant as defined in Sect. A.2.3. The much more complicated argument of Maulik and Pandharipande [31] in Sects. A.2–A.5 culminating in Propositions A.3 and A.4 achieves the same. $\square$

Suppose any invariant of the form (34) is nonzero. Then one summand in (35) is nonzero. Hence one summand in the corresponding sum (37) is nonzero, and this is possible only if one $V_{\lambda, P|_\lambda}$ is nonzero.

Clearly in each summand of (37) there is a vertex λ such that V_{λ, P_λ} contains a descendant point insertion. We call one D_0 -vertex λ a constant vertex if $B(\lambda) = \pi_*(B(\lambda))$ is zero, and a rubber vertex a constant vertex if $\pi_*(B(\lambda))$ is zero. By Lemma 4.3 below we can assume such a vertex is not a constant vertex.

Apply Lemma 4.2 to such a vertex. By the last statement about dual insertions and the zero curve class equation for ordinary GW invariants, we can conclude that for some $B_i \neq 0$ the corresponding invariant involves a point insertion. Thus we can express $\pi_*(B(\lambda))$ as an effective decomposition such that one of the summands is a uniruled class. This is a contradiction. $\square$

Lemma 4.3 *Suppose one summand in (37) is nonzero. If the vertex λ containing a point insertion is a constant vertex, then there is another vertex λ' in the graph such that $B(\lambda') \neq 0$ and $V_{\lambda', P_{|\lambda'}}$ contains a point insertion.*

Proof We have to show that the point insertion can always be applied to a nonconstant vertex. Suppose first that the point insertion is on a constant vertex in D_0 . We claim that any other D -insertions of this vertex must be the fundamental class. Suppose that the constant vertex contains absolute marked points $x_{j_1}, \dots, x_{j_h}$. There is an additional marked point from the edge connecting it to a rubber vertex. The moduli space is then $\overline{\mathcal{M}}_{0, h+1}$.

Via the diagonal insertion process it is easy to see that the contribution of the constant vertex under consideration is the summation of terms

$$\int_{\overline{\mathcal{M}}_{0, h+1}} \alpha_i H_i,$$

for some equivariant class H_i . This term is nonzero only if $\alpha_i = 1$. Hence, its dual $\check{\alpha}_i = [pt]$ is inserted into the contribution of the corresponding rubber index.

If the rubber vertex under consideration is also constant. We can apply the same argument to transport the point insertion into the next vertex along the graph until reaching a nonconstant vertex. $\square$

Recall that V is the area of the minimal uniruled class of D_0 . The above theorem implies that the first part of Theorem of 3.38.

Corollary 4.4 *Suppose that a non-fiber class B has the property that $\omega(B) < V$. Then,*

$$\langle \pi^* \iota^* \varpi, \tau_{i_1}([pt]_{D_0}[D_0]), \tau_{i_2}(\beta_2[D_0]), \dots, \tau_{i_k}(\beta_k[D_0]) | \mu \rangle_B^{P, D_\infty} = 0.$$

Proof We observe that $B = B_0 + |\mu|F$ for $B_0 = \pi_*(B) \in H_2(D_0, \mathbb{Z})$. Therefore, $\omega(\pi_*(B)) \leq \omega(B) < V$. By the definition of V , $\pi_*(B)$ cannot be the sum of an uniruled class and an effective class of D . $\square$

4.1.3 The Second Vanishing Theorem

Another case of interest is the special graph in the degeneration formula where the degeneration graph lies completely in P . This is an admissible invariant with no

relative insertions on D_∞ . The purpose of this subsection is to show that the corresponding relative invariant of (P, D_∞) is zero. Namely, we would like to calculate certain admissible invariants with empty relative insertion,

$$\langle \pi^* \iota^* \varpi, \tau_{i_1}([pt]_{D_0}[D_0]), \tau_{i_2}(\beta_2[D_0]), \dots, \tau_{i_k}(\beta_k[D_0]) | \emptyset \rangle_A^{P, D_\infty}, \quad (39)$$

with $m = \sum_j (i_j + 1) \leq D_0 \cdot A$. Since there are no relative insertions at D_∞ , we have $D_\infty \cdot A = 0$. It implies that $A \in H_2(D_0, \mathbb{Z})$. The dimension condition is

$$\begin{aligned} & 2(C_1(A) + D_0 \cdot A + 2 + n - 2 + k + l) \\ &= \sum_j i_j + 2k + 2n + \sum_t \deg(\beta_t) + \deg(\varpi). \end{aligned} \quad (40)$$

Recall here l is the number of α insertions in ϖ . Using the assumption $m \leq D_0 \cdot A$, we have

$$\begin{aligned} & 2(C_1(A) + n - 2 + k + l) \\ & \leq 2n + \sum_j \deg(\beta_j) + \deg(\varpi). \end{aligned} \quad (41)$$

Theorem 4.5 Suppose that $0 < \omega(\pi_*(A)) \leq V$ or $\pi_*(A)$ is not a sum of a uniruled class and nonzero effective class of D and $m = \sum_t (i_t + 1) \leq D_0 \cdot A$. Then,

$$\langle \pi^* \iota^* \varpi, \tau_{i_1}([pt]_{D_0}[D_0]), \tau_{i_2}(\beta_2[D_0]), \dots, \tau_{i_k}(\beta_k[D_0]) | \emptyset \rangle_A^{P, D_\infty} = 0.$$

Notice that unlike Theorem 4.1, $\pi_*(A)$ could be a uniruled class.

Proof If a constant vertex is a D_0 -vertex, it represents constant stable maps into D_0 . If a constant vertex is a rubber vertex, it represents a multiple of fiber classes in the rubber. Such a rubber stable map necessarily contains relative insertions which contradicts to our assumption. Therefore, rubber constant vertices do not exist in our situation.

There are three cases to consider.

Case 1: Suppose that our bipartite graph consists of no edges and only one D_0 -vertex. Since there are no rubber constant vertices, there are no other constant vertices either. This case has been analyzed in Sect. 4.1.1, in particular, the fixed point set is simply $\overline{\mathcal{M}}_{0,k+l}^D(A, pt, S_2, \dots, S_k)$.

By the localization formula, the contribution of this graph is

$$\int_{[\overline{\mathcal{M}}_{0,k+1}^D(A, pt, S_2, \dots, S_k)]^{\text{vir}}} \frac{\prod_t \bar{\psi}_j^{i_j} \varpi}{e(N^{\text{virt}})}.$$

The degree of the virtual fundamental class is

$$2(C_1(A) + n - 3 + k + l) - 2n - \sum_{i \geq 2} \deg(\beta_i).$$

Both absolute and relative insertions have no equivariant parameter. By (41) the total degree of insertions is straightly larger than the degree of the virtual class. Therefore, the contribution is zero.

Case 2: There is only one rubber vertex with homology class B , several D_0 -vertices including some constant D_0 -vertices. The constant D_0 -vertices must be connected to the rubber vertex by edges. But there may be other edges not connected to any constant vertex. Suppose that the partition for the rubber vertex is μ . It is easy to see that

$$|\mu| = D_0 \cdot A, \quad A = B + |\mu|F.$$

To simplify the notation, we first assume that there are only relative absolute insertions, i.e., $\varpi = \emptyset$. Then all the marked points lies either on M_0 or on an edge.

Suppose that the vertex under consideration contains only marked points $i_1, \dots, i_l$ from relative absolute insertions. It has an additional marked point from the edge which is a degree m multiple cover of the fiber. The moduli space is clearly

$$\overline{\mathcal{M}}_{0,l+1} \times (S_{i_1} \cap \dots \cap S_{i_l}).$$

Suppose that the number of constant vertices is l'_μ . We also need to consider the edge containing a marked point, say x_i . Since the image of x_i has to be in S_i , it contributes a factor

$$2n - \deg(\beta_i)$$

to the dimension of fixed point loci. We treat it as part of M_0 . Let k' be the number of such edges. It is clear that

$$l'_\mu + k' \leq l_\mu.$$

The dimension of M_0 is

$$2n(l'_\mu + k') - 2n - \sum_j \deg(\beta_j) + 2(k - k') - 4l'_\mu.$$

Let us compute the virtual dimension of the rubber moduli space $\widetilde{\overline{\mathcal{M}}}_{0,0}(B, \mu|\emptyset)$. A rubber stable map can be thought as a $\mathbb{C}^*$ -equivalence class of stable maps to P relative to both D_0 and D_∞ . The relative condition determines a unique lift of $B \in H_2(D_\infty, \mathbb{Z})$ to a class of P . A moment of thought tells us that the lift is A itself. Then,

$$\begin{aligned} \dim_{\mathbb{C}} [\widetilde{\overline{\mathcal{M}}}_{0,0}(B, \mu|\emptyset)]^{\text{vir}} &= C_1(A) + D_0 \cdot A + n + 1 - 3 + l_\mu - |\mu| - 1 \\ &= C_1(A) + n - 3 - l_\mu. \end{aligned} \quad (42)$$

Adding back ϖ and using the fact that $m \leq D \cdot A = |\mu|$, the dimension of fixed loci is

$$\begin{aligned} & 2(C_1(A) + n - 3 + l - l_\mu) + 2(k - k') - 4l'_\mu - 2n - \sum_t \deg(\beta_t) \\ &= 2(C_1(A) + n - 2 + k + l) - 2n - \sum_j \deg(\beta_j) - 2(k' + l_\mu + 2l'_\mu - 1) \\ &< \deg(\varpi). \end{aligned} \quad (43)$$

Hence, the contribution is zero.

General Case: The general case contains at least two vertices with non-fiber homology classes. Let us consider the marked point corresponding to the point class. There are two cases, either on a nonconstant vertex or on a constant vertex.

By Lemma 4.3 we can transport the point D -insertion on a constant vertex to one on a nonconstant vertex. Suppose that the homology class of this nonconstant vertex is B . Then, $\omega(\pi_*(B)) < \omega(\pi_*(A))$. By the argument of Theorem 4.1, the contribution is zero. $\square$

4.2 A Non-vanishing Theorem

In previous section, we prove a vanishing theorem for the admissible invariant of (P, D_∞) with empty relative insertion on D_∞ . In this section, we consider a sup-admissible invariant as in (39) with

$$k = D_0 \cdot A + 1. \quad (44)$$

Consider the divisor invariant

$$\langle l^* \varpi, \tau_{i_1}([pt]_{D_0}[D_0]), \tau_{i_2}(\beta_2), \dots, \tau_{i_k}(\beta_k) \rangle_A^D. \quad (45)$$

Its dimension condition is

$$2(C_1(A) + n + k + l - 3) = 2 \sum_t i_t + 2n + \sum_t \deg(\beta_t) + \deg(\varpi).$$

By (40) both invariants (39) and (45) are well-defined only under the assumption (44).

Theorem 4.6 *Suppose that $0 < \omega(A) \leq V$ and $k = D_0 \cdot A + 1$. Then,*

$$\begin{aligned} & \langle \pi^* l^* \varpi, [pt]_{D_0}[D_0], \beta_2[D_0], \dots, \beta_k[D_0] | \emptyset \rangle_A^{P, D_\infty} \\ &= c \langle l^* \varpi, [pt], \beta_2, \dots, \beta_k \rangle_A^D \end{aligned} \quad (46)$$

for some $c \neq 0$.

Proof The proof is similar to that of the second vanishing theorem. Applying localization formula, the relative invariant is the non-equivariant limit of

$$\sum_F \int_{[F]^{\text{virt}}} \frac{\varpi}{e(N^{\text{virt}})}.$$

We again divide into three cases.

Case 1: Suppose that our bipartite graph consists of only one vertex corresponding to the ordinary stable map into D_0 and no edges. In this case, the fixed point set is simply $\overline{\mathcal{M}}_{0,k+l}^D(A, pt, S_2, \dots, S_k)$. By the localization formula, the contribution of this graph is

$$\int_{[\overline{\mathcal{M}}_{0,k+l}^D(A, pt, S_2, \dots, S_k)]^{\text{virt}}} \frac{\varpi}{e(N^{\text{virt}})}.$$

Since ϖ have no equivariant parameter, we only need to take the non-equivariant limit of $\frac{1}{e(N^{\text{virt}})}$. The latter is precisely the Euler class e_L appearing in twisted GW invariants. Therefore, the contribution is just the twisted GW invariant

$$\langle \iota^* \varpi, [pt], \beta_2, \dots, \beta_k \rangle_A^{D,L}.$$

In the case of $k = D_0 \cdot A + 1$,

$$\dim[\overline{\mathcal{M}}_{0,k+l}^D(A, pt, S_2, \dots, S_k)]^{\text{virt}} = 2 \sum_j i_j + \deg(\varpi).$$

We immediately obtain the contribution as

$$c \langle \iota^* \varpi, [pt], \beta_2, \dots, \beta_k \rangle_A^D,$$

where c is the constant term of $\frac{1}{e(N^{\text{virt}})}$. Note that N^{virt} has rank zero and hence $\frac{1}{e(N^{\text{virt}})}$ is a degree zero equivariant class. Therefore, it has an expression

$$c + t^{-1} \gamma_1 + t^{-2} \gamma_4 + \dots + t^{-m} \gamma_{2m} + \dots,$$

where γ_{2m} has degree $2m$. On the other hand, the equivariant Euler class $e(N^{\text{virt}})$ is an invertible element after inverting equivariant parameter t . It implies that $c \neq 0$. The precise value of c can also be worked out by the method of [3]. However, we do not need it in our paper.

Case 2: There is only one rubber vertex with homology class B , several D_0 -vertices including some constant vertices. Using the computation in the same c of the second vanishing theorem, the dimension condition and the assumption $k = D_0 \cdot A + 1$, the virtual dimension of fixed loci is seen to be

$$\begin{aligned} & 2(C_1(A) + n - 3 + k + l) - 2n - \sum_j \deg(\beta_j) - 2(k' + l_\mu + l'_\mu) \\ &= \deg(\varpi) - 2(k' + l_\mu + 2l'_\mu) \\ &< \deg(\varpi). \end{aligned} \tag{47}$$

The contribution in this case is therefore equal to zero.

General Case: The general case contains at least two vertices with non-fiber homology classes. The proof is identical to that of the second vanishing theorem. We omit it. $\square$

Remark 4.7 However, there is still a well-defined twisted GW invariant in all the sup-admissible cases. Our expectation is that the invariant of (P, D_∞) is still dominated by the twisted one. We do not know yet how to carry out a correspondence with twisted GW invariants and we hope to come back to it in the future.

Remark 4.8 The vanishing and non-vanishing results in Sects. 4.1 and 4.2 are closely related to the virtual push forward property studied in [4].

4.3 The Proof of Main Theorem

We are finally able to give the proof of Theorem 1.1, which we restate here for the convenience of readers.

Theorem 4.9 *Suppose D is uniruled and A is a minimal uniruled class of D such that*

$$\langle \iota^* \alpha_1, \dots, \iota^* \alpha_l, [pt], \beta_2, \dots, \beta_k \rangle_A^D \neq 0 \quad (48)$$

for $k \leq D \cdot A + 1$, $\beta_i \in H^(D; \mathbb{R})$, and $\alpha_j \in H^*(X; \mathbb{R})$. Then (X, ω) is uniruled.*

Proof Notice that all the insertions of the invariant (1) are non-descendant. By linearity we can assume that each $\beta_i \in \Theta$ and each $\alpha_j \in \mathcal{E}$. Recall that Θ is the chosen self dual basis of $H^*(D; \mathbb{R})$, and $\mathcal{E}$ is the chosen basis of $H^*(X; \mathbb{R})$.

Assume first that the number k of insertions of (1) satisfies

$$k = D \cdot A + 1.$$

Then, as noticed in Sect. 4.2, the following relative invariant of (P, D_∞) :

$$\langle \pi^* \iota^* \alpha_1, \dots, \pi^* \iota^* \alpha_l, [pt], \tilde{\beta}_2, \dots, \tilde{\beta}_k \rangle_A^{P, D_\infty}, \quad (49)$$

is well defined, as the dimension difference of the moduli spaces

$$2D \cdot A + 2$$

matches with the difference of the total cohomology degree $2k$. Moreover, by the minimality of the class A and Theorem 4.6, the invariant (49) is nonzero.

Notice that the invariant (49) is the relative invariant associated to the following standard sup-admissible graph $\Gamma(\varpi | \mu)$ of the symplectic cut with

$$\varpi = (\alpha_1, \dots, \alpha_l), \quad \mu = ((1, [pt]), (1, \beta_2), \dots, (1, \beta_k)). \quad (50)$$

Hence the vector $v_{D\text{-pt}}^{\text{rel}}$ is nonzero.

By Theorem 3.37, the vector $v_{D\text{-pt}}^{\text{abs}}$ is nonzero as well. Notice that all the relevant invariants of X have a point insertion (possibly descendant). Thus X is uniruled by Theorem 2.4.

For the general case that $k \leq D \cdot A + 1$, notice that we can always increase the number of D -insertions by adding divisor insertions of the form $PD(\omega_D)$ to achieve the equality. $\square$

Remark 4.10 We think it is possible to weaken the assumption by the existence of a minimal descendant invariant of the form

$$\langle \iota^* \varpi, \tau_{i_1}([pt]), \tau_{i_2}(\beta_2), \dots, \tau_{i_k}(\beta_k) | \mu \rangle_A^D \neq 0,$$

with $\sum_i (i_i + 1) \leq D \cdot A + 1$. What is missing is an analogue of Theorem 4.6 in this set up.

We are able to weaken the assumption in a different direction in the following subsection.

4.4 Weakening the Minimal Condition

The minimality condition on the uniruled class A is a symplectic condition and is in agreement with the main theme of the paper conceptually. However, given a tamed almost complex structure, it is possible to weaken somewhat the minimal condition.

If J is a tamed almost complex structure, a uniruled class A is called J -indecomposable or just indecomposable if it is not the sum of another uniruled class and a nonzero J -effective class. For example, a uniruled class which is primitive and lies on an extremal ray of the J -effective cone is an indecomposable uniruled class. A minimal uniruled class is obviously indecomposable. Unfortunately, our argument does not extend to the indecomposable situation. Instead, it applies to the following intermediate situation.

We call a uniruled class A of D a globally indecomposable uniruled class of (X, D) if A is not a sum of a uniruled class of D and a nonzero effective class of X . Certainly a globally indecomposable class uniruled class of (X, D) is an indecomposable uniruled class of D . The converse is not true, as a class of D is not necessarily effective in D even if it is effective in X .

Notice that a minimal uniruled class is globally indecomposable. Thus the following is a slightly stronger version of Theorem 4.11.

Theorem 4.11 *Suppose A is a globally indecomposable uniruled class of D such that*

$$\langle \iota^* \alpha_1, \dots, \iota^* \alpha_l, [pt], \beta_2, \dots, \beta_k \rangle_A^D \neq 0$$

for $k \leq D \cdot A + 1$. Then X is uniruled.

Proof We shall sketch the necessary modification of the proof while leaving the details for the interested readers. First step is to modify the partial order where we define $B \leq A$ if A is a sum of B and an effective class. Next, we modify the correspondence.

Definition 4.12 We consider the following subset $I_{D\text{-pt},0}$ of colored standard graphs of X :

- (1) $g(\Gamma) = 0$,
- (2) the fixed class A ,
- (3) admissible graphs with a D -point insertion,

(4) sup-admissible graphs of the form Γ_0

$$\sum_{t=1}^k (i_t + 1) = D \cdot A + 1, \quad (51)$$

(5) empty graph excluded.

To prove the correspondence, we apply the degeneration formula as before. The point of attention is the nonzero term involving non-trivial graphs for both Γ^+ and Γ^- . In this case, we express $A = A_1 + A_2$, where $A_2 = \pi_*(A(\Gamma^+))$ is the effective class of D , and A_1 is an effective class of X . By Theorem 4.1, $A(\Gamma^-)$ is either a fiber class or A_2 is a sum of a uniruled class and an effective class of D . If A_2 is a sum of a uniruled class and an effective class of D , A itself is a sum of a uniruled class of D and an effective class of X . This contradicts the assumption. Therefore, only the case of $A(\Gamma^-)$ being a fiber class can appear. But then $A(\Gamma^+) = A$, which is exactly what we want.

The other situation is of course the special graph. The rest of proof goes through without change. $\square$

5 Applications

As mentioned in the introduction, our main theorem can be applied as an existence theorem of uniruled manifolds. In this section, we apply our main theorem to construct uniruled symplectic manifolds inductively, generalizing several early results of McDuff.

5.1 4-Dimensional Uniruled Divisors in 6-Manifolds

A deep result in dimension 4 is that being uniruled is a smooth property. More precisely, a 4-manifold (M, ω) is uniruled if and only if M is diffeomorphic to a connected sum of $\mathbb{P}^2$ or a S^2 -bundle over a surface with a number of $\overline{\mathbb{P}}^2$. Moreover, the isotopy class of ω is determined by $[\omega]$.

To apply Theorem 4.11 to construct 6-dimensional uniruled manifolds we need to further analyze uniruled classes and the corresponding insertions for these uniruled 4-manifolds.

Proposition 5.1 *If A is a uniruled class of a 4-manifold, then*

- (i) A is represented by an embedded symplectic surface,
- (ii) $C_1(A) \geq 2$,
- (iii) $A^2 \geq 0$,
- (iv) $A \cdot B \geq 0$ for any class B with a non-trivial GW invariant.

Proof The point is that 4-manifolds are semi-positive. Thus, for a generic tamed almost complex structure J and a generic point, only somewhere injective J -holomorphic curves with a smooth domain S^2 contribute to the relevant GW invariant (see [30]). Such a curve can be smoothed to an embedded symplectic surface.

The genus 0 moduli space of a class A has dimension $C_1(A) - 1$. Since there is at least a point insertion, we have (ii).

Now the adjunction inequality in [29], together with (i) and (ii), implies that $A \cdot A \geq 0$.

(iv) follows from (i), (iii) and positivity of intersection in [29]. By (i) we have an embedded J -holomorphic sphere C in the class A for some tamed J . The class B is represented by a union $\cup m_i D_i$ of possibly singular irreducible J -holomorphic curves with multiplicities. If an irreducible component D_i is distinct from C , then apply the positivity of intersection, if $D_i = C$ then apply (iii). $\square$

5.1.1 Simple Case-Proportional

For $\mathbb{P}^2$, let H be the generator of H_2 with positive area. H is a uniruled class and any uniruled class of the form aH with $a > 0$. Obviously, H is the minimal uniruled class. The relevant insertion is (pt, pt) . As pt is a restriction class, i.e., an α class, we can take $k = 1$.

Similarly, for the blow-up of an S^2 -bundle over a surface of positive genus, the fiber class is a uniruled class, and any uniruled class is a positive multiple of the fiber class. The relevant insertion for the fiber class is pt . Thus again we can take $k = 1$.

It is easier to apply Theorem 4.11 in this case.

Corollary 5.2 *Suppose (X^6, ω) contains a divisor D which is diffeomorphic to $\mathbb{P}^2$ or the blow-up of an S^2 -bundle over a surface of positive genus. If the normal bundle N_D is non-negative on a uniruled class, then X is uniruled.*

5.1.2 General Case

For other M^4 , the uniruled classes are not proportional to each other. Thus the minimality condition depends on the class of the symplectic form on M .

We first analyze the easier case of an S^2 -bundles over S^2 . For $S^2 \times S^2$, by uniqueness of symplectic structures, any symplectic form is of product form. Let A_1 and A_2 be the classes of the factors with positive area. It is easy to see that any uniruled class is of the form $a_1 A_1 + b_1 A_2$ with $a_1 \geq 0, a_2 \geq 0$. Thus either A_1 or A_2 has the minimal area.

For the non-trivial bundle $S^2 \tilde{\times} S^2$, let F_0 be the class of a fiber with positive area and E be the unique -1 section class with positive area. If $aF_0 + bE$ is a uniruled class then $b \geq 0$ by (iv) of Proposition 5.1, since $F_0 \cdot E = 1, F_0 \cdot F_0 = 0$. And if $b > 0$, then, by (ii) of Proposition 5.1, $a \geq 1$ as $C_1(E) = 1$. Thus F_0 is always the minimal uniruled class no matter what the symplectic structure is.

Since the relevant insertion for A_1, A_2 and F_0 is just pt , we have the following corollary.

Corollary 5.3 *Suppose $D = S^2 \times S^2$ and the restriction of the normal bundle N_D to the factor with the least area is non-negative, then X is uniruled.*

In the case of the non-trivial bundle, X is uniruled if the restriction of the normal bundle N_D to F_0 is non-negative.

Remark 5.4 The following restatement is related to the contraction criterion in [21]. Suppose (X^6, ω) is a non-uniruled 6-manifold containing an S^2 -bundle D . Then, if $D = S^2 \tilde{\times} S^2$, the normal bundle is negative along the S^2 -fibers, and in the case of $S^2 \times S^2$ where there are two S^2 -directions, the normal bundle is negative along the one with the least area.

The remaining uniruled 4-manifolds are $(\mathbb{P}^2 \# k \overline{\mathbb{P}}^2, \tau)$ with $k \geq 2$. Let H, E_i be the generators of H^2 of the summands with positive area. Then $[\tau]$ is of the form $uH - \sum_{i=1}^k v_i E_i$ with $u, v_i > 0$.

H, E_i all have non-trivial GW invariants. Thus, by (iv) of Proposition 5.1, a class $\xi = aH - \sum_{i=1}^k b_i E_i$ is uniruled only if

$$a \geq 0, \quad b_i \geq 0. \quad (52)$$

The condition that ξ has square 0 is

$$a^2 = \sum_i b_i^2. \quad (53)$$

If (53) is satisfied, then the condition that ξ is represented by an embedded sphere of genus 0 is given by the adjunction formula

$$3a = \sum_i b_i + 2. \quad (54)$$

Definition 5.5 Any primitive class ξ satisfying (52)–(54) is called a fiber class.

Theorem 5.6 Suppose (D, τ) is a symplectic $\mathbb{P}^2 \# k \overline{\mathbb{P}}^2$ with $k \geq 1$. Then a fiber class is a uniruled class. In addition, any uniruled class is the sum of a positive multiple of a fiber class and another class with positive symplectic area.

Consequently, if (D, τ) is a symplectic divisor of a symplectic 6 manifold (X, ω) and the normal bundle N_D is non-negative on a fiber class with the minimal τ -area, then X is uniruled.

Proof By [14], a fiber class is equivalent to the indecomposable uniruled class $H - E_1$ via diffeomorphisms preserving the canonical class.

By Proposition 5.1, a uniruled class is represented by an embedded symplectic surface with non-negative self-intersection. By [14] such a class is equivalent to a reduced class via diffeomorphisms preserving the canonical class. A class $\xi = aH - \sum_{i=1}^k b_i E_i$ is called reduced if

$$a \geq b_1 + b_2 + b_3, \quad b_i \geq b_{i+1} \geq 0.$$

It is also shown in [14] if the surface is actually a sphere, then the class is equivalent to

$$2H, \quad H - E_1, \quad (l+1)H - lE_1, \quad (l+1)H - lE_1 - E_2, \quad \text{where } l \geq 2.$$

The effective curve cone is generated by -1 classes.

Hence it suffices to show that a reduced class with non-negative self-intersection is the sum of a positive multiple of a fiber class and another class with positive symplectic area. $\square$

To enumerate fiber classes, we notice that

$$H - E_1 = (H - E_1 - E_2) + E_2,$$

i.e., it is the sum of two -1 classes whose intersection number is equal to 1. Thus a fiber class is the sum of two -1 classes.

When $k \leq 8$, there are only finitely many -1 classes. So it is straightforward though tedious to list all the fiber classes.

Any class of the form $H - E_i$, $1 \leq i \leq k$ is such a class, and when $k \leq 3$, there are no other classes.

When $k = 4$, there is a new class with the coefficient of H being 2,

$$2H - E_1 - E_2 - E_3 - E_4.$$

By choosing any four distinct numbers between 1 and $k \geq 4$, we get many such classes for higher k . When $k = 4, 5$, there are no other new classes.

When $k = 6$, there are six new classes with the coefficient of H being 3,

$$3H - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6,$$

and its permutations in the E_i . For higher k , there are similar classes.

When $k = 7$, there are additional classes,

$$(4|2, 2, 2, 1, 1, 1, 1), \quad (5|2, 2, 2, 2, 2, 1)$$

and their permutations in the E_i .

When $k = 8$, there are additional classes,

$$(4|3, 1, 1, 1, 1, 1, 1, 1), \quad (5|3, 2, 2, 2, 1, 1, 1, 1),$$

$$(6|3, 3, 2, 2, 2, 2, 1, 1), \quad (7|4, 3, 2, 2, 2, 2, 2, 1),$$

and their permutations in the E_i .

5.2 Higher-Dimensional Case

We start with the proof of Corollary 1.2.

Proof The homologically injective case, which is equivalent to being cohomology surjective, is clear.

For the case of a projective uniruled divisor, according to Theorem 2.6, there is a nonzero invariant $I_{p,q}$ for a minimal uniruled class A . We now only need to observe that $[\omega|_D]^p = [\omega^p|_D]$. $\square$

As we already mentioned that a Fano manifold is projectively uniruled. In particular, a hypersurface of $\mathbb{P}^n$ (for $n \geq 4$) of degree at most n is Fano and hence projectively uniruled.

Corollary 5.7 *Suppose $n \geq 4$ and D is a Fano hypersurface symplectic divisor of X . If $N_D = \lambda[\omega_D]$ with $\lambda \geq 0$, then X is uniruled.*

Proof Since $n \geq 4$, by the Lefschetz hyperplane theorem, D has $b_2 = 1$. As $N_D = \lambda[\omega_D]$ for some λ for any uniruled class of D , and in particular a minimal uniruled class, the statement follows from Corollary 1.2. $\square$

Of course a particular case is $D = \mathbb{P}^{n-1}$ discussed in Sect. 2. In general case we still need to verify the minimal condition. Of course the uniruled divisor needs not to be a projective manifold. For instances, the divisor could be a rather general uniruled fibration discussed in Sect. 2. Let us treat the case of a symplectic $\mathbb{P}^k$ -bundle. Since the line class in the fiber is uniruled, and the relevant insertions can be taken to be $(pt, \omega|_D^k)$, we have the following corollary.

Corollary 5.8 *Suppose D is a symplectic divisor of X . If D is a projective space bundle with the fiber class being the minimal uniruled class and normal bundle N_D is non-negative along the fibers, then X is uniruled.*

McDuff also considered the case of product $\mathbb{P}^k$ -bundles in [27]. A natural source of such D is from blowing up a ‘non-negative’ $\mathbb{P}^k$ with a large trivial neighborhood. Suppose $\mathbb{P}^k \subset X$ has trivial normal bundle. Then the blow-up along $\mathbb{P}^k$ has a divisor $D = \mathbb{P}^k \times \mathbb{P}^{n-k-1}$. The normal bundle of D along a line in $\mathbb{P}^k$ is trivial. Similar to the case of $S^2 \times S^2$, we can argue that the area of this line is minimal among all uniruled class of D . In particular, as observed by [27], a symplectic $\mathbb{P}^1$ with a sufficiently large product symplectic neighborhood can only exist in a uniruled manifold.

In fact, we can prove more.

Corollary 5.9 *Suppose S is a uniruled symplectic submanifold whose minimal uniruled class has area η and insertions all being restriction classes. If S has trivial symplectic neighborhood of radius at least η . Then X is uniruled.*

We will treat the more general case of ‘non-negative’ normal bundle in another paper on uniruled submanifolds.

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Appendix A: Relative GW Invariants of (P, D_∞)

In this appendix we collect several important variations and techniques in Gromov–Witten theory in suitable forms so they can be readily applied in Sect. 4.

A.1 Twisted Invariants of D

Let L be a Hermitian line bundle over D and $P = \mathbb{P}(L \oplus \mathbb{C})$ with a Hermitian connection. Then $\pi : P \rightarrow D$ is a $\mathbb{P}^1$ -bundle with zero section D_0 and infinity section D_∞ .

There is an important twisted Gromov–Witten theory treated by Farber and Pandharipande in [3], and Coates and Givental in [2].

Suppose that $f : \Sigma \rightarrow D$ is a stable map. Then there is a twisted operator over D

$$\mathcal{D}_f^L : \Omega^0(f^*L) \rightarrow \Omega^{0,1}(f^*L). \quad (55)$$

We thus have a virtual vector space

$$H^1(f^*L) - H^0(f^*L), \quad (56)$$

from the cokernel and kernel of $\mathcal{D}_f^L$. They piece together to a virtual bundle over the moduli space $\overline{\mathcal{M}}_{0,k+l}^{D_0}(A)$ of stable maps denoted the same way. Let e_L denote its Euler class. One way to define e_L is to use the S^1 -action on the fiber to define the equivariant Euler class of (56) and then take the non-equivariant limit (cf. [2, Sect. 4]).

Associated to the relative invariant (27) we have the GW invariant of $D_0 = D$ twisted by e_L defined as

$$\begin{aligned} & \langle \iota^* \varpi, \tau_{i_1}([pt]), \tau_{i_2}(\beta_2), \dots, \tau_{i_k}(\beta_k) \rangle_{0,A}^{D, e_L} \\ &= \int_{[\overline{\mathcal{M}}_{0,k+l}^D(A)]^{\text{vir}}} \prod_s \iota_s^* \alpha_s [pt] \psi_1^{i_1} \prod_{t=2} \beta_t \psi_t^{i_t} \wedge e_L. \end{aligned} \quad (57)$$

The above twisted invariant has been studied in [2, 3] via marking an additional point. We collect their results and summarize in the following form.

Proposition A.1 *A twisted invariant with curve class A can be expressed in terms of a similar invariant replacing e_L by a descendant at the additional marked point together with some products of ordinary invariants. The total curve class of each product of invariants is still equal to A , each insertion of the original invariant appears as an insertion of a factor invariant, and the factor invariants are linked by dual insertions from the diagonal class.*

These products have the stated properties because they correspond to boundary contributions from nodal curves and are obtained via diagonal splitting.

A.2 Type I/II Relative Invariants and Twisted Rubber Invariants

Let Y , $\mathcal{L}$ and $\mathcal{R}$ all denote the $\mathbb{P}^1$ bundle $\mathbb{P}(N_D \oplus \mathbb{C})$. Here, $\mathcal{L}$ and $\mathcal{R}$ stand for left and right rubbers.

For any non-negative integer m , construct $\mathcal{R}_m$ by gluing together m copies of $\mathcal{R}$, where the infinity section of the i th component is glued to the zero section of the $(i+1)$ th component for $1 \leq i \leq m-1$. Denote the zero section of the i th component by $D_{i,0}$, and the infinity section by $D_{i,\infty}$, so $\text{Sing } \mathcal{R}_m = \bigcup_{i=1}^{m-1} D_{i,\infty}$. Define Y_m

by gluing Y along its infinity section denoted by $D_{0,\infty}$ to $\mathcal{R}_m$ along $D_{1,0}$. Thus $\text{Sing } Y_m = \bigcup_{i=0}^{m-1} D_{i,\infty}$. $Y_0 = Y$ will be referred to as the level 0 component and the $\mathcal{R}_i$ will be called the level i component. We will also sometimes denote $D_{m,\infty}$ by D_∞ if there is no confusion.

Let $\text{Aut}_D \mathcal{R}_m$ be the group of automorphisms of $\mathcal{R}_m$ preserving each $D_{i,0}$, $D_{i,\infty}$, $1 \leq i \leq m$, and the morphism to $D_{1,0}$. And let $\text{Aut}_D Y_m$ be the group of automorphisms of Y_m with restriction to $\mathcal{R}_m$ being contained in $\text{Aut}_D \mathcal{R}_m$ and to Y_0 being the identity. Clearly, $\text{Aut}_D Y_m = \text{Aut}_D \mathcal{R}_m \cong (\mathbb{C}^*)^m$, where each factor of $(\mathbb{C}^*)^m$ dilates the fibers of the $\mathbb{P}^1$ -bundle $\mathcal{R}_i \rightarrow D_{i,0}$. Denote by $\pi[m]: Y_m \rightarrow Y$ the map which is the identity on the root component Y_0 and contracts all the bubble components to $D_{1,0}$ via the fiber bundle projections.

Similarly we can form $\mathcal{L}_n$ and glue it to the left of Y_m . We denote the resulting chain of $\mathbb{P}^1$ -bundle by ${}_n Y_m$. If $m = 0$ we simply write ${}_n Y$ for ${}_n Y_0$. Of course Y_m is the same as ${}_0 Y_m$. $\mathcal{L}_j$ is regarded as the level $-j$ component. The automorphism group for $\mathcal{L}_n$ is defined in the same way as for Y_m , and the extension to ${}_n Y_m$ is obvious.

A.2.1 Type I Invariants and Type II Invariants

Given a relative insertion μ at D_∞ , the relative moduli space for (Y, D_∞) consists of the union over m of equivalence classes of marked relative stable maps into $(Y_m, D_{m,\infty})$ satisfying the relative constraint. It comes with a virtual fundamental homology class.

There are also natural cohomology classes associated to marked points: the pull back classes via the evaluation maps at the marked points, as well as descendant classes. Thus we can integrate these classes over the virtual class to define relative invariants, called type I invariants in [31].

In the same way we can define relative invariants for (Y, D_0) by considering marked relative stable maps into $({}_n Y, D_{-n,0})$ for various n . These invariants are also called type I invariants.

Type II invariants are relative invariants for Y relative to both D_0 and D_∞ . They are defined via moduli spaces of equivalence classes of marked stable maps into $({}_n Y_m, D_{-n,0}, D_{m,\infty})$.

A.2.2 Distinguished Type II Invariants and Their Orders

A distinguished invariant of type II is an invariant of $(Y, D_0 \cup D_\infty)$ with a distinguished insertion of the form $[D_\infty]\delta$ (notice that our definition is different from that in [31] where $[D_0]$ is in place of $[D_\infty]$).

Distinguished type II invariants are ordered in a similar way as in Definition 3.9. The new features are that parts (3) and (4) of that order are replaced successively by the number of non-distinguished insertions, the total degree of D_0 -relative insertions, the total degree of D_∞ -relative insertions, the degree of the distinguished insertion.

A.2.3 Non-rigid Targets and Rubber Invariants with Ψ Insertions

Let us denote by ${}_n Y_m \sim$ the fiberwise collapsing of Y_0 in ${}_n Y_m$. ${}_n Y_m \sim$ is called a non-rigid target, due to the $\mathbb{C}^*$ action at each level.

Given relative insertions μ and ν on the two sides $D_{-n,0}$ and $D_{m,\infty}$, there is the rubber moduli space $\overline{\mathcal{M}}_{0,l}^Y(B; \mu|\nu)^\sim$ consisting of the union over m, n of equivalence classes of stable pseudo-holomorphic maps from l -marked genus 0 curve into ${}_n Y_m^\sim$ with class $B \in H_2(Y; \mathbb{Z})$. Notice that here the stability is the rubber stability and the equivalence is the rubber equivalence under enlarged rubber automorphisms.

As for the ordinary moduli space, a rubber moduli space also comes with the virtual fundamental (homology) class. Evaluation classes and descendant classes are invariant under enlarged rubber automorphisms and hence define classes on the rubber moduli space. Thus we can integrate these classes over the virtual class to define invariants, which are called rubber invariants in [31].

In fact, there are two additional degree 2 cohomology classes on the rubber moduli space coming from two classes on the relative moduli space, Ψ_0 and Ψ_∞ . They are the cotangent classes at D_0 and D_∞ , respectively. We give the description of Ψ_0 here following [4, p. 132], the construction for Ψ_∞ is the same. Given a relative map f with a marked point x_i mapping to D_0 , if the multiplicity of f at x_i is α_i , then define

$$\Psi_0 = \alpha_i \psi_i + e v_{x_i}^* c_1(N_{D_0|Y}) = \alpha_i \psi_i + e v_{x_i}^* c_1(L). \quad (58)$$

It is independent of the choice of x_i as long as it is mapped to D_∞ .

An important new feature is that these classes are not generated by evaluation classes. We call rubber invariants with Ψ insertions twisted rubber invariants. We will see very soon that twisted rubber invariants appear in the relative virtual localization formula. They are generally very difficult to evaluate explicitly.

A.3 Relative Virtual Localization on $\mathbb{P}^1$ Bundle

In this subsection we follow [4, pp. 135–138].

We switch back to the notation P , i.e., $P = Y$ and ${}_n P_m = {}_n Y_m$. P carries a natural S^1 -action by rotating the fibers. The fixed point loci are precisely D_0 and D_∞ . It induces a natural action on the moduli space of relative stable maps $\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)$.

A.3.1 Fixed Point Components Labeled by Bipartite Graphs

The set of connected components of fixed point locus is indexed by bipartite graphs. Each vertex corresponds to either the ordinary (connected) stable maps into D_0 or stable maps into rubber over D_∞ . The first type of vertex is called an ordinary vertex, and the second type is called a rubber vertex.

Each edge corresponds to Galois covers of fibers of P totally ramified over $D_{0,0}$ and $D_{0,\infty}$ with no marked points away from $D_{0,0}$ and $D_{0,\infty}$.

The connection data of the Galois covers is described by a sum over cohomology weighted partitions specifying the rubber relative conditions on the connecting divisor.

Given each bipartite graph, each component F is then a finite quotient of a fiber product $M_0 \boxtimes \mathcal{M}_1$ by a finite group G . Here

- M_0 is the product over vertices on D_0 of ordinary moduli spaces $\overline{\mathcal{M}}_{0,l}^D(B)$ to D_0 , adding the nodes y_i where the fibers are attached,

- $\mathcal{M}_1$ is the product of the rubber moduli space $\overline{\mathcal{M}}_{0,l}^P(B; \mu|v) \cong$ over vertex on D_∞ , adding the nodes y_i where the fibers are attached,
- $\boxtimes$ denotes a fiber product over evaluation maps to $D_{0,\infty} = D_{1,0}$ at the gluing points y_i (in the intersection of levels 0 and 1),
- G is the group of permutations of the points y_i that preserves the multiplicities.

The virtual fundamental class of F is the one induced by this product structure,

$$[F]^{\text{virt}} = \frac{1}{|G|} ([M_0]^{\text{virt}} \boxtimes [\mathcal{M}_1]^{\text{virt}}).$$

A.3.2 Equivariant Euler Class

For each component $F \subset \overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)$, the equivariant Euler class of the virtual normal bundle $N_{F/\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)}^{\text{virt}}$ is also a fiber product,

$$e(N_{F/\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)}^{\text{virt}}) = e_0(N_{F/\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)}^{\text{virt}}) e_1(N_{F/\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)}^{\text{virt}}).$$

e_0 is itself a product with one factor being the equivariant Euler class of the virtual bundle

$$H^0(f^*N_{D_0/P}) - H^1(f^*N_{D_0/P}) = H^0(f^*L) - H^1(f^*L) \quad (59)$$

on M_0 . This is why we need to consider the twisted invariants as in Sect. A.1. There are two other types of factors of e_0 . Denote the generator of $H_{\mathbb{C}^*}^*(pt)$ by t . Every node y_i in $D_{0,0}$ connecting to a multiple cover of degree m contributes a product

$$\left(\frac{t + ev_{y_i}^* c_1(L)}{m} - \psi_{y_i} \right) \prod_{k=1}^{m-1} \frac{k(t + ev_{y_i}^* c_1(L))}{m}.$$

Here the first term corresponds to part (iii) in [4, p. 137], and the second term corresponds to (ii)(b) there.

Concerning e_1 , each marked point x_i in D_0 on a multiple cover of degree m contributes to e_1 a product

$$(-t - \psi_0) \prod_{k=1}^m \frac{k(t + ev_{x_i}^* c_1(L))}{m}.$$

Here the first term corresponds to part (iv) in [4, p. 137], and the second term corresponds to (ii)(a) there. This is why we need to consider twisted rubber invariants in Sect. A.2.3.

A.3.3 Virtual Localization Formula

The virtual localization formula equates

$$[\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)]^{\text{virt}} = \sum_F \frac{[F]^{\text{virt}}}{e(N_{F/\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)}^{\text{virt}})} \quad (60)$$

in the equivariant cohomology of $\overline{\mathcal{M}}_{0,k+l}^{P,D_\infty}(A)$.

Thus, as mentioned, to apply the virtual relative localization we need to evaluate an ordinary twisted invariants over M_0 and certain twisted rubber invariants. As mentioned in Sect. 4.1, ordinary twisted invariants have been treated in [3]. As for twisted rubber invariants, a nice algorithm, the rubber calculus, has been developed in [31].

A.4 Rubber Calculus

In this subsection we review the rubber calculus and in addition, we keep track how the curve class behaves.

A.4.1 The First Reduction—Removing Ψ_0

The first step is to remove Ψ insertions from rubber invariants, i.e., express twisted rubber invariants in terms of (ordinary) rubber invariants.

It involves the dilaton equation, the divisor equation and the topological recursion relation. We only describe them in the form needed.

Dilation Equation: if $c = 2g - 2 + n + l(\mu) + l(v) \neq 0$ for a rubber invariant, then it is $c^{-1} \times$ the rubber invariant with an extra absolute insertion $\tau_1(1)$. The curve class is again preserved.

Divisor Equation: if a divisor is nonzero on the curve class, a Ψ_0^k -rubber invariant is the sum of the rubber invariant with the divisor added as an absolute insertion, rubber invariants with smaller descendant powers, and Ψ_0^{k-1} -rubber invariants. The curve classes are still preserved.

Finally we describe the application of the topological recursion relation to a rubber invariant with a Ψ_0 insertion and at least one absolute insertion. The starting point is that, by (58), Ψ_0 over the rubber moduli space can be expressed as $ev_p^*(c_1(L)) + \alpha^*\psi_3$, where p is the marked point carrying one absolute insertion. α is the canonical map to the Artin stack of 3-pointed genus 0 curves, sending a rubber map f to the fiber C_f containing p with marked points $D_0 \cap C_f$, $f(p)$, $D_\infty \cap C_f$.

Topological Recursion: ψ_3 is expressed as a sum of boundary divisor, hence a Ψ_0^k -rubber invariant with at least one absolute insertion is the sum of a Ψ_0^{k-1} -invariant, and Ψ_0^{k-1} -rubber invariants multiplied by a rubber invariant with no Ψ_0 insertion and at least one absolute insertion. The curve classes are possibly smaller.

Remark A.2 To be able to apply topological recursion, we need at least one absolute insertion. This can be achieved either by the dilaton equation or the divisor equation.

For a fiber class Ψ_0^k -rubber invariant, the target stability ensures that the relevant c in the dilaton equation is nonzero, hence it can be turned into a Ψ_0^k -rubber invariant with an extra dilaton insertion, and hence at least one absolute insertion. Apply the topological recursion to reduce the dilaton rubber invariant to rubber invariants with fewer Ψ_0 insertions. Repeating the cycle yields rubber invariants without Ψ_0 insertions.

For a non-fiber class Ψ_0^k -rubber invariant, first add a divisor insertion $\pi^*\omega_D$. As $\pi^*\omega_D$ pairs positively with a non-fiber class, the divisor equation can be used to express the original invariant in terms of the new invariant which has the divisor insertion and hence at least one absolute insertion, together with either Ψ_0^{k-1} -rubber invariants or Ψ_0^k -rubber invariants with smaller descendant powers.

Notice that invariants with a negative power descendant insertion are automatically zero. Thus we can employ the topological recursion repeatedly until there are no Ψ_0 insertions.

A.4.2 The Second Reduction—Rubber to Type II

This is achieved through the rigidification process. A rubber invariant with at least one absolute insertion is turned into a type II invariant with one absolute insertion replaced by its product with the divisor D_∞ . The curve class is preserved.

A fiber class rubber integral with no Ψ_0 insertions is turned into type II invariants through rigidification after adding a dilaton insertion.

While a non-fiber class rubber integral with no Ψ_0 insertions is expressed in terms of type II invariants through rigidification after adding a divisor insertion of the form $\pi^*\omega_D$.

In summary, the following is proved in [31].

Proposition A.3 *Any Ψ_∞^k -rubber invariant can be expressed in terms of type II invariants with the same class, or products of type II invariants. The total curve class of each product of invariants is still equal to A , each insertion of the original invariant appears as an insertion of a factor invariant, and the adjacent factor invariants are linked by dual insertions.*

Moreover, if the curve class is non-fiber and the domain is connected, the type II invariants with the same curve class are distinguished and their relative insertions are no bigger than those of the rubber invariant, the number of non-distinguished insertions is not bigger than the number of absolute insertion of the rubber invariant.

A.5 The Reduction Algorithm—Relative to Divisor

Let us now finally describe the algorithm in [31] to determine relative invariants of (P, D_∞) from invariants of D . This step involves both virtual localization and degeneration.

For a fiber class, the moduli space of relative stable maps fibers over D with fiber isomorphic to the moduli space of stable maps to $\mathbb{P}^1$ relative to ∞ . Thus a fiber class invariant can be expressed in terms of the classical cohomology of D and equivariant GW invariant of $(\mathbb{P}^1, \infty)$.

A.5.1 Three Relations and the Final Reduction

Now consider a class A with $\pi_*(A) \neq 0$, i.e., a non-fiber class.

By first applying localization a type I relative invariant of (P, D_∞) can be expressed in terms of twisted rubber invariants (with Ψ_0 insertions) and twisted integrals in the GW theory of D .

Then Proposition A.3 expresses the involved twisted rubber invariants in terms of distinguished type II invariants. And Proposition A.1 expresses the involved twisted integrals in the GW theory of D in terms of ordinary invariants of D .

Thus it suffices to show every distinguished type II invariants can be expressed in terms of invariants of D . This is achieved through the following three relations.

For relative conditions

$$\mu = \{(\mu_i, \delta_{r_i})\}, \quad \nu = \{(v_j, \delta_{s_j})\},$$

consider a (Y, D_∞) type I invariant of the form

$$\left\langle \prod_i \tau_{\mu_i-1}([D_0]\delta_{r_i}) \omega \tau_0([D_\infty]\delta) \Big|_A \nu \right\rangle^{P, D_\infty}, \quad (61)$$

and the distinguished type II invariant

$$\langle \mu | \omega \tau_0([D_\infty]\delta) | \nu \rangle_A. \quad (62)$$

Relation (1) in [31], proved by the degeneration formula for (61) relative to D_0 , expresses the distinguished type II invariant (62) in terms of the type I invariant (61), undistinguished type I invariants of (P, D_∞) with class A , type I invariants of (P, D_∞) with smaller curve classes, and distinguished type II invariants lower than (62).

Relation (2) in [31], proved again by virtual localization, Propositions A.1 and A.3, expresses the type I invariant (61) in terms of distinguished type II invariants with class A but strictly lower than (62), type II invariants with class smaller than A and invariants of D . We briefly review the argument.

In the virtual localization formula, there are contributions from M_0 and $\mathcal{M}_1$, respectively. The contribution from M_0 is a twisted integral in the GW theory of D , which can be reduced to ordinary GW invariants of D by Proposition A.1.

Thus the principal terms of the localization formula come from fixed loci with constant D_0 vertices. Apply Proposition A.3 to the rubber invariants, which have the curve class A and a Ψ_0 insertion, we obtain distinguished type II invariants with curve class A , together with products of type II invariants with the same total curve class and including each original insertion.

Similarly, Relation (2') in [31] expresses undistinguished type I invariants with class A in Relation (1) in [31] in terms of distinguished type II invariants with class A which are lower than (62), products of type II invariants with the same total curve class, and products of GW invariants of D the same total curve class.

By Relations (1), (2), and (2') in [31], a non-fiber class distinguished type II invariant can be inductively computed from products of lower order distinguished type II invariants together with invariants of D with the total curve class. Here the primary induction is on the pair (g, A) , and the secondary induction is on the ordering.

In summary, the following is proved in [31].

Proposition A.4 *Any type I/II invariant can be expressed in terms of invariants of D with the same class, or products of invariants of D . The total curve class of each product of invariants is still equal to A , each insertion of the original invariant appears as an insertion of a factor invariant, and the adjacent factor invariants are linked by dual insertions.*

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